(1) Consider the initial value problem \( y' = Ay, y(0) = y_0 \), where \( A = \begin{pmatrix} -11 & 9 \\ 9 & -11 \end{pmatrix} \).

(a) Let \( u_n \) be the numerical solution given by Euler’s method with step size \( h \) and \( u_0 = y_0 \). What condition on \( h \) (if any) ensures that \( u_n \) remains bounded as \( n \to \infty \)?

(b) Consider the initial condition \( y(0) = (1, 0)^\top \). Give an explicit expression for the numerical solution \( u_n \). Assume that \( t > 0 \) is fixed and \( t = nh \).

(c) Find the limit of the numerical solution \( u_n \) as \( n \to \infty \) under the same assumption that \( t > 0 \) is fixed and \( t = nh \).

(2) Consider the 2-step Adams-Bashforth method \( u_{n+1} = u_n + \frac{h}{2} (3f(u_n) - f(u_{n-1})) \) for the initial value problem \( y' = f(y), y(0) = y_0 \). Assume that \( u_0 = y_0, u_1 = y(h) \).

(a) The local truncation error has the form \( \tau_n = ch^p y_n^{(q)} + O(h^r) \), where \( c, p, q, r \) are constants and \( y_n = y(nh) \) is the exact solution. Find \( c, p, q, r \).

(b) Assume that \( t > 0 \) is fixed and \( t = nh \). If the step size \( h \) is reduced by a factor of \( \frac{1}{2} \), then the error in the numerical solution \( |u_n - y_n| \) is reduced by what factor?

(c) Is the negative real axis contained in the region of absolute stability?

(3) Consider the system of linear equations \( 2x - y = 1, 2y - x = 1 \).

(a) Starting from initial iterate \((x, y) = (0, 0)\), take one step each of Jacobi’s method, the Gauss-Seidel method, and SOR with relaxation parameter \( \omega = 3/2 \). Which method gives the smallest error after one step?

(b) Which of these methods (if any) converge as the number of iterations tends to infinity?

(4) Consider the heat equation \( v_t = v_{xx} \) with initial condition \( v(x, 0) = v_0(x) \) on the real axis. Let \( u^n_j \) be the numerical solution given by the finite-difference scheme

\[
\frac{u^n_{j+1} - u^n_j}{h} = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{h^2}, \quad u^0_j = v_0(jh),
\]

where \( h, k \) are the space step and time step, respectively, and \( u^n_j \approx v(jh, nk) \). Set \( \lambda = k/h^2 \) and show that the scheme is stable in the \( \infty \)-norm if and only if \( \lambda \leq \frac{1}{2} \).

(5) Consider the initial value problem for the scalar wave equation \( v_t + cv_x = 0 \) on the real line. Let \( u^n_j \) be the numerical solution given by the Lax-Wendroff scheme with space step \( h \) and time step \( k \), and assume that \( h, k \) are chosen so that \( \lambda = k/h \) is fixed.

(a) Find the amplification factor of the scheme.

(b) Show that the scheme is stable in the 2-norm under the assumption that \( |c|\lambda \leq 1 \).