Analysis Qualifying Review. January 7, 2017

Morning Session, 9:00 am - 12:00 pm

1. (a) Let \( f_n \) be a sequence of continuous real-valued functions on \([0, 1]\) which converges uniformly to \( f \). Prove that \( \lim_{n \to \infty} f_n(x_n) = f(1/2) \) for any sequence \( \{x_n\} \) that converges to \( 1/2 \).

(b) Suppose the convergence \( f_n \to f \) is only pointwise. Does the conclusion still hold? Explain.

Solution

(a) Fix \( \epsilon > 0 \) and let \( N_0 \in \mathbb{N} \) be such that \( n \geq N_0 \) implies \( |f_n(x) - f(x)| < \epsilon / 2 \) for all \( x \in [0, 1] \).

Since the convergence is uniform, \( f \) is continuous, so we can pick \( \delta > 0 \) such that \( |f(x) - f(1/2)| < \epsilon / 2 \) for all \( x \in [0, 1] \) with \( |x - 1/2| < \delta \). Let \( N_1 \in \mathbb{N} \) be such that \( n \geq N_1 \) implies \( |x_n - 1/2| < \delta \). Then \( n \geq \max\{N_0, N_1\} \) implies \( |f_n(x_n) - f(1/2)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(1/2)| < \epsilon / 2 + \epsilon / 2 = \epsilon \).

(b) The conclusion is false, as the following counterexample shows: Define

\[
  f_n(x) = \begin{cases} 
  0 & \text{if } 0 \leq x < 1/2 - 1/2n \\
  2nx - (n-1) & \text{if } 1/2 - 1/2n \leq x \leq 1/2 \\
  1 & \text{if } 1/2 \leq x \leq 1 
  \end{cases} \tag{1}
\]

Let \( x_n = 1/2 - 1/n \). Then \( x_n \to 1/2 \) but \( f(1/2) = 1 \neq 0 = \lim_n f_n(x_n) \).

2. Show that

\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \tag{2}
\]

for \(-\pi \leq x \leq \pi\).

Solution: Consider the periodic function \( f : \mathbb{R} \to \mathbb{R} \) of period \( 2\pi \) and defined by \( f(x) = x^2 \) for \(-\pi \leq x \leq \pi\). Its Fourier series converges uniformly since \( f \) is Lipschitz continuous (for example). Now the \( n \)th Fourier coefficient is

\[
  \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} \, dx 
\]

A direct calculation shows that \( \hat{f}(0) = \pi^2 / 3 \) and \( \hat{f}(n) = \frac{2(-1)^n}{n^2} \) for \( n \neq 0 \). Hence

\[
f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx},
\]

1
which yields the desired formula since $e^{inx} + e^{-inx} = 2 \cos nx$.

3. Let $R$ be the unit square $[0, 1] \times [0, 1]$ in the plane, and let $\mu$ be the usual Lebesgue measure on the real Cartesian plane. Let $N$ be the function that assigns to each real number $x$ in the unit interval the positive integer that indicates the first place in the decimal expansion of $x$ after the decimal point where the first 0 occurs. If there are two expansions, use the expansion that ends in a string of zeroes. If 0 does not occur, let $N(x) = \infty$. For example, $N(0.0) = 1$, $N(0.5) = 2$, $N(1/9) = \infty$, and $N(0.4763014\ldots) = 5$. Evaluate $\int \int_{R} y^{-N(x)} d\mu$.

**Solution:** The interval $[0, 1]$ consists of one interval of length $\frac{1}{10}$ where $N = 1$, 9 intervals of length $10^{-2}$ where $N = 2$, and, in general, $9^{k-1}$ intervals of length $10^{-k}$ where $N = k$. It follows that for $y$ fixed,

$$
\int_{0}^{1} y^{-N(x)} \, dx = \sum_{k=1}^{\infty} 9^{k} 10^{-k} y^{-k} = \frac{y}{10} \left( 1 - \frac{9}{10y} \right) = \frac{y}{10 - 9y}.
$$

Hence, by the Fubini-Tonelli theorem,

$$
\int \int_{R} y^{-N(x)} \, d\mu = \int_{0}^{1} \frac{y}{10 - 9y} \, dy = \int_{0}^{1} \left( -\frac{1}{9} + 10 \frac{10}{81} \frac{1}{y} - y \right) \, dy = \frac{10}{81} \log 10 - \frac{1}{9}.
$$

4. Let $(f_{n})_{1}^{\infty}$ be a sequence in $L^{p}(\mu)$, where $1 \leq p < \infty$. Show that if $\lim ||f_{n} - f||_{p} = 0$, where $f \in L^{p}(\mu)$, then $(f_{n})$ converges to $f$ in measure.

**Solution:** Pick $\epsilon > 0$ and consider the measurable set

$$
E_{n, \epsilon} := \{ x \in X \mid |f_{n}(x) - f(x)| \geq \epsilon \}.
$$

for $n \geq 1$. Then

$$
\int |f_{n} - f|^{p} \, d\mu \geq \int \chi_{E_{n, \epsilon}} |f_{n} - f|^{p} \, d\mu \geq \epsilon^{p} \int \chi_{E_{n, \epsilon}} \, d\mu = \epsilon^{p} \mu(E_{n, \epsilon}). \tag{3}
$$

Since the left hand side tends to zero as $n \to \infty$, we see that $\lim_{n\to\infty} \mu(E_{n, \epsilon}) = 0$ for every $\epsilon > 0$, which precisely means that $f_{n} \to f$ in measure.
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Afternoon Session, 2:00 pm - 5:00 pm

1. Let $f(z)$ and $g(z)$ be entire functions for which there exists a constant $C > 0$ such that $|f(z)| \leq C|g(z)|$ for all $z$. Prove that there exists a constant $c$ such that $f(z) = cg(z)$ for all $z$.

