1. Show that there is no simple group of order 84. (Note: $84 = 2^2 \times 3 \times 7$.)

2. Let $A$ be the ring $\mathbb{Z} \times \mathbb{Z}$.
   (1) List, with proof, all prime ideals of $A$.
   (2) Which of them are maximal?

3. Let $A$ be a $5 \times 5$ matrix such that: The characteristic polynomial of $A$ is $(\lambda - 1)^5$, the minimal polynomial is $(\lambda - 1)^3$ and the rank of $A$ – Id is 2. What is the Jordan normal form of $A$?

4. Give examples of pairs $(G, N)$, with $G$ a group and $N$ a normal subgroup, such that
   (1) $G$ is the direct product of $N$ and $G/N$.
   (2) $G$ is a semi-direct product of $N$ and $G/N$, but not the direct product.
   (3) $G$ is not a semi-direct product of $N$ and $G/N$.

5. Let $p$ be a prime number.
   (1) Show that the equation $x^3 = 1$ has either 1 or 3 roots in $\mathbb{Z}/p\mathbb{Z}$.
   (2) For which primes $p$ are there 3 roots? (Give a short, non-tautological, answer.)
   (3) How many roots does $x^3 = 1$ have in the ring $\mathbb{Z}/1001\mathbb{Z}$? (Note: $1001 = 7 \times 11 \times 13$.)
1. List the isomorphism classes of abelian groups of order \(108 = 2^2 \times 3^3\). Do not list two groups which are isomorphic to each other.

2. For which real numbers \(s\) is the matrix \(
\begin{pmatrix}
 s & -1 & 0 \\
 -1 & s & -1 \\
 0 & -1 & s
\end{pmatrix}
\) positive definite?

3. Let \(V\) be a \(\mathbb{C}\)-vector space of dimension \(n \geq 2\) with basis \(e_1, \ldots, e_n\).
   (1) Show that there exists a unique linear map \(\phi : V \otimes V \otimes V \to V \otimes V \otimes V\) such that \(\phi(x \otimes y \otimes z) = y \otimes z \otimes x\).
   (2) What are the eigenvalues of \(\phi\)? Give a basis for each of the eigenspaces.
   (3) What is the minimum polynomial of \(\phi\)? What is the characteristic polynomial of \(\phi\)?

4. Let \(G\) be the group \(GL_3(\mathbb{Z}/7\mathbb{Z})\). You may use without proof that \(|G| = (7^3 - 1)(7^3 - 7)(7^3 - 7^2)\). Let \(g\) be the element
   \(
   \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 2 & 0 \\
   0 & 0 & 4
   \end{pmatrix}
   \) of \(G\).
   (1) How many elements of \(G\) commute with \(g\)?
   (2) How large is the conjugacy class of \(g\)?

5. Let \(\zeta\) be a primitive 11-th root of unity. Let \(K = \mathbb{Q}(\zeta)\). You may assume the following facts: The minimal polynomial of \(\zeta\) is \(x^{10} + x^9 + \cdots + x^2 + x + 1\), and the Galois group of \(K/\mathbb{Q}\) is cyclic of order 10, generated by the symmetry
   \[
   \zeta \mapsto \zeta^2 \mapsto \zeta^4 \mapsto \zeta^8 \mapsto \zeta^5 \mapsto \zeta^{10} \mapsto \zeta^9 \mapsto \zeta^7 \mapsto \zeta^3 \mapsto \zeta^6 \mapsto \zeta.
   \]
   Define
   \[
   \gamma := \zeta - \zeta^2 - \zeta^4 - \zeta^8 - \zeta^5 - \zeta^{10} + \zeta^9 - \zeta^7 + \zeta^3 - \zeta^6.
   \]
   (1) Show that \(\gamma^2\) is rational.
   (2) Show that \(\gamma\) is not rational.