Problem 1: Suppose $f(\cdot)$ is holomorphic on the punctured unit disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and satisfies the inequality

$$|f(z)| \leq \log \frac{1}{|z|}, \quad 0 < |z| < 1.$$ 

Show that $f(\cdot) \equiv 0$.

Solution: Since $\lim_{z \to 0} z f(z) = 0$, the point $z = 0$ is a removable singularity, whence $f(\cdot)$ is holomorphic on the unit disk. For $0 < R < 1$ let $D_R = \{z \in \mathbb{C} : |z| < R\}$. Since $f(\cdot)$ is holomorphic on the closure of $D_R$ the maximum principle implies that

$$\max\{|f(z)| : z \in D_R\} = \max\{|f(z)| : z \in \partial D_R\} \leq \log \frac{1}{R}.$$ 

The result follows by letting $R \to 1$.

Problem 2: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{x^\alpha}{(x+2)^2} \, dx \quad \text{for } -1 < \alpha < 1.$$ 

Sketch the contour you use and show all estimates.

Solution: Let $R, \varepsilon > 0$ be such that $0 < \varepsilon < R/2$ and define the contour $\gamma_{\varepsilon,R}$ as follows: the union of the half circle $C_\varepsilon = \varepsilon e^{i\theta}, \pi/2 < \theta < 3\pi/2$, the line segments $L_{+,R}^+ = \{x + i: 0 < x < R\}$ and $L_{-,R}^- = \{x - i: 0 < x < R\}$, and the arc of the circle $C_R = \{|z| = \sqrt{R^2 + \varepsilon^2}\}$, which lies outside the region $\{z : \Re z > 0, |3z| < \varepsilon\}$. The function $f(z) = z^\alpha/(z+2)^2$ is analytic inside $\gamma_{R,\varepsilon}$ except for a pole at $z = -2$. By the residue theorem we have

$$\frac{1}{2\pi i} \int_{\gamma_{\varepsilon,R}} f(z) \, dz = \text{Res} f(z)|_{z=-2} = \alpha z^{\alpha-1}|_{z=-2} = -\alpha 2^{\alpha-1} e^{i\alpha \pi}.$$ 

We also have

$$\int_{C_\varepsilon} f(z) \, dz \leq \frac{\pi \varepsilon^{1+\alpha}}{4}, \quad \left| \int_{C_R} f(z) \, dz \right| \leq \frac{2\pi R^{1+\alpha}}{(R-2)^2}.$$ 

Since $-1 < \alpha < 1$ these integrals converge to 0 as $\varepsilon \to 0$ and $R \to \infty$. We also have

$$\lim_{\varepsilon \to 0, R \to \infty} \int_{L_{+,R}^+} f(z) \, dz = I, \quad \lim_{\varepsilon \to 0, R \to \infty} \int_{L_{-,R}^-} f(z) \, dz = -e^{2\alpha i} I,$$
where \( I \) is the integral

\[
I = \int_0^\infty \frac{x^\alpha}{(x+2)^2} \, dx.
\]

We conclude that

\[
[1 - e^{2\alpha \pi i}]I = -2\pi \alpha i 2^{\alpha-1} e^{i \alpha \pi}, \quad \text{so} \quad I = \frac{\pi \alpha 2^{\alpha-1}}{\sin \pi \alpha},
\]

provided \( \alpha \neq 0 \). We can get the \( \alpha = 0 \) case by letting \( \alpha \to 0 \) in the formula using \( \lim_{x \to 0} \sin x/x = 1 \), yielding \( I = 1/2 \).

**Problem 3:** Find the Laurent series expansion about the origin of the function

\[
f(z) = \frac{z^2 + 3z + 5}{2z^2 - 5z - 3}
\]

which converges in the annulus \( 1 < |z| < 2 \).

**Solution:** We have

\[
\frac{1}{2z^2 - 5z - 3} = \frac{1}{(2z + 1)(z - 3)} = \frac{1}{7} \left[ \frac{1}{z - 3} - \frac{2}{2z + 1} \right].
\]

Now

\[
\frac{1}{z - 3} = -\frac{1}{3} [1 - z/3]^{-1} = -\frac{1}{3} \sum_{n=1}^{\infty} (z/3)^n \quad \text{converges for} \quad |z| < 3.
\]

Similarly we have that

\[
\frac{2}{2z + 1} = \frac{1}{z} [1 + 1/2z]^{-1} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n (2z)^n \quad \text{converges for} \quad |z| > 1/2.
\]

The Laurent series for \( f(z) \) is therefore

\[
f(z) = -\frac{1}{7} (z^2 + 3z + 5) \left[ \frac{1}{3} \sum_{n=1}^{\infty} (z/3)^n + \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n (2z)^n \right].
\]

**Problem 4:** Is there a conformal mapping from \( \mathbb{C} \setminus \{0\} \) onto the punctured disk \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \)? Either prove no such mapping exists or exhibit one.

