AM.1 Let $G$ have order 84. The number of 7-Sylow subgroups of $G$ must divide 84, and must be 1 mod 7. The only divisor of 84 which is 1 mod 7 is 1. (The complete list is 1, 2, 3, 4, 6, 7, 12, 14, 21, 42 and 84.) So the 7-sylow of $G$ is normal, and $G$ is not simple.

AM.2 The complete list of prime ideals is $\mathbb{Z} \times \{0\}$, $\{0\} \times \mathbb{Z}$, $\mathbb{Z} \times p\mathbb{Z}$ for $p$ a prime and $p\mathbb{Z} \times \mathbb{Z}$. One can check that each of these are prime by seeing that the corresponding quotient rings – namely $\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ are all domains. The maximal ideals are $\mathbb{Z} \times p\mathbb{Z}$ and $p\mathbb{Z} \times \mathbb{Z}$, since these are the ones for which the quotient ring is a field.

To see that these are all the ideals, define $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Since $e_1 e_2 = 0$, any prime ideal $p$ must contain one of $e_1$ and $e_2$. Consider the case that $p$ contains $e_1$; the case that it contains $e_2$ is similar. The ideal $e_1 A$ is $\mathbb{Z} \times \{0\}$ and $A/e_1 A \cong \mathbb{Z}$. So the ideals of $A$ containing $e_1$ are in bijection with the ideals of $\mathbb{Z}$. Specifically, the prime ideals containing $e_1$ are in bijection with the prime ideals of $\mathbb{Z}$, which are known to be $\{0\}$ and $p\mathbb{Z}$.

AM.3 Let $J_k(\lambda)$ be the $k \times k$ Jordan block with $\lambda$’s on the diagonal, 1’s immediately above the diagonal, and 0’s everywhere else. Since the only root of the characteristic polynomial is 1, the Jordan canonical form of $A$ must look like $\bigoplus J_{k_i}(1)$ with $\sum k_i = 5$. Since the minimal polynomial of $A$ is $(\lambda - 1)^3$, the largest of the $k_i$ must be 3. The means that $A$ is either $J_3(1) \oplus J_1(1) \oplus J_1(1)$ or $J_3(1) \oplus J_2(1)$. The ranks of $A - \text{Id}$ for these two possibilities are 2 and 3 respectively, so $A$ has Jordan canonical form $J_3(1) \oplus J_1(1) \oplus J_1(1)$.

AM.4 There are many examples. Here is one set of possibilities.

(1) Take $G = \mathbb{Z}/2 \times \mathbb{Z}/3$ and $N = \mathbb{Z}/2$.
(2) Take $G = S_3$ and $N = A_3 \cong \mathbb{Z}/3$, so $G/N \cong \mathbb{Z}/2$. We can see that $G \not\cong (\mathbb{Z}/2) \times (\mathbb{Z}/3)$ because the right hand side is abelian and the left is not.
(3) Take $G = \mathbb{Z}/4$ and $N = \mathbb{Z}/2$. Note that $N$ has no trivial automorphisms, so the only semidirect product of $N$ by $G/N$ is the direct product; we have $\mathbb{Z}/4 \not\cong (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ because the left hand side has elements of order 4, and the right hand side only has elements of orders 1 and 2.

AM.5 Recall that, when $p$ is prime, $\mathbb{Z}/p\mathbb{Z}$ is a field.

(1) **Solution 1:** In a field, a polynomial of degree $d$ can have at most $d$ roots, so the number of roots of $x^3 - 1$ is either 0, 1, 2 or 3. Since $1^3 = 1$, there is always at least one root. If $\omega$ is a root which is not 1, then $\omega^{-1}$ is another. Moreover, $\omega \neq \omega^{-1}$ because, if they were equal, than $\omega^2 = 1$ and, combining with $\omega^3 = 1$, we deduce that $\omega = 1$. So, if $x^3 - 1$ has a root other than 1, it has at least 2. Thus, there are 1 or 3 roots.

**Solution 2:** The unit group of a finite field is always cyclic. So it either contains 1 or 3 elements of order 3.

(2) As in Solution 2 to the previous part, the unit group of $\mathbb{Z}/p\mathbb{Z}$ is cyclic of order $p - 1$. So it has 1 or 3 elements of order 3 according to whether or not $p \equiv 1 \mod 3$.

