QR Exam Algebra, January 2015, Morning

Please justify all your answers, and label which solutions apply to which problems. We write \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) and \( \mathbb{F}_p \) for the integers, the rational numbers, the reals, the complex numbers and the field with \( p \) elements, respectively.

**Problem 1** Let \( V \) be the vector space of real \( 2 \times 3 \) dimensional matrices. Let \( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Define a symmetric bilinear form \( \langle \, , \, \rangle \) on \( V \) by
\[
\langle A, B \rangle = \text{trace}(A^T J B).
\]
Compute the signature of \( \langle \, , \, \rangle \).

**Problem 2** Let \( p \) be a prime and let \( G \) be a subgroup of \( S_p \) acting transitively on \( \{1, 2, \ldots, p\} \). Let \( N \) be a nontrivial normal subgroup of \( G \). Show that \( N \) also acts transitively on \( \{1, 2, \ldots, p\} \).

**Problem 3** Let \( p \) be a prime which is 3 mod 4; let \( R \) be the ring \( \mathbb{F}_p[x]/(x^2(x - 2)(x^2 + 1)) \); let \( F \) be the endomorphism \( \phi(u) = u^p \) of \( R \); note that \( \phi \) is an \( \mathbb{F}_p \)-linear map. Compute the characteristic polynomial of \( \phi \) as an \( \mathbb{F}_p \)-linear automorphism of \( R \).

**Problem 4** Suppose that \( \alpha \in \mathbb{C} \) is not algebraic over \( \mathbb{Q} \), and let \( g(x) \in \mathbb{Q}[x] \) be a nonconstant polynomial of degree \( d \) and define \( \beta = g(\alpha) \in \mathbb{Q}(\alpha) \). Show that the polynomial \( g(x) - \beta \in \mathbb{Q}(\beta)[x] \) is irreducible. (Hint: view \( g(x) - \beta \) as a polynomial in \( \mathbb{Q}[\beta][x] \).)

**Problem 5** Let \( p \) be a prime number, let \( n \) and \( m \) be positive integers and let \( A \) be an \( n \times n \) matrix with integer entries such that \( A^m = p \, I_n \) (\( I_n \) is the identity matrix). Show that \( m \leq n \).
QR Exam Algebra, January 2015, Afternoon

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**Problem 1** Let $R$ and $S$ be commutative rings and let $\phi : R \to S$ be a ring homomorphism taking 1 to 1.
(a) Show that, if $p$ is a prime ideal of $S$, then $\phi^{-1}(p)$ is a prime ideal of $R$.
(b) Give an example to show that, if $m$ is a maximal ideal of $S$, then $\phi^{-1}(m)$ need not be a maximal ideal of $R$.

**Problem 2** Let $V$ be an $n$ dimensional complex vector space and let $\phi : V \to V$ be a complex linear map. Let $W$ be $V$ considered as a real vector space (of dimension $2n$) and let $\psi$ be the same map $\phi$ considered as an $\mathbb{R}$-linear map $W \to W$.
What is the relationship between the complex number $\det \phi$ and the real number $\det \psi$? (Prove your answer to be correct.)

**Problem 3** Show that there is a matrix in $GL_n(\mathbb{F}_p)$ (the general linear group of invertible $n \times n$ matrices with entries in the field $\mathbb{F}_p$) of multiplicative order $p^n - 1$.

**Problem 4** Show that there are no simple groups of order 300. (Hint: A short route is through the 5 Sylows.)

**Problem 5** Let $f(x)$ be a degree $n$ polynomial over $\mathbb{Q}$ such that the splitting field $K$ of $f$ has $[K: \mathbb{Q}] = n!$. Let $g$ be a polynomial in $\mathbb{Q}[x]$ with $0 < \deg g < n$. Show that there is a nonzero polynomial $h \in \mathbb{Q}[x]$ so that $f(x)$ divides $h(g(x)) - x$. 