**Solution:** Let \( f : \mathbb{C} \setminus \{0\} \to \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) be the conformal mapping. Since \( \lim_{z \to 0} zf(z) = 0 \) the mapping \( f \) extends to a holomorphic function \( f : \mathbb{C} \to \{ z \in \mathbb{C} : |z| < 1 \} \). Since \( f(\cdot) \) is an entire function which is bounded Liouville’s theorem implies \( f(\cdot) \equiv \text{constant} \). Hence \( f(\cdot) \) cannot be conformal.

**Problem 5:** Let \( f : \mathbb{C} \to \mathbb{C} \) be the function \( f(z) = z^3 - \exp[z^3 - 4] + 1 \).
(a) Find how many solutions counted according to multiplicity there are to the equation $f(z) = 0$ in the disk $|z| < 3/2$.

(b) Find the number of distinct solutions to $f(z) = 0$ in $|z| < 3/2$.

**Solution:** (a) Taking $g(z) = z^3$ we have that

$$|f(z) - g(z)| \leq 1 - e^{-(3/2)^3 - 4} < |g(z)| \quad \text{for } |z| = 3/2.$$ 

By the Rouché theorem $f(\cdot)$ and $g(\cdot)$ have the same number of zeros in the disk $|z| < 3/2$, counted according to multiplicity. Hence $f(\cdot)$ has 3 zeros.

(b) We have that

$$f'(z) = 3z^2 \{1 - \exp[z^3 - 4]\},$$

so $f'(\cdot)$ has exactly one zero in the disk $|z| < 3/2$ at $z = 0$. Since $z = 0$ is not a zero of $f(\cdot)$ it follows that all three zeros of $f(\cdot)$ are distinct.
Problem 1: Let $E \subset (0, 1)$ be a measurable set such that for any interval $(a, b) \subset (0, 1)$, there exists an interval $(c, d) \subset (a, b) \setminus E$ with 
\[ d - c \geq \frac{a}{10}(b - a). \]
Prove that $m(E) = 0$.

Solution: Applying the Lebesgue Differentiation Theorem to the indicator function $1_E$, we see that 
\[ \lim_{h \to 0^+} \frac{m(E \cap (x-h, x+h))}{2h} = 1_E(x) \]
for a.e. $x \in (0, 1)$. Therefore, it is enough to show that 
\[ \limsup_{h \to 0^+} \frac{m(E \cap (x-h, x+h))}{2h} < 1 \]
for all $x \in (0, 1)$. Fix $x \in (0, 1)$ and apply the assumption on $E$ with $(a, b) = (x-h, x+h)$. This yields 
\[ m(E \cap (x-h, x+h)) \leq 2h - (d - c) \leq \left(1 - \frac{x}{20}\right) \cdot 2h \]
for any $h < \frac{x}{2}$. The result follows.

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that 
\[ f(y) \leq f(x) + (x^2 + y^2)(x-y) \quad \text{for} \quad -\infty < y < x < \infty. \]
Show that the derivative function $x \to f'(x)$ exists a.e. on $\mathbb{R}$.

Solution: Let $N \in \mathbb{N}$. It is enough to prove that $f$ is a.e. differentiable on the interval $(-N, N)$. Note that the assumption on $f$ implies 
\[ f(y) \leq f(x) + 2N^2(x-y) \quad \text{for} \quad -N < y < x < N, \]
and so the function $g(x) = f(x) + 2N^2x^2$ is increasing on the interval $(-N, N)$. As any increasing function is a.e. differentiable, $f(x) = g(x) - x^2$ is a.e. differentiable as well.

