1. Let $E$ be the subset of the interval $[0, 1]$ consisting of the points $x$ that has a decimal expansion

$$x = 0.a_1a_2a_3a_4 \cdots$$

with $a_n \neq 5$ for all $n = 1, 2, 3, \cdots$. (For example, both 0.5 and 0.6 are in $E$ since 0.5 has an expansion $0.5 = 0.4999 \cdots$ and 0.6 has an expansion $0.6 = 0.6000 \cdots$.) Show that $E$ is Lebesgue measurable and evaluate the Lebesgue measure of $E$.

2. The Fourier transform of a complex valued function $f$ on $\mathbb{R}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx, \quad \xi \in \mathbb{R}.$$ 

Prove that if $f \in L^1(\mathbb{R})$, then $\hat{f}$ is continuous on $\mathbb{R}$ and

$$\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0.$$ 

3. Let $p > 0$ and let $E$ be a measurable subset of $\mathbb{R}^d$. Suppose that $f_n, f \in L^p(E)$, and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

(a) Show that for every $\epsilon > 0$,

$$\lim_{n \to \infty} m(\{x \in E : |f_n(x) - f(x)| > \epsilon\}) = 0.$$ 

Here $m$ denotes the Lebesgue measure.

(b) Show that there exists a subsequence $f_{n_j}$ such that $f_{n_j}(x) \to f(x)$ for almost every $x \in E$.

(There is a theorem states (a) implies (b). You need to prove this theorem instead of quoting the theorem.)

4. Let $f \in L^1[a, b]$. Prove that if

$$\lim_{h \to 0} \frac{1}{h} \int_a^b |f(x + h) - f(x)| \, dx = 0,$$

then there is a constant $c$ such that $f(x) = c$ for almost every $x \in (a, b)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported $C^1$ function. Show that there is a constant $C > 0$, independent of $f$, such that for all $x \in \mathbb{R}$,

$$\int \frac{(f(x) - f(y))^4}{(x-y)^4} \, dy \leq C \|f'\|_{L^4(\mathbb{R})}^4.$$ 

(Hint: You may like to use integration by parts.)
1. Prove that if an entire function \( f \) satisfies \( \text{Re}(f(z)) > 0 \) for all \( z \in \mathbb{C} \), then \( f \) is a constant function.

2. Let \( f \) be a non-vanishing analytic function on \( D = \{ z : |z| < 1 \} \) which is continuous on \( \bar{D} = \{ z : |z| \leq 1 \} \). Suppose that \( |f(e^{2\pi it})| = e^{t(1-t)} \), for \( t \in [0, 1] \). Find \( |f(0)| \).

3. Evaluate the integral
\[
\int_0^\infty \frac{\ln x}{x^2 - 1} \, dx.
\]

4. Let
\[
f(z) = z^3 + \frac{1}{(z - 1)^2}.
\]

   (a) How many times does \( f(z) \) wind around the origin as \( z \) moves along the circle \( |z| = 2 \) counterclockwise?

   (b) How many zeros, counting multiplicity, does \( f \) have inside the circle \( |z| = 2 \)?

5. Let \( D = \{ z : |z| < 1 \} \). Suppose \( f \) is an analytic function on \( D \setminus \{ 0 \} \) satisfying
\[
\int \int_D |f(x + iy)|^2 \, dx \, dy < \infty
\]
where the integral is the usual \( \mathbb{R}^2 \)-area integral. Prove that \( f \) has a removable singularity at \( z = 0 \).