1. List, by giving generators for them, the ideals of \( \mathbb{Z}[x]/\langle 4, x^2 \rangle \).

2. Let \( p \) be prime and let \( \mathbb{F}_{p^6} \) be the field with \( p^6 \) elements.
   (a) For \( x \in \mathbb{F}_{p^6} \), what are the possible degrees of the minimal polynomial of \( x \) over \( \mathbb{F}_p \)?
   (b) For how many elements of \( \mathbb{F}_{p^6} \) does each of these minimal degrees occur?

3. Let \( V \) be an \( n \)-dimensional vector space, let \( w \) be a vector in \( V \), and let \( \phi \) be in the dual space \( V^* \). Define \( T: V \to V \) by the formula \( T(v) = \phi(v)w \). Find the characteristic polynomial of \( T \).

4. Let \( G \) be a finite group acting transitively on a set \( X \) with \( |X| > 1 \).
   (a) Show that there is some element of \( G \) which fixes no element of \( X \).
   (b) Give a counter-example to this claim when \( G \) and \( X \) are infinite.

5. Let \( p \) be prime. For which abelian groups \( G \) is there a short exact sequence

   \[
   0 \to \mathbb{Z}/p^2\mathbb{Z} \to G \to \mathbb{Z}/p^2\mathbb{Z} \to 0
   \]
1. Let $D$ be a commutative integral domain. Let $A$ be an $n \times n$ matrix with entries in $D$. Show that, if $A^N = 0$ for some positive integer $N$, then $A^n = 0$. 

2. Let $f(x)$ be a degree 4 polynomial with rational coefficients and let $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ be the roots of $f$. Let $\beta = \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4 + \theta_1\theta_4$. Show that $\beta$ is the root of a cubic polynomial with rational coefficients.

3. Consider the bilinear form $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 - x_3y_3$ on $\mathbb{R}^3$. Let $V$ be a 2-dimensional subspace of $\mathbb{R}^3$. What are the possible signatures of the restriction to $V$ of the form $\langle , \rangle$?

4. If $G$ is a group, and $H$ a subgroup of $G$, the normalizer $N(H)$ is defined to be $\{g \in G : gHg^{-1} = H\}$. Let $p$ be prime; let $G$ be the symmetric group $S_p$ and let $H$ be the cyclic subgroup of $G$ generated by the cycle $(1 \ 2 \ 3 \ \cdots \ p)$. What is the order of $N(H)$?

5. Let $K$ be the splitting field of $x^{420} - 1$ over $\mathbb{Q}$. Show that $\text{Gal}(K/\mathbb{Q})$ is abelian. (Do not simply quote the classification of Galois groups of cyclotomic extensions.)