1. Let $X$ be the topological space obtained from the disjoint union of two Moebius bands by gluing their boundaries by a homeomorphism. Compute the homology of $X$.

2. Recall that $SL_n\mathbb{R}$ denotes the set of real $n \times n$ matrices with determinant 1 which we regard as a subset of $\mathbb{R}^{n^2}$ in the usual way. You may assume that $SL_n\mathbb{R}$ is a smooth submanifold of $\mathbb{R}^{n^2}$. Define $\pi : SL_n\mathbb{R} \to \mathbb{R}^n$ by setting $\pi(A)$ equal to the first column of $A$, for each matrix $A$ in $SL_n\mathbb{R}$. Show that $\pi$ is a submersion if $n > 1$.

3. Let $Y \subset \mathbb{R}^3$ be the subspace obtained by removing the $x$-axis and the circle given by the equations $x = 0, y^2 + z^2 = 1$. Compute $\pi_1(Y)$.

4. (a) Let $X$ denote $\mathbb{R}^\omega$ with the product topology. Show that $X$ is connected.

(b) let $Y$ denote $\mathbb{R}^\omega$ with the box topology. Thus $Y$ is the product $\prod_{i=1}^{\infty} R_i$, where each $R_i$ is homeomorphic to the real line $\mathbb{R}$, and a basis for the topology on $Y$ consists of all subsets of the form $\prod_{i=1}^{\infty} U_i$, where each $U_i$ is open in $R_i$.

i. Is $Y$ Hausdorff?

ii. Does $Y$ have a countable dense subset?

5. A graph $\Gamma$ consists of two sets $V, E$ (vertices and edges) and two maps $S, T : E \to V$ (source and target). The topological realization of $\Gamma$ is the quotient of

$$\bigcup_{e \in E} \{e\} \times [0, 1] \cup V$$

by identifying, for each $e \in E$, $(e, 0)$ resp. $(e, 1)$ with the source resp. target of $e$. A graph is called connected if its topological realization is connected. A connected graph is called a tree if its topological realization contains no subspace homeomorphic to $S^1$. A (discrete) group $G$ acts freely on a graph $\Gamma$ if it acts freely on the sets $V$ and $E$ in a way compatible with source and target. Note that then $G$ also acts freely on the topological realization of $\Gamma$.

(a) Characterize all groups $G$ for which there exists a connected graph $\Gamma(G)$ on which $G$ acts freely.

(b) Characterize all groups $G$ for which there exists a tree $\Gamma(G)$ on which $G$ acts freely.
1. Let $X$ and $Y$ be metric spaces and let $X$ be compact. Let $f$ be an isometry of $X$ onto a subspace of $Y$, and let $g$ be an isometry of $Y$ onto a subspace of $X$. Show that $f$ must be onto.

[Hint: consider the images of the iterates of $g \circ f$.]

2. Prove or disprove the following statement: Let $X, Y$ be path-connected spaces, and let $H$ denote singular homology. Then

$$H_1(X \times Y) \cong H_1(X) \oplus H_1(Y).$$

3. Recall that $S^2$ is the unit sphere in $\mathbb{R}^3$, and let $X = \{(x, y, z) \in S^2 : y^2z = x^3 - xz^2\}$. Is $X$ a smooth submanifold of $\mathbb{R}^3$? Explain your answer.

4. Give an explicit example of a covering map $f : X \to Y$ with $X$ path-connected such that for some $* \in Y$, $|f^{-1}(*)| = 3$ and $f$ is not a regular covering.

5. Let $M$ and $N$ be smooth manifolds, and suppose that $M$ is connected. Let $f : M \to N$ be a smooth map such that for each $x$ in $M$, the differential $df_x$ is zero. Give a careful proof that $f$ must be a constant map.