1. If $M$ is a manifold with boundary, then the double of $M$ is defined by identifying two copies of $M$ along their boundaries by the identity map. Let $M = D^2 - \bigcup_i D_\epsilon(x_i)$ where $\{D_\epsilon(x_i)\}$ are $n$ mutually disjoint open disc of radius $\epsilon$ in the interior of $D^2$ centered at $\{x_i\}$. Let $W$ be the double of $M$. Determine the fundamental group and Euler characteristic of $W$.

2. Let $U_1, U_2, \cdots$ be a countable open covering of a metric space $X$. A refinement is an open covering $V_1, V_2, \cdots$ of $X$ such that for each $i$, $V_i \subset U_j$ for some $j$. Show that there exists a refinement $V_1, V_2, \cdots$ which is star-finite i.e., for each $i$, $V_i \cap V_j \neq \emptyset$ for at most finitely many values of $j$.

3. Let $M$ be a compact smooth manifold of dimension $n$, and let $f : M \to \mathbb{R}^n$ be a smooth map. Show that $f$ has a singular point.

4. Let $RP^2$ and $T$ denote, in this order, the real projective plane and the torus $S^1 \times S^1$. Prove that any map $f : RP^2 \to T$ is homotopic to a constant map.

5. Consider the covering map $f : S^2 \to RP^2$.

Let $X$ be the homotopy pushout of the diagram

$$
\begin{array}{ccc}
S^2 & \xrightarrow{f} & RP^2 \\
\downarrow f & & \downarrow \\
RP^2 & & \\
\end{array}
$$

Calculate the homology groups of $X$. (Recall that a homotopy pushout of a diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ is $(X \times [0,1]) \bigsqcup Y \bigsqcup Z/\sim$ with the quotient topology, where $\sim$ is the smallest equivalence relation satisfying $(x,0) \sim f(x)$, $(x,1) \sim g(x)$ for every $x \in X$.)
1. Let $X$ be obtained by gluing two solid tori $D^2 \times S^1$ along their boundary via the map $f : \partial D^2 \times S^1 \to \partial D^2 \times S^1$ given by $f(x, y) = (y^p x, y)$ where $p$ is a fixed positive integer.

(a) For which values of $p$ can $X$ be given the structure of topological manifold?

(b) Compute $\pi_1(X)$.

2. Consider the space $O_{n+1, 2} = \{(x_1, x_2) | <x_1, x_2> = 0\} \subset S^n \times S^n$ where $S^n$ is the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ with standard inner product. Denote by $p : O_{n+1, 2} \to S^n$ the projection on the first factor. Prove that there is a section $s : S^n \to O_{n+1, 2}$ (i.e. a continuous map $s$ such that $ps = Id$) if and only if $n$ is odd.

3. Let $X, Y$ be topological spaces with $Y$ compact. Let $p : X \times Y \to X$ be the projection to the first factor. Show that $p$ maps each closed subset of $X \times Y$ onto a closed subset of $X$.

4. Let $X$ be the union of the three coordinate axes in $\mathbb{R}^3$. Calculate the homology of $\mathbb{R}^3 - X$.

5. Let $S^2 \subset \mathbb{R}^3$ be the standard unit sphere and $X = \{(x, y, z) \in S^2 : y^2 z = x^3 - xz^2\}$.

Is $X$ a smooth submanifold of $\mathbb{R}^3$?