1. Let $X$ be a topological space, and $U$ an open subset of $X$. Let $A$ denote $\overline{U} - U$. Show that the interior of $A$ is empty.

2. (a) Show that $H_1(S^1 \times P^2) \cong \mathbb{Z} \times \mathbb{Z}_2$.
(b) Suppose that $S^1 \times P^2$ covers some space, and let $h$ be a covering translation. Show that the induced isomorphism $h_*$ of $H_1(S^1 \times P^2)$ must be the identity.

3. Let $K$ be a 3–dimensional simplicial complex, and let $f : K \to \mathbb{R}P^2 \times S^3$ and $g : \mathbb{R}P^2 \times S^3 \to S^1 \times S^4$ be continuous maps. Show that the composite map $g \circ f$ is null homotopic.

4. Let $f : X \to Y$ be a continuous map. Suppose that $Y$ is connected, and that $f^{-1}(y)$ is also connected, for each $y$ in $Y$.
   (a) Show that if $f$ is a quotient map, then $X$ is connected.
   (b) If $f$ is not a quotient map, must $X$ necessarily be connected? Explain your answer.

5. A space $X = \bigcup_{k=1}^{\infty} X_k$, where each $X_k$ is a simply connected subset of $X$, and $X_k \subset X_{k+1}$, for each $k \geq 1$.
   (a) Show that if each $X_k$ is open in $X$, then $X$ is simply connected.
   (b) Show that if each $X_k$ is closed in $X$ (but not necessarily open), then $X$ need not be simply connected.
1. A space $X$ is constructed by gluing a Moebius band $M$ and an annulus $A$ as follows. One boundary component of $A$ is glued to $\partial M$ by a homeomorphism. The other boundary component of $A$ is glued to the core circle of $M$ by a homeomorphism. Calculate the homology groups of $X$.

2. Let $X$ denote the cone on the real line $\mathbb{R}$. Decide whether $X$ is locally compact. [The cone on a space $Y$ is the quotient of $Y \times I$ obtained by identifying $Y \times \{0\}$ to a point.]

3. Let $M$ be a connected smooth compact $n$–manifold without boundary, and let $N$ be a connected smooth $n$–manifold. Let $f : M \to N$ be a smooth immersion.
   
   (a) Show that $f$ is a covering map.
   
   (b) Show that $S^2 \times S^2$ cannot be immersed into $\mathbb{R}^4$, but can be embedded in $\mathbb{R}^5$.

4. Let $X$ and $Y$ be closed connected oriented $2$–manifolds, and let $f : X \to Y$ be a map of degree $m \neq 0$. Let $A$ denote the image of $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$.

   By considering the cover of $Y$ corresponding to $A$, show that the index of $A$ in $\pi_1(Y, f(x))$ is finite and divides $m$.

5. If $A$ is a subspace of a topological space $X$, a map $f : X \to A$ is a retraction if the restriction of $f$ to $A$ is the identity map. Prove that

   (a) if $X$ is a compact smooth manifold, there is no retraction of $X$ to its boundary.
   
   (b) there is no retraction of $\mathbb{R}P^2$ onto $\mathbb{R}P^1$.
   
   (c) there is no retraction of the plane $\mathbb{R}^2$ onto the "topologist’s sine curve" $W$, where $W = \{(x, \sin \frac{1}{x}) : x \neq 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$.