(3) Since 7, 11 and 13 are relatively prime, $\mathbb{Z}/1001\mathbb{Z} \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$. So the number of solutions to $x^3 = 1$ in $\mathbb{Z}/1001\mathbb{Z}$ is equal to the number of triples $(y_1, y_2, y_3)$ in $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z}$ for which $y_1^3$, $y_2^3$ and $y_3^3$ are all 1. By the previous parts, there are 3 options for $y_1$, 1 options for $y_2$ and 3 for $y_3$, so there are 9 solutions in $\mathbb{Z}/1001\mathbb{Z}$.
**PM.1** Every finite abelian group is a direct sum of cyclic groups of prime power order, and such direct sums are isomorphic if and only if the multisets of prime powers are identical. Thus, we just need to find all ways to write \(2^23^3\) as a product of prime powers. There are 2 ways to break up \(2^2\), and 3 ways to break up \(3^3\), leading to 6 solutions. The complete list is:

\[
\begin{align*}
\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/27\mathbb{Z} & \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} & \quad \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^3 \\
(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/27\mathbb{Z} & \quad (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} & \quad (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3.
\end{align*}
\]

**Remark:** Some people prefer to write their solutions in Smith normal form, so they would have given the isomorphic list:

\[
\begin{align*}
\mathbb{Z}/108\mathbb{Z} & \quad \mathbb{Z}/36\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} & \quad \mathbb{Z}/12\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2 \\
\mathbb{Z}/54\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \quad \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} & \quad (\mathbb{Z}/6\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}.
\end{align*}
\]

**PM.2** Answer: The matrix is positive definite for \(s > \sqrt{2}\). There are several ways to prove this. One is to use Sylvester’s criterion, which says that \(\begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix}\) will be positive definite if and only if the determinants

\[
\det(s), \quad \det\left(\begin{smallmatrix} s & -1 \\ -1 & s \end{smallmatrix}\right) \quad \text{and} \quad \det\left(\begin{smallmatrix} s & -1 & 0 \\ -1 & s & -1 \\ 0 & -1 & s \end{smallmatrix}\right)
\]

are all positive definite. These determinants are \(s\), \(s^2 - 1\) and \(s^3 - 2s = s(s^2 - 2)\). The last quantity is positive if \(s\) is either \(> \sqrt{2}\) or in \((-\sqrt{2}, 0)\). The other two quantities are positive in the first case, but not the second.

**PM.3** (1) Note that \(V \otimes V \otimes V\) is spanned by elements of the form \(u \otimes v \otimes w\), so any linear map is specified by what it does on such an element. We have required that \(\phi(u \otimes v \otimes w) = v \otimes w \otimes u\), so \(\phi\), if it exists, is unique.

Attempt to define \(\phi\) by \(\sum c_i u_i \otimes v_i \otimes w_i \mapsto \sum c_i v_i \otimes w_i \otimes u_i\). In order to check that this is well defined, we must check that it respects the linear relations in \(V \otimes V \otimes V\): Namely that this recipe assigns the same image to \((pu_1 + qu_2) \otimes v \otimes w\) and to \(pu_1 \otimes v \otimes w + qu_2 \otimes v \otimes w\), the same in the other two tensor factors. This is obvious. **Remark:** We have not used that \(V\) is finite dimensional.

(2) Let \(e_i\) be a basis for \(V\). Then \(e_i \otimes e_j \otimes e_k\) is a basis for \(V \otimes V \otimes V\). Write \(V \otimes V \otimes V\) as the direct sum of the \(n\) subspaces of the form \(\text{Span}_\mathbb{C}(e_i \otimes e_j \otimes e_k)\) and the \((n^3 - n)/3\) subspaces of the form \(\text{Span}_\mathbb{C}(e_i \otimes e_j \otimes e_k, e_j \otimes e_k \otimes e_i, e_k \otimes e_i \otimes e_j)\), where \(i, j, k\) are not all equal.

