N.B.: $D$ below denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

(1) Suppose that we have
   (a) simply-connected domains $\Omega_1, \Omega_2 \subset \mathbb{C}$;
   (b) distinct points $z_1, w_1 \in \Omega_1$;
   (c) distinct points $z_2, w_2 \in \Omega_2$.
Show that there is an analytic map $f: \Omega_1 \to \Omega_2$ satisfying $f(z_1) = z_2, f(w_1) = w_2$ or an analytic map $f: \Omega_2 \to \Omega_1$ satisfying $f(z_2) = z_1, f(w_2) = w_1$ (or both).

(2) Let $\Sigma$ be the strip $\{z \in \mathbb{C} : |\text{Im}(z)| < 1\}$, and let $F$ be analytic on $\Sigma$, continuous on $\Sigma$, and verifying $|F(z)| \leq 1$ on $\partial \Sigma$.
   (a) Show that $|F(z)|$ is not necessarily $\leq 1$ on $\Sigma$.
   (b) Show that if, in addition, $F$ verifies the hypothesis $|F(z)| \leq Ce^{b|z|^\rho}$, for some constants $C, b > 0$ and $0 < \rho < 2$, then $|F(z)| \leq 1$ on $\Sigma$.

   Hint: Consider $F_\epsilon(z) := e^{-\epsilon z^2}F(z)$, for all $\epsilon > 0$.

(3) Let $f$ be an analytic function on $D$ which is continuous on $\overline{D}$ with $|f(z)| \equiv 1$ on $\partial D$. Show that $f$ is the restriction to $D$ of a rational function on $\mathbb{C}$.

(4) Let $D^* := D \setminus \{0\}$ be the punctured unit disk. Let $f: D^* \to \mathbb{C}$ be analytic and injective.
   (a) Show that $\{f(z): 0 < |z| < 1/2\}$ is not dense in $\mathbb{C}$.
   (b) Show that $f$ has a meromorphic extension to $D$. (Do not quote Picard’s theorem here.)

(5) Suppose that $g, h$ are continuous, $\mathbb{C}$-valued and nowhere vanishing functions on $\{z \in \mathbb{C} : |z| < 2\}$, $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively. Suppose that $f = g/h$ is analytic on the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.
   (a) Show that there are continuous, single-valued functions $\log g$ on $\{z \in \mathbb{C} : |z| < 2\}$, and $\log h$ on $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$.

(continued over)
(b) Show that $U = \log g - \log h$ is analytic on the annulus $A$.

(c) Show that $f$ can be written as $f(z) = G(z)/H(z)$ where $G, H$ are nowhere vanishing analytic functions on $\{z \in \mathbb{C} : |z| < 2\}, \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively.