1. Prove that if $X$ is a (non-empty) countable compact Hausdorff space, then $X$ is not connected. (You may use the fact that an intersection of countably many dense open sets in a compact Hausdorff space is dense.)

2. Let $P$ be a polygon with an even number of sides. Suppose that the sides are identified in pairs in any way whatsoever. Prove that the quotient space is a manifold.

3. Prove that if $M$ is a non-empty compact smooth manifold with boundary, then there is no smooth retraction from $M$ to its boundary $\partial M$. (You may use Sard’s theorem.)

4. Let $X$ be a path-connected topological space. For $n > 1$ an integer, denote by $S_n$ the symmetric group on $n$-letters. State and prove a bijective correspondence between degree $n$ covering spaces of $X$ and group homomorphisms $\pi_1(X) \to S_n$. (Note that finding an accurate statement is part of the problem.)

5. For integers $k, n$ with $1 \leq k \leq n$, let

$$S^n = \{(x_1, ..., x_{n+1}) | x_1^2 + \cdots + x_{n+1}^2 = 1 \} \subset \mathbb{R}^{n+1}$$

and let

$$D_k = \{(x_1, ..., x_{n+1}) | x_1^2 + \cdots + x_k^2 \leq 1, x_{k+1} = \cdots = x_{n+1} = 0\}.$$

Calculate the homology of $X_{k,n} = S^n \cup D_k$. 
1. Prove that the one point compactification $X \cup \{\infty\}$ is Hausdorff if and only if $X$ is locally compact and Hausdorff.

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. The point $(x, y) \in \mathbb{R}^2$ is the stereographic projection of the point $(\xi, \eta, \zeta) \in S^2$ if and only if the three points $(0, 0, 1), (x, y, 0)$, and $(\xi, \eta, \zeta)$ are collinear; this defines a map $\sigma : \mathbb{R}^2 \to S^2, \sigma(x, y) = (\xi, \eta, \zeta)$. Show that $\sigma$ maps $\mathbb{R}^2$ diffeomorphically onto the complement of a point in $S^2$.

3. By definition, a topological group is a set $G$ with both a topology and a group structure, such that the map $G \to G$ sending $x$ to $x^{-1}$ and the map $G \times G \to G$ sending $(x, y)$ to $xy$ are both continuous. Let $1 \in G$ denote the identity of this topological group $G$. Show that $\pi_1(G, 1)$ is abelian.

4. Show that the map $\phi : S^1 \times S^1 \to \mathbb{R}^3$ defined by

$$
\phi(u, v) = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
2 + \cos \beta \\
0 \\
\sin \beta
\end{pmatrix}
$$

for $u = (\cos \alpha, \sin \alpha)$ and $v = (\cos \beta, \sin \beta)$ is an embedding.

5. Let $X$ be a finite simplicial complex of dimension 1. Prove that either $\pi_1X \cong \mathbb{Z}$, or every continuous map $f : X \to X$ homotopic to the identity has a fixed point.