Write \(\zeta\) for \(e^{2\pi i/3}\). On a subspace of the first form, \(\phi\) acts with eigenvalue 1. On a subspace of the second form, \(\phi\) acts with matrix \(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\), hence with eigenvalues 1, \(\zeta\) and \(\zeta^2\). An explicit list of eigenvectors is

\[
\begin{align*}
e_i \otimes e_j \otimes e_k & \quad + \quad e_j \otimes e_k \otimes e_i & \quad + \quad e_k \otimes e_i \otimes e_j \\
e_i \otimes e_j \otimes e_k & \quad + \quad \zeta e_j \otimes e_k \otimes e_i & \quad + \quad \zeta^2 e_k \otimes e_i \otimes e_j \\
e_i \otimes e_j \otimes e_k & \quad + \quad \zeta^2 e_j \otimes e_k \otimes e_i & \quad + \quad \zeta e_k \otimes e_i \otimes e_j.
\end{align*}
\]

In total, we have eigenvalues 1, \(\zeta\) and \(\zeta^2\), with multiplicities \((n^3 - n)/3 + n\), \((n^3 - n)/3\) and \((n^3 - n)/3\).

(3) The minimal polynomial of \(\phi\) is \(X^3 - 1\). Using the eigenvalues computed in the previous part, the characteristic polynomial is \((X^3 - 1)^{(n^3 - n)/3}(X - 1)^n\).
PM.4 (1) We claim that there are $6^3 = 216$ elements of $G$ commuting with $g$, namely, the diagonal matrices $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$ with $x$, $y$ and $z$ nonzero. Clearly, all of these elements commute with $g$.

Conversely, suppose that $\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ commutes with $g$. So

$$
\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} =
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}
$$

or

$$
\begin{pmatrix} x_{11} & 2x_{12} & 4x_{13} \\ x_{21} & 2x_{22} & 4x_{23} \\ x_{31} & 2x_{32} & 4x_{33} \end{pmatrix} =
\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 2x_{21} & 2x_{22} & 2x_{23} \\ 4x_{31} & 4x_{32} & 4x_{33} \end{pmatrix}.
$$

We see that $(2 - 1)x_{12}$, $(1 - 2)x_{21}$, $(4 - 1)x_{13}$, $(1 - 4)x_{31}$, $(4 - 2)x_{23}$ and $(2 - 4)x_{32}$ are all 0. None of these coefficients is a zero divisor in $\mathbb{Z}/7\mathbb{Z}$, so $x_{ij} = 0$ whenever $i \neq j$. In short, only diagonal matrices can commute with $g$, and a diagonal matrix is in $G$ if and only if its entries are nonzero.

(2) $G$ acts on the conjugacy class of $g$ by conjugation and has one orbit (by definition). The stabilizer of an element is the $6^3$-element group found above, so the size of an orbit is $(7^3 - 1)(7^3 - 7)(7^3 - 7^2)/6^3$. For the curious, this is $(7^2 + 7 + 1)(7 + 1)7^3 = 156408$.

PM.5 The key to both parts is the following computation

$$
\sigma(\gamma) = \sigma(\zeta - \zeta^2 + \zeta^4 - \zeta^8 + \zeta^5 - \zeta^9 + \zeta^7 + \zeta^3 - \zeta^6) = \zeta^2 - \zeta^4 + \zeta^8 - \zeta^5 + \zeta^{10} - \zeta^9 + \zeta^7 - \zeta^3 + \zeta^6 - \zeta = -\gamma
$$

(1) We have $\sigma(\gamma^2) = \sigma(\gamma)^2 = (-\gamma)^2 = \gamma^2$. So $\gamma^2$ is fixed by the Galois group, and is rational.

(2) Solution 1: If $\gamma$ were rational, then $\sigma(\gamma) = \gamma$. However, we also know that $\sigma(\gamma) = -\gamma$, so this implies that $\gamma = 0$. But then the equation $\zeta - \zeta^2 + \zeta^4 - \zeta^8 + \zeta^5 - \zeta^{10} + \zeta^9 - \zeta^7 + \zeta^3 - \zeta^6 = 0$ would be a degree 10 polynomial obeyed by $\zeta$, with rational coefficients, contradicting that we already know the minimal polynomial of $\zeta$ to be a different polynomial of degree 10.

Solution 2: Suppose that $\gamma$ equals the rational number $r$. Then $\zeta - \zeta^2 + \zeta^4 - \zeta^8 + \zeta^5 - \zeta^{10} + \zeta^9 - \zeta^7 + \zeta^3 - \zeta^6 - r = 0$ is a degree 10 polynomial obeyed by $\zeta$, with rational coefficients, and we obtain a contradiction as before.

Remark: In fact, $\gamma^2 = -11$. The quantity $\gamma$ is an example of a Gauss sum, which are important in number theory.