Justify all of your answers. We write $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}$ for the complex numbers, the field with $p$ elements, the rational numbers, the real numbers and the integers respectively.

**Problem 1.** How many isomorphism classes of abelian groups of order $6^4$ are there?

**Problem 2.** Let $\zeta_n = e^{2\pi i/n}$ be a primitive $n^{th}$ root of unity.
   (a) For which positive integers $n$ does $\mathbb{Q}(\zeta_n)$ contain $\sqrt{2}$?
   (b) For which positive integers $n$ does $\mathbb{Q}(\zeta_n)$ contain $\sqrt[3]{2}$?

**Problem 3.** Suppose that $A$ and $B$ are complex, invertible $n \times n$ matrices with $AB + BA = 0$. Show that there exists a complex, invertible $n \times n$ matrix $C$ such that $A + CAC = 0$.

**Problem 4.** Let $V$ be the set of $2 \times 2$ real matrices, thought of as a 4-dimensional real vector space. For a real number $\lambda$, define a symmetric bilinear form $\langle \ , \ \rangle$ on $V$ by
   $\langle A, B \rangle = \text{Tr}(AB) + \lambda \text{Tr}(AB^t)$
Here Tr is trace and $B^t$ is the transpose of $B$. For which $\lambda$ is this form positive definite?

**Problem 5.** Let $p$ be a prime number and let $n$ be a positive integer.
   (a) Show that there is a positive integer $m$, depending on $p$ and $n$, such that if $A$ is an invertible $n \times n$ matrix with entries in $\mathbb{F}_p$ that is diagonalizable over the algebraic closure $\overline{\mathbb{F}}_p$ then $A^m = \text{id}_n$.
   (b) Determine the minimal positive $m$ in (a) when $p = 3$ and $n = 4$. 
May 2017, Qualifying Review Algebra, Afternoon

Justify all of your answers. We write $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}$ for the complex numbers, the field with $p$ elements, the rational numbers, the real numbers and the integers respectively.

**Problem 1.** Let $G$ be a finite group and let $p$ be a prime number. Show that the following conditions are equivalent:

(a) The group $G$ acts transitively on a set $X$ such that the cardinality of $X$ is at least 2 and relatively prime to $p$.

(b) The order of $G$ is not a power of $p$.

**Problem 2.** Suppose that $R$ is a commutative ring with 1, and $p$ and $q$ are prime ideals of $R$ such that every element of $R \setminus (p \cup q)$ is a unit. Show that at least one of $p$ or $q$ is maximal.

**Problem 3.** Suppose that $K$ is a field of characteristic $\neq 2$ and $L = K(\beta)$ is a field extension of $K$ with $\beta^2 + \beta^{-2} \in K$. Show that $L/K$ is a Galois extension.

**Problem 4.** Suppose that $V$ is a real vector space of dimension $n$.

(a) Show that there exists a linear map $\varphi: \bigwedge^2 V \to \text{Hom}(V^*, V)$ such that

$$\varphi(a \wedge b)(f) = f(a)b - f(b)a$$

for all $a, b \in V$.

(b) Suppose $n$ is odd. Show that no element of the image of $\varphi$ is invertible.

**Problem 5.** Let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$. A *matching* on $V$ is a set $\{E_1, E_2, E_3, E_4\}$ where each $E_i$ is a two-element subset of $V$ such that $V = E_1 \cup E_2 \cup E_3 \cup E_4$. Let $\mathcal{M}$ be the set of matchings. The group $S_8$ naturally acts on $\mathcal{M}$, and the action is transitive. Let $G \subset S_8$ be the stabilizer of some matching. How many orbits does $G$ have on $\mathcal{M}$?