Problem 1. Let $V$ be the $\mathbb{R}$-vector space of all polynomials $ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$. For $p(x), q(x) \in V$, define

$$\langle p(x), q(x) \rangle = p'(0)q(0) + p(0)q'(0),$$

where $p'(x)$ denotes the derivative of $p(x)$.

(a) Verify that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form.
(b) Find an orthogonal basis of $\langle \cdot, \cdot \rangle$.
(c) What is the signature of $\langle \cdot, \cdot \rangle$?

Problem 2. List all the groups of order $2012 = 2^2 \cdot 503$ (503 is prime). List each of the groups exactly once, and also include the abelian groups.

Problem 3. What is the Galois group of $p(x) = x^3 - x + 4$, considered over the ground fields

(a) $\mathbb{Z}/3\mathbb{Z}$;
(b) $\mathbb{R}$;
(c) $\mathbb{Q}$.

Problem 4. Suppose that $V$ is an $n$-dimensional $\mathbb{C}$-vector space of dimension $n$.

(a) Show that there exists a unique $\mathbb{C}$-linear map $\psi : \bigwedge^3 V \to (\bigwedge^2 V) \otimes V$ such that

$$\psi(a \wedge b \wedge c) = (b \wedge c) \otimes a + (c \wedge a) \otimes b + (a \wedge b) \otimes c.$$

(b) Show that $\psi$ is injective.

Problem 5. Suppose that $G$ is a finite group such that the automorphism group $\text{Aut}(G)$ is nilpotent. Show that $G$ is nilpotent.
Problem 1. Let $T$ the $\mathbb{C}$-linear endomorphism of $\mathbb{C}[x]/((x^3 - x)^2)$ given by multiplication by $x^2$.
   
   (a) What is the minimum polynomial of $T$?
   (b) What is the Jordan canonical form of $T$?

Problem 2. List all prime ideals in the ring $\mathbb{Z}[x]/(30, x^2 + 1)$. List each ideal exactly once. Which of these prime ideals are maximal?

Problem 3. Let $\zeta \in \mathbb{C}$ be a primitive $8^{th}$ root of unity.
   
   (a) What is the Galois group $\text{Gal}(\mathbb{Q}(\zeta) : \mathbb{Q})$? You do not have to prove your answer, but indicate how the Galois group acts.
   (b) How many subfields $L$ does $\mathbb{Q}(\zeta)$ have with $[L : \mathbb{Q}] = 2$?
   (c) For each field $L$ in part (b), find an element $\alpha$ such that $\mathbb{Q}(\alpha) = L$. You may express $\alpha$ in terms of $\zeta$.

Problem 4. Suppose that $S$ is a unique factorization domain, and $R$ is a subring of $S$ with the following property: if $f \in S$ and $g \in R$ such that $f$ divides $g$, then $f \in R$. Show that $R$ must also be a unique factorization domain.

Problem 5. Let $G$ be a group with $8 \cdot 7^m$ elements.
   
   (a) Show that the number of $7$-Sylow subgroups of $G$ is 1 or 8.
   (b) Show that if $S_8$ is the symmetric group, and $\varphi : G \to S_8$ is a group homomorphism, then the image of $\varphi$ has $< 60$ elements.
   (c) Show that $G$ is solvable.