1. Let $f : U \to \mathbb{C}$ be analytic, $f(x + iy) = u + iv$. Show that $u^2 - v^2$ is harmonic. How about $uv$? Recall that $g(x + iy)$ is harmonic if $g_{xx} + g_{yy} = 0$.

2. Let $(X, \mu)$ be a measure space. Find a necessary and sufficient condition on $(X, \mu)$ that $L^q(E) \subset L^p(E)$ for all $1 \leq p < q \leq \infty$.

3. Let $U$ be the region in the first quadrant bounded by the unit circle and the straight line from 1 to $i$. Find a conformal map from $U$ to the first quadrant.

4. Let $\{f_n\}$ be a sequence of measurable real-valued functions on $[0, 1]$. Show that the set of $x$ for which $\lim_{n \to \infty} f_n(x)$ exists is measurable.

5. (a) Suppose that $f(z)$ is analytic for $1 \leq |z| \leq 4$. Assume that $|f(z)| \leq 1$ for $|z| = 1$ and $|f(z)| \leq 16$ for $|z| = 4$. Prove that $|f(3i)| \leq 9$.

(b) Prove that there is no non-constant analytic function on the Riemann sphere.
6. Let \( \{ f_n(x) \} \) be a sequence of continuous, strictly positive functions on \( \mathbb{R} \) which converges uniformly to the function \( f(x) \). Suppose that all the functions \( \{ f_n \}, f \) are integrable. Is
\[
\lim_{n \to \infty} \int f_n(x) \, dx = \int f(x) \, dx.
\]
Justify your answer.

7. Let \( f \) be an entire function in the complex plane. Suppose that there is a positive integer \( N \) such that \( \frac{f(z)}{z^N} \to 0 \) as \( |z| \to \infty \). Prove that \( f(z) \) is a polynomial of degree at most \( N - 1 \).

8. a) State and prove the Hölder inequality for real measurable functions on \([0,1]\) in suitably related classes \( L^p \) and \( L^q \).
   b) For \( 1 \leq r < p < \infty \) prove the continuous injection of \( L^p([0,1]) \) into \( L^r([0,1]) \).

9. Compute the integral
\[
\int_0^\infty \frac{\ln x}{x^2 + 1} \, dx.
\]

10. For a (real-valued) function \( f \) on \( \mathbb{R} \), define
\[
f^y(x) = f(x - y), \quad y \in \mathbb{R}.
\]
   a) Suppose \( f \) is a continuous function on \( \mathbb{R} \) with a compact support. Show that \( \| f^y - f \|_{L^\infty(\mathbb{R})} \to 0 \) as \( y \to 0 \).
   b) Show that if \( f \in L^p(\mathbb{R}) \) for some \( p \in [1, \infty) \), then \( \| f^y - f \|_{L^p(\mathbb{R})} \to 0 \) as \( y \to 0 \).
   c) Give an example of a function \( f \) such that \( \| f^y - f \|_{L^\infty(\mathbb{R})} \to 0 \) as \( y \to \infty \).