Problem 1. How many elements does each of the following groups have?
   (1) $\text{Hom}_Z(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/10\mathbb{Z})$
   (2) $\mathbb{Z}/2\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$
   (3) $(\mathbb{Z} \times \mathbb{Z})/M$, where $M$ is the subgroup generated by $(2, 4)$ and $(4, 2)$

Problem 2. Suppose that $K/\mathbb{Q}, L/\mathbb{Q}$ are Galois extensions with $\text{Gal}(K/\mathbb{Q}) = \text{Gal}(L/\mathbb{Q}) = \mathbb{Z}/6\mathbb{Z}$ and $[K \cap L : \mathbb{Q}] = 2$.
   (1) What is $[KL : \mathbb{Q}]$? Is $KL/\mathbb{Q}$ a Galois extension?
   (2) How many subfields does $KL$ have?

Problem 3. Suppose that $X$ and $Y$ are skew-symmetric $n \times n$ matrices with entries in $\mathbb{R}$. For $A, B \in \text{Mat}_{n,n}(\mathbb{R})$, define $\langle A, B \rangle = \text{Tr}(A^t XBY)$ where $\text{Tr}$ denotes the trace and $A^t$ is the transpose of $A$.
   (1) Show that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form.
   (2) If $n = 2$ and $X = Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, what is the signature of $\langle \cdot, \cdot \rangle$?

Problem 4. Suppose that $G$ is a group of order $p^n$ where $p$ is prime, and suppose that $g \in G$ is not in the center of $G$. Show that there exists an element $h \in G$ such that $h \neq g$, $h$ is conjugate to $g$ and $h$ commutes with $g$.

Problem 5. Let $T$ be a linear endomorphism of a finite-dimensional real vector space $V$. Let $I$ denote the identity endomorphism of $V$. Suppose $\text{rank}(T) + \text{rank}(I - T) = \text{dim}(V)$. Show that $T^2 = T$.
Problem 1. Let $R$ be a commutative ring containing 1, $R^*$ be the set of invertible elements and $m = R \setminus R^*$.

(1) Show that if $m$ is an abelian group under addition, then it is the unique maximal ideal of $R$.

(2) Conversely, suppose that $R$ has a unique maximal ideal. Show that this maximal ideal is equal to $m$.

Problem 2.

(1) List all monic irreducible polynomials of degree 2 and 3 over the field $\mathbb{F}_2$.

(2) Write down all possible rational canonical forms in Mat$_3(\mathbb{F}_2)$.

(3) How many conjugacy classes does the group $GL_3(\mathbb{F}_2)$ have?

Problem 3. Suppose that

$\alpha_1 = \sqrt{4 + 2\sqrt{5}}, \quad \alpha_2 = \sqrt{\frac{9}{2} + 2\sqrt{5}}, \quad \alpha_3 = \sqrt{5 + 2\sqrt{5}}, \quad \alpha_4 = \sqrt{6 + 2\sqrt{5}}$

and let $K_i$ be the smallest normal extension of $\mathbb{Q}$ containing $\alpha_i$ for $i = 1, 2, 3, 4$. The Galois groups Gal($K_i/\mathbb{Q}$), $i = 1, 2, 3, 4$ are equal to

$\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $D_8$ (dihedral group with 8 elements),

but not necessarily in that order. Match each of the 4 normal extensions with its Galois group.

Problem 4. Classify the groups of order $68 = 2^2 \cdot 17$. List each group exactly once.

Problem 5. Suppose that $A : V \to V$ is a linear endomorphism of an $n$-dimensional $\mathbb{C}$-vector space $V$.

(1) Prove that there exists a unique linear map $B : \wedge^2 V \to \wedge^2 V$ such that

$B(v \wedge w) = Av \wedge w + v \wedge Aw$

for all $v, w \in V$.

(2) If $A$ has $n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, what are the eigenvalues of $B$?