1. Which of the following groups are fundamental groups of compact surfaces without boundary? For those which are, classify the surface:
   (a) \(\langle a, b, c \mid abca^{-1}b^{-1}c \rangle\)
   (b) \(\langle a, b, c, d \mid abcdab^{-1}c^{-1}d^{-1} \rangle\)
   (c) \(\langle a, b, c \mid abcb^{-1}a^{-1}c \rangle\).

2. Describe a set of free generators of the subgroup of the free group on two generators \(a, b\) generated by \(b\) and all the conjugates of \(a^2, b^2, (ab)^3\). Is this a normal subgroup?

3. Let \(S^1\) be the unit circle in \(\mathbb{C}\). Let a space \(X\) be obtained from \(S^1 \times [0, 1]\) by identifying \((x, \epsilon) \sim (ix, \epsilon)\) for all \(x \in S^1, \epsilon = 0, 1\), with the quotient topology. Compute \(\pi_1(X)\) in terms of generators and defining relations.

4. Let \(T\) be the set of all 4-tuples \((x, y, z, t) \in \mathbb{C}^4 \setminus \{(0, 0, 0, 0)\}\) satisfying \(xy + zt = 0\) and let \(X\) be the quotient of \(T\) by identifying \((x, y, z, t) \sim (\lambda x, \lambda y, \lambda z, \lambda t)\) for all \(\lambda \in \mathbb{C} \setminus \{0\}\). Let \(Y\) be the subset of \(X\) consisting of points represented by tuples of the form \((x, y, z, 0)\) and let \(Z\) be the one point subset represented by \((0, 0, 1, 0)\).
   (a) Prove that \(\emptyset \subset Z \subset Y \subset X\) (possibly with some identity inclusions inserted) is a CW filtration of \(X\).
   (b) Compute the homology of \(X\).

5. Let \(Q\) be the quotient of \(\mathbb{C}^n\) by identifying \((x_1, \ldots, x_n) \sim (\lambda x_1, \ldots, \lambda x_n)\) for all \(\lambda \in \mathbb{C} \setminus \{0\}\) with \(|\lambda| = 1\). For what values of \(n \in \mathbb{N}\) is \(Q\) a topological manifold without boundary?
1. The following questions are True/False. State whether the given statement is always true (“True”) or only sometimes or never true (“False”), and give a brief reason why it is true, or a counter-example if your answer is “False”.

   a) If $M$ is a compact manifold with boundary $\partial M = N$, and $N$ is orientable, then so is $M$.

   b) If $f : M^5 \rightarrow N^3$ is a smooth mapping of compact, differentiable manifolds of the indicated dimensions, and $f(M)$ contains an open set of $N$, then for some $y \in N$, $f^{-1}(y) \subset M$ is a differentiable manifold of dimension 2.

   [10]

2. Let $f$ be a smooth function in a neighborhood of $0 \in \mathbb{R}^2$. Suppose $df(0) = 0$, i.e., \( \frac{\partial f}{\partial x_i}(0) = 0, i = 1, 2 \). Show that the number of positive, negative and zero eigenvalues of the Hessian of $f$ at $0$, $Hess(f)(0) = (\frac{\partial^2 f}{\partial x_i \partial x_j})(0)$, are independent of the choice of local coordinates $(x_1, x_2)$ at $0$.

   [10]

3. Let $M = T^2 = S^1 \times S^1$, the 2-torus. Calculate the first DeRham cohomology group $H^1_{DR}(M)$ of $M$. (Do not just quote the DeRham isomorphism to do this.)

   [10]
4. A smooth two form $\beta$ on a manifold $M$ is called non-degenerate if the $2n$ form $\beta^n = \beta \wedge \cdots \wedge \beta$ (n factors) is nowhere 0, where $\dim M = 2n$.

a) Show that the form

$$\omega = \frac{dx \wedge dy}{(1 + |z|^2)^2},$$

on $\mathbb{R}^2 = \mathbb{C}$, where $z = x + iy$, has a smooth extension $\Omega$ to $S^2$ which is non-degenerate.

b) If $f(z) := z^3 + 2\bar{z}$, what is $\int_{S^2} f^*\Omega$?

c) Show that there is no closed non-degenerate two form on $S^4$.
(You may use that $H^2_{DR}(S^{2n}) = 0$, for $n > 1$.)

5. Consider $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates $z_j = x_j + iy_j, j = 1, 2$, and consider the vector-fields

$$\xi_1 = y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2},$$

$$\xi_2 = -x_2 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2},$$

$$\xi_3 = -y_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2}.$$

a) Show that each $\xi_j, j = 1, 2, 3$, is tangent to the unit sphere

$$S^3 = \{ (x_1, y_1, x_2, y_2) | x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 \} \subset \mathbb{R}^4.$$  
(Please just show that $\xi_1$ is tangent to $S^3$; the other cases are similar.)

b) Show that $[\xi_1, \xi_2] = -2 \xi_3, [\xi_2, \xi_3] = -2 \xi_1, [\xi_3, \xi_1] = -2 \xi_2$.
(Please just show that $[\xi_1, \xi_2] = -2 \xi_3$; the other cases are similar.)

c) Show that $\mathfrak{g} = \mathbb{R}\xi_1 + \mathbb{R}\xi_2 + \mathbb{R}\xi_3$ is a Lie algebra.

d) What is a Lie group whose Lie algebra is $\mathfrak{g}$?
(No proof necessary.)