Solution: If $g$ is identically zero then so is $f$ and any $c$ will do. Otherwise, $g$ has isolated zeros. The function $h(z) = f(z)/g(z)$ is holomorphic outside the zeros of $g$ and satisfies $|h(z)| \leq C$. It suffices to prove that $h$ extends to an entire function, since then $h$ will be a bounded entire function, and hence constant by Liouville’s Theorem. Now consider any zero $z_0$ of $g$. We can write $f(z) = (z - z_0)^m f(z)$ and $g(z) = (z - z_0)^n g(z)$, where $m$ and $n$ are nonnegative integers and where $\tilde{f}$, $\tilde{g}$ are holomorphic near $z_0$ (in fact, they are entire) and do not vanish at $z_0$. The estimate $|f(z)| \leq C|g(z)|$ implies that $m \geq n$. Thus $h(z) = (z - z_0)^{m-n} \tilde{f}(z)/\tilde{g}(z)$ in a punctured neighborhood of $z_0$, and the right-hand side is a holomorphic function in a neighborhood of $z_0$. Thus $h(z)$ extends to a holomorphic function in a neighborhood of $z_0$. This completes the proof since $z_0$ was an arbitrary zero of $g$.

2. Find a conformal mapping $w = f(z)$ that takes the first quadrant in the $z$-plane onto the unit disc in the $w$-plane, and such that $f(0) = 1$, $f(1 + i) = 0$.

Solution: First set $\zeta = z^2$. This takes the first quadrant onto the upper half plane, $z = 0$ to $\zeta = 0$ and $z = 1 + i$ to $\zeta = 2i$. Now set $w = \frac{2i - \zeta}{\zeta + 2i}$. This takes the upper half plane to the unit circle, $\zeta = 0$ to $w = 1$, and $\zeta = 2i$ to $w = 0$. Thus we can set

$$w = f(z) = \frac{2i - z^2}{z^2 + 2i}.$$

3. Find all analytic functions on the unit disc that satisfy $f'(\frac{1}{n}) = f(\frac{1}{n})$ for $n = 2, 3, 4, \ldots$. Justify your answer.

Solution: The function $g(z) := f'(z) - f(z)$ has zeros at the points $z = \frac{1}{n}$, $n \geq 2$, and these points accumulate at the origin, so we must have $g(z) \equiv 0$, that is, $f'(z) = f(z)$. This implies $\frac{d}{dz}(e^{-z}f(z)) = 0$, so $f(z) = ce^z$ for some complex number $c$. Conversely, if $f(z) = ce^z$, then it is clear that $f' = f$, and in particular $f'(\frac{1}{n}) = f(\frac{1}{n})$ for $n \geq 2$. 


4. Let \( a \in \mathbb{C} \) with \( |a| \neq 1 \). Evaluate the integral
\[
\oint_{|z|=1} \frac{z}{a - z^{100}} \, dz.
\]

**Solution:** Using \( z \bar{z} = 1 \) for \( |z| = 1 \), the integral is
\[
\oint_{|z|=1} \frac{1}{z(a - z^{100})} \, dz = \oint_{|z|=1} f(z) \, dz.
\]

If \( |a| > 1 \), then \( f \) has a exactly one simple pole at \( z = 0 \) in \( |z| < 1 \). and the residue of \( f \) is \( 1/a \) there, so the integral is equal to \( \frac{2\pi i}{a} \).

If instead \( |a| < 1 \), then there are 101 poles in \( |z| < 1 \) and no poles on \( |z| > 1 \). We can therefore replace the countour \( |z| = 1 \) by \( |z| = R \), for \( R \gg 1 \). As \( R \to \infty \), it follows from the Cachy estimates (ML bound) that the integral is zero.

5. Let \( f(z) \) be an analytic function in the unit disc \( \{|z| < 1\} \). Prove that there exists a sequence \((z_n)_{n=1}^{\infty}\) in the disc such that \( \lim_{n \to \infty} |z_n| = 1 \) and such that \( \sup_n |f(z_n)| < \infty \).

**Solution:** Suppose no such sequence exists. Then \( f \) only have finitely many zeros in the disc, say at \( z = a_i \), \( 1 \leq i \leq r \), with multiplicities \( m_i \), \( 1 \leq i \leq r \). Set \( p(z) = \prod_{i=1}^{r} (z - a_i)^{m_i} \). Then the function \( h(z) = p(z)/f(z) \) is analytic on the unit disc and tends to zero at the boundary. It then follows from the Cauchy estimates (or the maximum principle) that \( h \) is identically zero, a contradiction.
5. Suppose that \( f \in L^p([-1,1]) \) for all \( 1 \leq p < \infty \). Prove that the integral

\[
\int_{-1}^{1} \frac{|f(x)|}{|x|^s} \, dx
\]

is finite for all \( 0 < s < 1 \).

**Solution:** This follows from Hölder's inequality. Pick \( p \) sufficiently large so that \( q = \frac{p}{p-1} < s^{-1} \). Then

\[
\int_{-1}^{1} \frac{|f(x)|}{|x|^s} \, dx \leq \|f\|_p \left( \int_{-1}^{1} |x|^{-qs} \, dx \right)^{1/q} < \infty.
\]