(1) Suppose that \( R \) is a commutative ring with 1 with only finitely many ideals. Suppose that \( m_1, m_2, \ldots, m_d \) are all maximal ideals.
   (a) Show that if \( a \in m_1 \cap m_2 \cap \cdots \cap m_d \) then \( a \) is nilpotent.
   (b) Show that if the number of distinct ideals of \( R \) is not a power of 2, then \( R \) contains a nonzero nilpotent element.

(2) Suppose that \( G \) is group of order \( 2^4 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \) with a normal 2-Sylow subgroup. Show that the center of \( G \) contains more than 1 element.

(3) We denote the field with \( q \) elements by \( \mathbb{F}_q \). Let \( \psi : \mathbb{F}_q^9 \to \mathbb{F}_3^9 \) be the map defined by \( \psi(a) = a^3 - a \). For which positive integers \( d \) is the kernel of \( \psi^d \) a subfield of \( \mathbb{F}_3^9 \)?

(4) Let \( D_4 \) be the dihedral group with 8 elements. Construct a Galois extension \( K/\mathbb{Q} \) with Galois group \( D_4 \). In your example, describe explicitly all intermediate fields \( L \) with \( \mathbb{Q} \subset L \subset K \) such that \( L/\mathbb{Q} \) is an extension of degree 2.

(5) (a) Give an example of a nonzero finitely generated \( \mathbb{Z}[X] \)-module \( M \) which is torsion-free, but not free.
   (b) Give an example of a nonzero finitely generated \( \mathbb{Z}[X] \)-module \( M \) and two irreducible elements \( f_1, f_2 \in \mathbb{Z}[X] \) such that \( f_1 f_2 \) kills \( M \), but \( M \) does not decompose as a product \( M_1 \times M_2 \) such that \( f_1 \) kills \( M_1 \) and \( f_2 \) kills \( M_2 \).
(1) Fix a field $k$ and $A$ be the ring $k[X]/(X^p - 1)$. Classify all simple $A$-modules in the following two cases:
(a) $k = \mathbb{Q}$;
(b) $k = \mathbb{F}_p$, the field with $p$ elements.
(An $A$-module $M$ is simple if it has exactly 2 submodules, namely 0 and $M$ itself.)

(2) Let $K$ be a separably closed field, so $K$ does not have any finite separable field extension other than $K$ itself. Let $L/K$ be a finite nontrivial extension of fields.
(a) Show that the trace map $\text{Tr} : L \to K$ is the zero map.
(b) Give an example of such a field extension $L/K$.

(3) Let $V_n$ be the space of polynomials in $x$ of degree at most $n$ with real coefficients. Define a linear map $\phi : V_n \to V_n$ by $\phi(f) = xf' + f''$. Show that there exists $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and a basis $\{f_0, f_1, \ldots, f_n\}$ of $V_n$ such that $\phi(f_i) = \lambda_i f_i$ for all $i = 0, 1, \ldots, n$.

(4) Suppose that $V$ is a finite dimensional real vector space equipped with a symmetric bilinear form $(\cdot, \cdot)$.
(a) Show that there exists a bilinear form $(\cdot, \cdot)_\ast$ on $\bigwedge^2 V$ with the property
$$(v_1 \wedge v_2, w_1 \wedge w_2)_\ast = (v_1, w_2)(v_2, w_1) - (v_1, w_1)(v_2, w_2).$$
(b) Give the signature of $(\cdot, \cdot)_\ast$ in terms of the signature of $(\cdot, \cdot)$.

(5) Show that an abelian group of order 100 cannot act faithfully on a set with 13 elements.