Problem 1: Find all entire functions $f$ with the property that $|f|$ is harmonic.

Problem 2:
(a) Find a quadratic function $Q(z)$ such that
$$5 - 4 \cos \theta = Q(z)/z \quad \text{for } z = e^{i\theta}, \theta \in \mathbb{R}.$$ 
(b) Use your representation in (a) and contour integration to compute the value of the integral
$$\int_0^{2\pi} \frac{d\theta}{5 - 4 \cos \theta}.$$ 

Problem 3: Let $\mathcal{F}$ denote the set of functions that are analytic on a neighborhood of the closed unit disk $|z| \leq 1$. Find
$$\sup \{|f(0)| : f \in \mathcal{F} \text{ with } f(1/2) = 0 = f(1/3) \text{ and } |f(z)| \leq 1 \text{ when } |z| = 1\}.$$ 
Is the supremum attained?

Hint: The function $\frac{z^{1/2} - z^{-1/3}}{1 - z^{1/2} z^{-1/3}}$ is useful.

Problem 4: Let $\Omega$ be a bounded open subset of $\mathbb{C}$ whose boundary $\partial \Omega$ is a $C^1$ simple closed curve and let $p$ be a monic polynomial of degree $n$ with distinct roots $w_1, \ldots, w_n$ all contained in $\Omega$. Let $f$ be an analytic function on a neighborhood of $\overline{\Omega}$. Let $q$ be the entire function defined by
$$q(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{p(\zeta) - p(z)}{p(\zeta)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$ 
(a) Show that $q$ is a polynomial of degree $\leq n - 1$.
(b) Evaluate $q(w_j)$ for $j = 1, \ldots, n$.

Problem 5: Let $F$ be a finite subset of $\mathbb{C}$ and let $h : \mathbb{C} \setminus F \to \mathbb{C} \setminus F$ be an analytic bijection.
(a) Suppose $z_j \in \mathbb{C} \setminus F, z_j \to z_+ \in F \cup \{\infty\}, h(z_j) \to w_+$. Show that $w_+ \in F \cup \{\infty\}$.
(b) Show that $h$ must be a rational function.
(c) Must $h$ be a linear fractional transformation?
Department of Mathematics, University of Michigan
Analysis Qualifying Exam, January 8, 2019
Afternoon Session, 2.00-5.00 PM

Note: Lebesgue measure is assumed throughout.

Problem 1: Let \( f_n : (0,1) \rightarrow \mathbb{R}, \ n = 1,2, \ldots, \) be a sequence of measurable functions and consider the sequence \( g_n : \mathbb{R} \rightarrow \mathbb{R}, \ n = 1,2, \ldots, \) of functions defined by
\[
g_n(x) = \frac{f_1(x) + \cdots + f_n(x)}{n}.
\]
(a) Suppose that there is a function \( f : (0,1) \rightarrow \mathbb{R} \) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) for almost all \( x \in (0,1) \). Show that \( \lim_{n \to \infty} g_n(x) = f(x) \) for almost all \( x \in (0,1) \).
(b) Let \( f_n(x) = \sin \pi nx, \ 0 < x < 1 \). Show that \( \lim_{n \to \infty} g_n(x) = 0 \) for almost all \( x \in (0,1) \), but also that for almost all \( x \in (0,1) \) the limit \( \lim_{n \to \infty} f_n(x) \) fails to exist.

Problem 2: Let \( p \) satisfy \( 1 < p < \infty \) and \( f : (0,\infty) \rightarrow \mathbb{R} \) be in \( L^p(0,\infty) \). Prove that
\[
\int_0^\infty \left( \int_0^1 |f(sx)| \, ds \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p \, dx.
\]
Hint: You may quote Minkowski's integral inequality
\[
\left( \int_T \left( \int_Y h(t,y) \, dy \right)^p \, dt \right)^{1/p} \leq \int_Y \left( \int_T |h(t,y)|^p \, dt \right)^{1/p} \, dy.
\]

Problem 3: Let \( c : (0,\infty) \rightarrow \mathbb{R} \) be non-negative and measurable such that the function \( x \mapsto (1+x)c(x) \) is integrable.
(a) Prove that the function \( w : [0,\infty) \rightarrow \mathbb{R} \) defined by \( w(x) = \int_x^\infty c(x') \, dx' \) is continuous and decreasing with \( \lim_{x \to \infty} w(x) = 0 \).
(b) Show that the function \( w(\cdot) \) is integrable.

Problem 4: Let \( f : [0,1] \rightarrow \mathbb{R} \) be defined by \( f(0) = 0 \) and \( f(x) = x^2 \sin(x^{-2}) \) for \( 0 < x \leq 1 \). Is \( f \) of bounded variation?

Problem 5: Let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be an integrable function with integral equal to 1. For \( N = 1,2, \ldots, \) define the functions \( \phi_N : \mathbb{R} \rightarrow \mathbb{R} \) by \( \phi_N(x) = N \phi(Nx), \ x \in \mathbb{R} \).
Prove that the Fourier transforms
\[
\hat{\phi}_N(\xi) = \int_{-\infty}^\infty e^{i\xi x} \phi_N(x) \, dx
\]
converge uniformly on any finite interval as \( N \to \infty \) to the function that is identically 1 on \( \mathbb{R} \). Is convergence also uniform on all of \( \mathbb{R} \)? Explain your answer.