Problem 1. For which positive integers $n$ is $S_n$ isomorphic to a quotient group of $S_{n+1}$?

Problem 2. Show that a finite integral domain $R$ is a field.

Problem 3. Suppose $A$ and $B$ are diagonalizable $n$ by $n$ complex matrices. Show that $AB = BA$ if and only if $A$ and $B$ are simultaneously diagonalizable (i.e., there is an invertible $n$ by $n$ matrix $P$ so $P^{-1}AP$ and $P^{-1}BP$ are diagonal).

Problem 4. Find the minimal polynomial over $\mathbb{Q}$ of each of $\alpha = e^{2\pi i/3} + \sqrt{2}$ and $\beta = 1 + \sqrt{2}$, and describe the relationship between the splitting fields of these two polynomials.

Problem 5. If one knows the characteristic polynomial of an endomorphism $T: V \to V$ of a finite dimensional vector space $V$, can one determine the characteristic polynomials of the exterior powers $\bigwedge^m T: \bigwedge^m V \to \bigwedge^m V$ for all $m > 1$? If so, give a proof; if not, give a counterexample.
All answers must be justified. Show your work on these pages.

Problem 1. For which positive integers $n$ does there exist an order-21 group $G$ having precisely $n$ order-3 subgroups?

Problem 2. Let $R$ be a commutative ring with identity, and let $J$ be the intersection of all the maximal ideals of $R$. Show that $J$ consists of the elements $x \in R$ with the property that, for each $y \in R$, the element $1 - xy$ is a unit in $R$.

Problem 3. Let $T : V \to V$ be an endomorphism of a finite-dimensional vector space. Show that there is a positive integer $m$ so that the kernel of $T^m$ intersects the image of $T^m$ only in the subspace $\{0\}$.

Problem 4. Let $G_1$ and $G_2$ be groups, each of order 128; let $X_1$ and $X_2$ be sets, each of cardinality 8. Suppose each $G_i$ acts faithfully, on the left, on $X_i$. (An action is faithful if no element of the group except the identity fixes every element of the set.) Show that there is a bijection $\psi : X_1 \to X_2$ and an isomorphism $\phi : G_1 \to G_2$ such that $\psi(g \cdot x) = \phi(g) \cdot \psi(x)$ for all $g$ in $G_1$ and all $x$ in $X_1$.

Problem 5. A 7 by 15 matrix of integers determines a $\mathbb{Z}$-linear homomorphism $\phi : \mathbb{Z}^{15} \to \mathbb{Z}^7$. Assume that all 6 by 6 minors of the matrix (determinants of submatrices obtained by choosing 6 of its rows and 6 of its columns) vanish, and that all of the 5 by 5 minors are divisible by 8, and one of these is equal to 8. Describe, as completely as possible, (a) the kernel, (b) the image, and (c) the cokernel (which is $\mathbb{Z}^7 / \text{Image}(\phi)$) of $\phi$. 