1. Let \((f_j)_{j=1}^{\infty}\) be a sequence of measurable functions on a measure space \((X, \mathcal{M}, \mu)\). Suppose that the series
\[
\sum_{j=1}^{\infty} \mu\{x \in X \mid |f_j(x)| \geq \epsilon\}
\]
converges for every \(\epsilon > 0\). Prove that \(f_j(x) \to 0\) almost everywhere on \(X\).

**Solution** Set \(E_{j,\epsilon} = \{|f_j| \geq \epsilon\}\) and \(A_\epsilon = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_{j,\epsilon}\). Then \(A_\epsilon\) is measurable and
\[
\mu(A_\epsilon) = \lim_{n \to \infty} \mu\left(\bigcup_{j=n}^{\infty} E_{j,\epsilon}\right) = \lim_{n \to \infty} \sum_{j=n}^{\infty} \mu(E_{j,\epsilon}) = 0.
\]
Set \(A = \bigcup_{k=1}^{\infty} A_{1/k}\). Then \(\mu(A) \leq \sum_k \mu(A_k) = 0\) and \(\lim_{j \to \infty} f_j(x) = 0\) for \(x \in A^c\).

2. Let \(E \subset [0, 1]\) be the middle-third Cantor set, i.e. \(E = [0, 1] \setminus \bigcup_{n=1}^{\infty} U_n\), where \(U_1 = (1/3, 2/3), U_2 = (1/9, 2/9) \cup (7/9, 8/9)\) etc. Find a function \(f \in C^\infty(\mathbb{R})\) such that \(f \geq 0\) and \(\{x \in \mathbb{R} \mid f(x) = 0\} = E\).

**Solution:** Let \(g(x)\) be the distance from a point \(x \in \mathbb{R}\) to \(E\). Then \(g\) is nonnegative with \(\{g = 0\} = E\). Further, \(g\) is continuous on \(\mathbb{R}\) and \(C^\infty\) on \(\mathbb{R} \setminus E\). Now consider the function \(\chi : \mathbb{R} \to \mathbb{R}\) defined by
\[
\chi(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
e^{-1/t} & \text{if } t > 0
\end{cases}
\]
Then \(f = \chi \circ g\) has the required properties.

3. Let \(\alpha < 1\). Prove the existence of the limit
\[
\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{1/n} e^{\alpha x} \, dx,
\]
and calculate it.

**Solution:** Consider the function \(f_n\) on \((0, \infty)\) defined by
\[
f_n(x) = \left(1 - \frac{x}{n}\right)^n x^{1/n} e^{\alpha x} \cdot \chi(0,n)
\]
We have \( \lim_{n \to \infty} f_n(x) = e^{-x} \cdot e^{\alpha x} = e^{-(1-\alpha)x} \) pointwise on \( \mathbb{R} \). To estimate \( f_n \) from above, first note that \( x^{1/n} \leq n^{1/n} \leq e^{x^{-1}} \) for \( x \in (0, n) \), where the last inequality follows by checking that the maximum of the function \( y^{1/y} \) on \((0, \infty)\) occurs at \( y = e \). Second, we have
\[
\log(1 - \frac{x}{n}) \leq -\frac{x}{n}
\]
for \( 0 < x < n \). Hence
\[
(1 - \frac{x}{n})e^{\alpha x} = \exp(n \log(1 - \frac{x}{n}) + \alpha x) \leq e^{-(1-\alpha)x}
\]
for \( 0 \leq x < n \), so that
\[
0 \leq f_n(x) \leq Ce^{-(1-\alpha)x}
\]
for all \( x \in \mathbb{R} \), where \( C = e^{e^{-1}} \). Since \( \int_0^\infty e^{-(1-\alpha)x} \, dx < \infty \), the dominated convergence theorem yields
\[
\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty \lim_{n \to \infty} f_n(x) \, dx = \int_0^\infty e^{-(1-\alpha)x} \, dx = \frac{1}{1 - \alpha}.
\]

4. Let \( \beta > 1 \) and \( C > 0 \). Find all functions \( f: \mathbb{R} \to \mathbb{R} \) such that \( |f(x) - f(y)| \leq C|x-y|^\beta \) for all \( x, y \in \mathbb{R} \).

**Solution:** For any \( x \), letting \( y \to x \) we see that \( f \) is differentiable at \( x \), with derivative 0. Thus \( f' \equiv 0 \), so that \( f \) is constant. Conversely, any constant function \( f \) clearly satisfies the condition.

5. Construct a function \( f \in L^1(\mathbb{R}^n) \) such that \( f \not\in L^p(U) \) for any open subset \( U \subset \mathbb{R}^n \) and any \( p > 1 \).

**Solution:** Pick a dense sequence \( (x_k)_{k=1}^\infty \) in \( \mathbb{R}^n \). For each \( k \), define a function \( f_k \) on \( \mathbb{R}^n \) by
\[
f_k(x) = \begin{cases} 
|x|^{-\frac{n k}{\beta+1}} & \text{if } |x| < 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Using polar coordinates we see that
\[
\int_{\mathbb{R}^n} f_k(x) \, dx = c'_n \int_0^1 r^{n-1} r^{-\frac{n k}{\beta+1}} \, dr = c_n(k + 1),
\]
where the constants \( c'_n \) and \( c_n \) only depend on the dimension \( n \). A similar computation also shows that \( f_k^p \) is not locally integrable at the origin for \( p \geq 1 + \frac{1}{k} \). Now set
\[
f(x) = \sum_{k=1}^\infty 2^{-k} f_k(x - x_k).
\]
Then
\[ \int_{\mathbb{R}^n} f(x) dx = c_n \sum_{k=1}^{\infty} (k+1)2^{-k} < \infty. \]

On the other hand, if \( p > 1 \) and \( U \subset \mathbb{R}^n \) is open, then \( x_k \in U \) for infinitely many \( k \), so there exists \( k \) with \( x_k \in U \) and \( p \geq 1 + \frac{1}{k} \). It then follows that \( f \notin L^p(U) \).
4. Prove that for any real number $a > 1$, the equation $ze^{a-z} = 1$ has exactly one solution in the unit disc, and that this solution is real and positive.

