1. Let $X$ be a topological space and let $d : X \times X \to \mathbb{R}$ be continuous. Suppose that $d$ satisfies the axioms for a metric, that is,

i) for all $x$ and $y$ in $X$, $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$
ii) for all $x$ and $y$ in $X$, $d(x,y) = d(y,x)$
iii) for all $x$, $y$ and $z$ in $X$, $d(x,y) + d(y,z) \geq d(x,z)$.

Prove or give a counterexample to the statements:

a) The topology of $X$ is given by the metric $d$.

b) $X$ is metrizable.

2. Find a CW complex $X$ such that $H_0(X) = \mathbb{Z}$, $H_5(X) = \mathbb{Z} \oplus \mathbb{Z}$, $H_n(X) = 0$, for $n \neq 0$ or 5.

3. Let $M = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^4 + y^4 + z^2 + w^2 = 1\}$ and let $f : M \to \mathbb{R}$ be given by $f(x,y,z,w) = x^3 - z$.

a) Show that $M$ is a manifold.

b) Find the critical points of $f$.

4. Let $X = \mathbb{R}P^2 \times S^1$. Calculate $\pi_1(X)$ and $H_*(X)$ without using the Kunneth theorem.

5. Let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve and let $A$ be the set of $r > 0$ such that the circle of radius $r$ about the origin is tangent to $\Gamma$ at some point. Show that the interior of $A$ is empty, that is, $A$ does not contain any open interval in $\mathbb{R}$. 
1. Let $R^n \subset R^{n+1}$ by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$ and let $R^\infty = \bigcup_{n=1}^{\infty} R^n$. Similarly, let $\Sigma = \bigcup_{n=1}^{\infty} S^n$. Define a topology on $\Sigma \subset R^n$ by $C \subset \Sigma$ is closed iff $C \cap S^n$ is closed in $S^n$ for every $n = 1, 2, \ldots$.

Prove or disprove the following statements:

a) $\Sigma$ is connected.

b) $\Sigma$ is compact.

c) $\Sigma$ is locally compact.

2. Let $f : S^1 \rightarrow X$ and $g : S^1 \rightarrow X$ be continuous maps and suppose that $f$ and $g$ are homotopic. Suppose also that $f(1) = g(1) = x_0$ where 1 is the basepoint in $S^1$ and $x_0$ is a point in $X$. Show that $[f]$ is conjugate to $[g]$ in $\pi_1(X, x_0)$.

3. Let $X$ be the set of pairs of unit vectors $(\vec{x}, \vec{y})$ in $\mathbb{R}^3$ such that $\vec{x} \cdot \vec{y} = \frac{1}{2}$, where $\cdot$ denotes the dot product. Thus $X = \{(\vec{x}, \vec{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 | \|\vec{x}\| = 1, \|\vec{y}\| = 1, \vec{x} \cdot \vec{y} = \frac{1}{2}\}$. Prove that $X$ is a manifold.

4. Let $K$ be a simplicial complex that is homeomorphic to $S^7$ and let $i : L \subset K$ be a subcomplex that is homeomorphic to $RP^2$. Let $X = K \cup L \times I/\sim$ where $(x, 0) \sim i(x)$ and $(x, 1) \sim (x', 1)$ for all $x$ and $x'$ in $L$. Calculate $H_*(X)$.

5. Let $X = R^\infty = \prod_{n=1}^{\infty} R$ with the product topology and let $Y = R^\infty = \prod_{n=1}^{\infty} R$ with the box topology. Prove that $X$ and $Y$ are not homeomorphic.