Problem AM1. Suppose that $A$ is a $3 \times 3$ complex matrix with $\text{tr}(A) = 1$, $\det(A) = 2$ and $\det(\text{Id} + A) = 3$. What is $\det(2\text{Id} + A)$? ($\text{tr}(A)$ and $\det(A)$ denote the trace and determinant of $A$ respectively, and $\text{Id}$ is the $3 \times 3$ identity matrix.)

Problem AM2. Let $A$ be a positive definite real symmetric $n \times n$ matrix.
(a) Suppose that $B$ is a real $n \times n$ matrix with $AB + BA = 0$. Show that $B = 0$.
(b) Show that, for every $n \times n$ real matrix $C$, there is an $n \times n$ real matrix $B$ with $AB + BA = C$.

Problem AM3. Let $\mathbb{F}_{256}$ be the field with $256 = 2^8$ elements.
(a) List all fields between $\mathbb{F}_2$ and $\mathbb{F}_{256}$.
(b) Let $\theta \in \mathbb{F}_{256}$ be such that $\mathbb{F}_{256} = \mathbb{F}_2(\theta)$. What are the possible orders of $\theta$ in the multiplicative group $\mathbb{F}_{256}^* = \mathbb{F}_{256} \setminus \{0\}$?

Problem AM4. Suppose that $L/K$ is a finite Galois extension of fields with Galois group $G$, and let $\theta$ be a primitive element, so $L = K(\theta)$. Let $H$ be a subgroup of $G$. Define elements $\alpha_j$ of $L$ by the polynomial equality

$$\prod_{h \in H} (x - h(\theta)) = \sum_{j=0}^{\left|H\right|} x^j \alpha_j$$

Show that

$$L^H = K(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{|H|}).$$

Problem AM5. Let $p$ be a prime number, let $G$ be a finite group and $H$ a subgroup of $G$. Show that the number of $p$-Sylow subgroups of $G$ is greater than or equal to the number of $p$-Sylow subgroups of $H$.
Problem PM1. Let $A$ be a $6 \times 6$ complex matrix with

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

(All omitted entries are zero.) What are the possible Jordan form(s) of $A$?

Problem PM2. How many subgroups of $\mathbb{Z}^2$ are there with index 10?

Problem PM3. Does there exist a finite group $G$ with normal subgroups $N_1$ and $N_2$ such that $N_1 \cong S_5$, $N_2 \cong S_7$, $G/N_1 \cong S_{42}$ and $G/N_2 \cong S_{41}$?

Problem PM4. Let $p$ be prime and let $k$ be a field of characteristic $p$. Let the symmetric group $S_p$ act on the vector space $k^p$ by permuting the coordinates. 

(a) Show that there is a one dimensional subspace $L$ of $k^p$ which is taken to itself by $S_p$.

(b) Show that there does not exist a subspace $W$ of $k^p$, taken to itself by $S_p$, for which $k^p = L \oplus W$.

Problem PM5. Let $f$ be a polynomial of degree $n$ over $\mathbb{Q}$, with distinct roots $\theta_1, \theta_2, \ldots, \theta_n$ and let $K = \mathbb{Q}(\theta_1, \ldots, \theta_n)$. Suppose that the Galois group of $K$ over $\mathbb{Q}$ is cyclic, with generator $\theta_1 \mapsto \theta_2 \mapsto \cdots \mapsto \theta_n \mapsto \theta_1$. Show that there is a polynomial $f \in \mathbb{Q}[x]$ such that $f(\theta_k) = \theta_{k+1}$ for all $k$ (with indices modulo $n$, so we require that $f(\theta_n) = \theta_1$.)