1. Let $f : X \to Y$ be a continuous closed surjection such that $f^{-1}(y)$ is compact for all $y \in Y$. Suppose that $X$ is Hausdorff. Prove that $Y$ is Hausdorff.

2. Let $X$ be the set of real $2 \times 2$ matrices with determinant $= 3$ considered as a subspace of $\mathbb{R}^4$. Is $X$ a manifold? Prove that your answer is correct.

3. Let $X$ be the space obtained by attaching a Moebius strip $M$ to a torus $T = S^1 \times S^1$ by a homeomorphism of the boundary circle of $M$ to the circle $S^1 \times \{(1, 0)\}$ in $T$ (where $(1, 0)$ denotes a point of the standard $S^1$ in $\mathbb{R}^2$). Calculate all the homology groups of $X$.

4. Let the topological space $X_n$ be obtained from $S^n$ by identifying three distinct points, i.e. $X_n = S^n / \{p, q, r\}$. Find the fundamental group of $X_n$.

5. Let $q : X \to Y$ be a quotient map of $X$ onto a connected space $Y$. Assume that $q^{-1}(y)$ is connected for each $y \in Y$.
   
a. Show that $X$ must be connected.
   b. Is $X$ necessarily connected if the map $q$ is only assumed to be continuous and onto and again assuming that $q^{-1}(y)$ is connected for each $y \in Y$? Explain.
1. (a) Show that a connected locally path connected space is path connected.

(b) Show that there are connected spaces which are not path connected.

2. a. Let \( f : S^2 \to \mathbb{R}^3 \) be a smooth embedding. Prove that there exist distinct points \( x, y \in S^2 \) such that the tangent planes to \( f(S^2) \) at \( f(x) \) and \( f(y) \) are parallel.

b. Exhibit a smooth proper embedding \( g : \mathbb{R}^2 \to \mathbb{R}^3 \) such that if \( x \) and \( y \) are any two distinct points in \( \mathbb{R}^2 \) then the tangent planes to \( g(\mathbb{R}^2) \) at \( g(x) \) and \( g(y) \) are not parallel. (A continuous map is proper if the pre-image of any compact set is compact.)

3. Let \( A \) and \( B \) be two round circles in \( \mathbb{R}^3 \) which intersect in a single point. Compute all the homology groups of \( \mathbb{R}^3 - (A \cup B) \).

4. Show that every continuous map \( f : \mathbb{RP}^2 \to \mathbb{RP}^2 \) has a fixed point.

5. Define \( f : \mathbb{R}^1 \to S^1 \) by \( f(x) = e^{i(x - \sqrt{2}\sin(x/\sqrt{2}))} \). Find all the regular points, all the regular values, all the critical points and all the critical values of \( f \).