1. Let $M$ be a smooth manifold, and let $f : M \to M$ be a diffeomorphism. Define the set of fixed points
\[ \text{Fix}_f := \{ x \in M | f(x) = x \} \]
Let $x \in \text{Fix}_f$, and suppose that $x$ is not isolated. Show that $df_x$ has 1 as an eigenvalue.

2. (a) Does there exist a continuous map $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ with no fixed points? Explain.
(b) Does there exist a continuous map $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ with no fixed points? Explain.

3. Show that every connected topological manifold is path-connected.

4. Denote by
\[ Y = S^2 \vee S^1 \]
the quotient space obtained from a disjoint union of $S^2$ and $S^1$ by choosing one point in each disjoint summand and identifying them.

(a) Describe the universal covering of $Y$.
(b) Is every (connected) covering of $Y$ regular? Prove your answer.

5. Let $O(n)$ be the set of $n \times n$ orthogonal matrices, viewed as a subset of all $n \times n$ matrices. Prove that $O(n)$ is a submanifold and describe its tangent space at each point.
1. In this problem, let $S^1$ be the unit sphere in $\mathbb{C}$. Let $T$ be the topological space obtained from $[0,1] \times S^1$ by identifying $(0,z)$ with $(1,z^2)$ for every $z \in S^1$, with the quotient topology. Compute $\pi_1(T)$ in terms of generators and defining relations.

2. (a) Let $f_a(x, y, z) = x^2 + y^2 - az^2$. For which values of $a$ is $X_a := f_a^{-1}(4)$ a smooth 2-dimensional manifold?

(b) Explicitly describe the tangent space of $X_1$ at the point $(2, 1, 1)$.

(c) For which values of $a$ is $X_a$ a smooth 2-manifold which intersects the unit 2-sphere transversely?

3. Let $X$ be a topological space. Write

$$Sym^2(X) = X \times X / \sim$$

with the quotient topology, where $\sim$ is the smallest equivalence relation such that $(x, y) \sim (y, x)$ for all $x, y \in X$. Calculate the homology of $Sym^2(S^1)$. ($S^n$ is the $n$-sphere.)

4. Let $X$ be a locally compact Hausdorff space, and let $Y$ be its one point compactification. Show that $Y$ is connected if $X$ is connected and noncompact.

5. Denote by $Z$ the space obtained from $\mathbb{R}^3$ by identifying each point $x \in \mathbb{R}^3$ with $-x$, with the quotient topology. Denote by $[x]$ the image of a point $x \in \mathbb{R}^3$ in $Z$.

(a) Compute $H_3(Z, Z \setminus \{[0]\})$.

(b) Is $Z$ a topological manifold? Explain.