Problem 3: Let $r_n$, $n = 1, 2, \ldots$, be an enumeration of the rationals in the interval $[0, 1]$ and consider the function $f : [0, 1] \to \mathbb{R} \cup \infty$ defined by 
\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x-r_n|^{1/3}}, \quad 0 \leq x \leq 1. \]
Show that $f \in L^2(0, 1)$. 
Solution: Denote

\[ f_n = \frac{1}{|x - r_n|^{1/3}}. \]

Then for any \( n \in \mathbb{N} \),

\[ \|f_n\|^2 = \int_0^1 \frac{1}{|x - r_n|^{2/3}} \, dx \leq 2 \int_0^1 \frac{1}{x^{2/3}} \, dx = 6. \]

Hence,

\[ \left( \int_0^1 (f(x))^2 \, dx \right)^{1/2} = \left( \lim_{N \to \infty} \int_0^1 \left( \sum_{n=1}^N \frac{1}{n^2} f_n(x) \right)^2 \, dx \right)^{1/2} \]

\[ = \lim_{N \to \infty} \left( \int_0^1 \left( \sum_{n=1}^N \frac{1}{n^2} f_n(x) \right)^2 \, dx \right)^{1/2} = \lim_{N \to \infty} \left\| \sum_{n=1}^N \frac{1}{n^2} f_n(x) \right\|_2 \]

\[ \leq \lim_{N \to \infty} \sum_{n=1}^N \left\| \frac{1}{n^2} f_n(x) \right\|^2 \]

\[ \leq \sqrt{6} \sum_{n=1}^\infty \frac{1}{n^2} < \infty, \]

where the first equality follows from the Monotone Convergence Theorem, and the second one from the continuity of \( g(x) = x^{1/2} \). The first inequality is the triangle inequality for \( \| \cdot \|_2 \), i.e. the Minkowski’s inequality.

Problem 4: Let \( f_n, \ n = 1, 2, \ldots, \) be the sequence of functions on \((0, \infty)\) defined by

\[ f_n(x) = \frac{1}{n} \left( 1 - \frac{x}{n} \right)^n e^{x/n}, \ 0 < x < n, \quad f_n(x) = 0, \ x \geq n. \]

Prove that the sequence \( a_n, \ n = 1, 2, \ldots, \) given by

\[ a_n = \int_0^\infty f_n(x) \, dx \]

converges and identify \( a_\infty = \lim_{n \to \infty} a_n. \)

Solution: Using a change of variables \( y = \frac{x}{n} \), we obtain

\[ a_n = \int_0^1 (1 - y)^n e^{ny} \, dy. \]

Note that \( g(y) = (1 - y)e^y < g(0) = 1 \) for any \( y \in (0, 1] \). Indeed, \( g'(y) = -ye^y < 0 \) on this interval, and so the previous assertion follows from the Mean Value Theorem. Therefore, \( (1 - y)^n e^{ny} < 1 \) for any \( y \in (0, 1] \), which means that \( h(y) = 1 \) is an integrable majorant for this sequence. As

\[ \lim_{n \to \infty} (1 - y)^n e^{ny} = 0 \quad \text{for any} \ y \in (0, 1], \]
Lebesgue Dominated Convergence Theorem yields
\[
\lim_{n \to \infty} \int_0^1 (1 - y)^n e^{ny} \, dy = \int_0^1 \lim_{n \to \infty} (1 - y)^n e^{ny} \, dy = 0.
\]

**Problem 5:** Suppose \( f \) is a continuously differentiable function on \( \mathbb{R} \) satisfying \( f(0) = 0, \ |f(x)| \leq |x|^{-1/2}, \ x \neq 0 \). Let \( g \) be in \( L^1(\mathbb{R}) \).

(a) Show there is a constant \( C \) such that \( m(\{|g| > \alpha\}) \leq C/\alpha \) for all \( \alpha > 0 \).

(b) Show that the function \( h(x) = f(g(x)) \) is in \( L^1(\mathbb{R}) \).

**Solution:**
(a) Let \( E_\alpha = \{x: |g(x)| > \alpha\} \). Then
\[
m(E) \leq \int_{E_\alpha} \frac{1}{\alpha} |g(x)| \, dx \leq \frac{1}{\alpha} \|g\|_1.
\]

(b) By the previous part, \( m(E_1) < \infty \). Since for any \( x \in E_1, \ |f'(g(x))| \leq |g(x)|^{-1/2} < 1, \)
\[
\int_{E_1} |f(g(x))| \, dx \leq m(E_1) < \infty.
\]
Also, since \( f \) is continuously differentiable and \( f(0) = 0, \)
\[|f(x)| \leq K|x| \quad \text{for any } x \in [-1, 1] \]
with \( K = \max_{x \in [-1,1]} |f'(x)| \). In view of the previous inequality,
\[
\int_{\mathbb{R} \setminus E_1} |f(g(x))| \, dx \leq \int_{\mathbb{R} \setminus E_1} K|g(x)| \, dx \leq K\|g\|_1 < \infty.
\]
As the integral over \( E_1 \) is also finite, the proof is complete.