Qualifying Exam Algebra May 2019
Morning

Instructions: Write your ID number in the upper right corner on each sheet that you hand in. Justify your answers.

(1) Suppose that $A$ is a complex $7 \times 7$ matrix that satisfies the relation $A^5 = 2A^4 + A^3$. Given that the rank of $A$ is 5 and the trace of $A$ is 4, what is the Jordan canonical form of $A$?

(2) Let $S$ be the set of all infinite sequences $(x_1, x_2, x_3, \ldots)$ in $\mathbb{R}$ for which the limit $\lim_{n \to \infty} x_n$ exists. We define an addition and a multiplication on $S$ by

$$(x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots)$$

$$(x_1, x_2, x_3, \ldots) \cdot (y_1, y_2, y_3, \ldots) = (x_1y_1, x_2y_2, x_3y_3, \ldots)$$

(a) Show that $S$ is a commutative ring with identity.
(b) Let $m \subseteq S$ be the set of all $(x_1, x_2, x_3, \ldots)$ with $\lim_{n \to \infty} x_n = 0$. Show that $m$ is a maximal ideal of $S$.

(3) Let $\alpha = \sqrt{2}$ and consider the field $K = \mathbb{Q}(\alpha)$. Suppose that $\beta = p + q\alpha + r\alpha^2$. What are the trace $\text{Tr}_{K/\mathbb{Q}}(\beta)$ and norm $\text{N}_{K/\mathbb{Q}}(\beta)$ of $\beta$? (Your answers should be polynomials in $p, q, r$ with coefficients in $\mathbb{Q}$.)

(4) A Hermitian complex matrix $H$ is said to have signature $(p, q, r)$ if there exists an invertible matrix $P$ so that $P^*HP$ is a real diagonal matrix whose diagonal has $p$ positive entries, $q$ negative entries and $r$ zeroes. Here $P^* = \overline{P}^t$ is the complex transpose matrix. Let $A$ be an $n \times n$ Hermitian matrix. Form the $2n \times 2n$ block matrix

$$M = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

and let $d$ be the nullity of $A$. Prove that the signature of $M$ is $(n - d, n - d, 2d)$.

(5) Suppose that $p > q$ are prime numbers, and that $G$ is a group of order $p^2q^2$.
(a) Show that $p = 3$ or $G$ has a normal subgroup of order $p^2$.
(b) If $p = 3$ (and therefore $q = 2$), show that $G$ has a normal subgroup of order 3 or 9.
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Afternoon

Instructions: Write your ID number in the upper right corner on each sheet that you hand in. Justify your answers.

(1) Suppose that $V$ is an $n$-dimensional $K$-vector space and $v \in V$ is nonzero.
   (a) Show that for $p$ with $0 \leq p < n$ there exists a unique linear map
       \[ \varphi_p : \bigwedge^p V \to \bigwedge^{p+1} V \]
       with the property that
       \[ \varphi_p(w_1 \wedge w_2 \wedge \cdots \wedge w_p) = v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_p. \]
   (b) What is the rank of $\varphi_p$?

(2) Suppose that $L/K$ is a field extension, and $A, B \in \text{Mat}_{n,n}(K)$ are $n \times n$ matrices
    with entries in $K$. Suppose that there exists an invertible matrix $C \in \text{Mat}_{n,n}(L)$
    with $CAC^{-1} = B$. Show that there exists an invertible matrix $D \in \text{Mat}_{n,n}(K)$ with
    $DAD^{-1} = B$. (Hint: Think about the invariant factors or the rational canonical
    form of the matrix $A$.)

(3) Let $a \in \mathbb{Q}$ and let $n \geq 2$ be an integer. Prove that the Galois group of $x^n - a$ over $\mathbb{Q}$
    is solvable. Prove also that its order is at most $n(n - 1)$.

(4) Suppose that $\mathcal{P}$ is a property of some groups. We say that a group is virtually $\mathcal{P}$ if
    it has a finite index subgroup which has property $\mathcal{P}$. Prove that if the group $G$ is
    virtually solvable, then so is every subgroup and every quotient group of $G$.

(5) Let $R$ be a commutative ring with identity and $M$ be an $R$-module (not necessarily
    finitely generated). Suppose that $a_1, a_2, \ldots, a_n \in R$ such that $(a_1, a_2, \ldots, a_n) = R$
    and $a_i a_j M = 0$ for $i \neq j$. Show that we have a direct sum decomposition
    \[ M = a_1 M \oplus a_2 M \oplus \cdots \oplus a_n M. \]