**Solution:** Set $f(z) = z - e^{z-a}$. When $|z| = 1$ we have $|e^{z-a}| = e^{\Re z-a} < 1 = |z|$, so by Rouché’s theorem, $f$ has the same number of zeros as the function $z$ in the unit disc, namely one. Now $f$ is real-valued on the real interval $[0,1]$, with $f(0) = -e^{-a} < 0$ and $f(1) = 1 - e^{1-a} > 0$, so, by continuity, $f$ has a zero on the interval $(0,1)$. 

3. Use residues to compute the integral $\int_0^\infty \frac{\sin tx}{x} dx$ for any $t \in \mathbb{R}$. Show all your steps.

**Solution:** Set $J(t) = \int_0^\infty \frac{\sin tx}{x} dx$. Clearly $J(0) = 0$ and $J(-t) = -J(t)$, so we may assume $t > 0$. In this case, the change of variables $x \to tx$ shows that the integral is independent of $t$, so we may assume $t = 1$. Now compute the integral $I = \int_\gamma \frac{\sin z}{z} dz$, where $\gamma$ consists of the following parts: $\gamma_1 := \{|z| = \epsilon, \Im z \geq 0\}$; $\gamma_2 := \{\epsilon, \Re \} \epsilon$; $\gamma_3 := \{R, R+iR\}$; $\gamma_4 := \{R+iR, R-iR\}$; and $\gamma_5 := \{R-iR, \epsilon\}$. The integral $I$ is zero since the integrand has no poles inside $\gamma$. The integral over $\gamma_1$ tends to $-\pi i$ as $\epsilon \to 0$. The integrals over $\gamma_2, \gamma_3$ and $\gamma_4$ tend to zero as $R \to \infty$. The sum of the integrals over $\gamma_1$ and $\gamma_5$ is equal to $2 \int_\epsilon^R \frac{\sin x}{x} dx$. Thus $J(t) = \pi/2$ for $t > 0$, $J(0) = 0$ and $J(t) = -\pi/2$ for $t < 0$.

1. Let $f(z)$ be an entire function such that $f(0) = 1 + \pi i$ and $\Re f(z) \geq 1$ when $|z| < 1$. Compute $f'(0)$.

**Solution:** The origin is a local maximum of $e^{-f}$. It follows from the maximum modulus principle that $e^{-f}$, and hence also $f$ is constant, so $f'(0) = 0$.

2. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc and $a \in \mathbb{D} \setminus \{0\}$ a point. Find all analytic functions $f(z)$ on $\mathbb{D}$ such that

- $|f(z)| < 1$ for all $z \in \mathbb{D}$;
- $f(a) = 0$ and $f(0) = a$.

**Solution:** Recall the Schwarz Lemma: if $g : \mathbb{D} \to \mathbb{D}$ is analytic and $g(0) = 0$, then $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Further, if $|g(a)| = |a|$ for some $a \neq 0$, then $g(z) = \lambda z$, where $|\lambda| = 1$.

Set $g(z) = f(\frac{a-z}{1+\overline{a}z})$. Then $g : \mathbb{D} \to \mathbb{D}$ is analytic, $g(0) = 0$, and $g(a) = a$. The Schwartz Lemma gives $g(z) = \lambda z$. Here $\lambda = 1$ since $g(a) = a$. Thus $g(z) = z$, i.e. $f(z) = \frac{a-z}{1-\overline{a}z}$.
5. Let \( f(z) \) be a complex-valued \( C^\infty \) function defined on a connected open subset \( \Omega \) of the complex plane. Assume that \( f(z) \) and \( f^2(z) \) are both harmonic (i.e. the real and imaginary parts of these functions are harmonic). Prove that either \( f(z) \) or \( \overline{f(z)} \) is analytic in \( \Omega \).

**Solution:** A direct computation shows that \( \Delta f^2 = 2f\Delta f + 2(f_x^2 + f_y^2) \), so the assumption \( \Delta f = \Delta f^2 = 0 \) gives \( 0 = f_x^2 + f_y^2 = (f_x + if_y)(f_x - if_y) \) in \( \Omega \). If \( f_x - if_y \equiv 0 \) in \( \Omega \), then \( \overline{f} \) is analytic in \( \Omega \). On the other hand, if \( f_x - if_y \not\equiv 0 \), then there exists an open subset \( D \subset \Omega \) where \( f_x + if_y \neq 0 \), and hence \( f_x - if_y = 0 \) on \( D \). Thus \( f \) is analytic on \( D \). We claim that \( f \) is in fact analytic on all of \( \Omega \). To see this, write \( f = u + iv \). Then \( u \) is harmonic on \( \Omega \), and hence admits a harmonic conjugate \( v' \) on \( \Omega \), that is, \( u + iv' \) is analytic on \( \Omega \). Now \( v' \) is unique up to a constant (since \( \Omega \) is connected) so we may assume \( v' = v \) on \( D \). Then \( v' - v \) is a real-valued harmonic function on \( \Omega \) that vanishes on \( D \), and hence must vanish everywhere. Thus \( f = u + iv = u + iv' \) is analytic on \( \Omega \).