(1) Suppose that $A$ is a complex $n \times n$ matrix. Show that $A$ is nilpotent if and only if $A$ and $2A$ are similar.

(2) Suppose that $G$ is a finite group of order $|G| = p^d n$ where $d$ and $n$ are positive integers and $p$ is a prime that does not divide $n$. Show that $G$ contains an element of order $p$ such that the cardinality of its conjugacy class divides $n$.

(3) For each of the following assertions concerning abelian groups, give an example of a nonzero abelian group $A$ satisfying this assertion.
   (a) $A \otimes_{\mathbb{Z}} A$ is isomorphic to $A \oplus A$;
   (b) $A$ is not finitely generated and $A \otimes_{\mathbb{Z}} A$ is isomorphic to $A$;
   (c) $A \otimes_{\mathbb{Z}} A = 0$.

(4) Determine whether the field extension $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ is Galois. If it is Galois, determine the Galois group.

(5) Suppose that $d$ is a positive integer and consider the ring $R = \mathbb{Z}[i]/(3^d)$. Here, $\mathbb{Z}[i]$ is the ring of Gaussian integers with $i^2 = -1$, and $(3^d)$ is the principal ideal in $\mathbb{Z}[i]$ generated by $3^d$.
   (a) How many elements does $R$ have?
   (b) How many elements does the group of units $R^\times$ have?
(1) Describe all prime ideals in \( \mathbb{C}[x, y, z]/(y^4 - z^3, y^2) \).

(2) Suppose that \( V \) is an \( n \)-dimensional complex vector space and \( A : V \to V \) is a linear map. Consider the linear map \( B = A \otimes I - I \otimes A \) from \( V \otimes V \) to itself, where \( I \) is the identity map. Show that the rank of \( B \) is at most \( n^2 - n \).

(3) For each of the following statements, prove that it is true or give a counterexample together with an explanation of why it is a counterexample.
   (a) If \( R \) is a commutative ring with \( 1 \neq 0 \) and every submodule of \( R \) (viewed as an \( R \)-module) is free, then \( R \) is a Principal Ideal Domain (PID).
   (b) Any PID must either have 1 or infinitely many prime ideals.

(4) In a finite group \( G \) we have \( g^2 h^2 = h^2 g^2 \) for all \( g, h \in G \). Show that the group \( G \) is solvable.

(5) Consider the polynomial \( p(x) = x^{44} - 1 \in \mathbb{F}_3[x] \).
   (a) Show that \( p(x) \) splits over the field \( \mathbb{F}_{3^{10}} \).
   (b) How many roots does \( p(x) \) have in each of the fields \( \mathbb{F}_3, \mathbb{F}_{3^2}, \mathbb{F}_{3^5} \)?
   (c) If we write \( p(x) \) as a product of irreducible polynomials, how many factors of degree 10 do we have? (You do not have to find an explicit factorization.)