Problem 1: Suppose \( f(\cdot) \) is holomorphic on the punctured unit disk \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) and satisfies the inequality

\[
|f(z)| \leq \log \frac{1}{|z|}, \quad 0 < |z| < 1.
\]

Show that \( f(\cdot) \equiv 0 \).

Problem 2: Use contour integration to evaluate the integral

\[
\int_0^\infty \frac{x^\alpha}{(x+2)^2} \, dx \quad \text{for} \quad -1 < \alpha < 1.
\]

Sketch the contour you use and show all estimates.

Problem 3: Find the Laurent series expansion about the origin of the function

\[
f(z) = \frac{z^2 + 3z + 5}{2z^2 - 5z - 3}
\]

which converges in the annulus \( 1 < |z| < 2 \).

Problem 4: Is there a conformal mapping from \( \mathbb{C}\{0\} \) onto the punctured disk \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \)? Either prove no such mapping exists or exhibit one.

Problem 5: Let \( f : \mathbb{C} \to \mathbb{C} \) be the function \( f(z) = z^3 - \exp[z^3 - 4] + 1 \).

(a) Find how many solutions counted according to multiplicity there are to the equation \( f(z) = 0 \) in the disk \( |z| < 3/2 \).

(b) Find the number of distinct solutions to \( f(z) = 0 \) in \( |z| < 3/2 \).
Problem 1: Let $E \subset (0, 1)$ be a measurable set such that for any interval $(a, b) \subset (0, 1)$, there exists an interval $(c, d) \subset (a, b) \setminus E$ with

$$d - c \geq \frac{a}{10}(b - a).$$

Prove that $m(E) = 0$.

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that

$$f(y) \leq f(x) + (x^2 + y^2)(x - y) \quad \text{for } -\infty < y < x < \infty.$$

Show that the derivative function $x \to f'(x)$ exists a.e. on $\mathbb{R}$.

Problem 3: Let $r_n, n = 1, 2, \ldots,$ be an enumeration of the rationals in the interval $[0, 1]$ and consider the function $f : [0, 1] \to \mathbb{R} \cup \{\infty\}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}}, \quad 0 \leq x \leq 1.$$

Show that $f \in L^2(0, 1)$.

Problem 4: Let $f_n, n = 1, 2, \ldots,$ be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{1}{n} \left(1 - \frac{x}{n}\right)^n e^x, \quad 0 < x < n, \quad f_n(x) = 0, \quad x \geq n.$$

Prove that the sequence $a_n, n = 1, 2, \ldots,$ given by

$$a_n = \int_0^\infty f_n(x) \, dx$$

converges and identify $a_\infty = \lim_{n \to \infty} a_n$.

Problem 5: Suppose $f$ is a continuously differentiable function on $\mathbb{R}$ satisfying $f(0) = 0, \, |f(x)| \leq |x|^{-1/2}, \, x \neq 0$. Let $g$ be in $L^1(\mathbb{R})$.

(a) Show there is a constant $C$ such that $m\{|g| > \alpha\} \leq C/\alpha$ for all $\alpha > 0$.

(b) Show that the function $h(x) = f(g(x))$ is in $L^1(\mathbb{R})$. 