Topology Qualified Exam (Differential Part) at May, 2018

May 8, 2018

1. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n - 2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

2. Let $\alpha$ be a closed differential two-form on $S^4$. Prove that $\alpha \wedge \alpha$ vanishes at some point.

3. Show that the real cubic surface defined by

$$S = \{ [x : y : z : w] \in \mathbb{R}P^3 : x^3 + y^3 + z^3 + w^3 = 0 \}$$

is an embedded submanifold of $\mathbb{R}P^3$, and compute its (real) dimension.

4. Consider the following subgroup $H$ of $GL_2(\mathbb{R})$:

$$H = \{ h \in GL_2(\mathbb{R}), \ h = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}, \ u > 0, v \in \mathbb{R} \}.$$ 

Show that the vector fields $u \frac{\partial}{\partial u}$ and $u \frac{\partial}{\partial v}$ form a basis of the Lie algebra $\mathfrak{h}$ of $H$.

5. Let $M$ be a smooth manifold and $C \subset O \subset M$, where $C$ is a closed subset and $O$ is an open subset. Let $f : C \to \mathbb{R}$ be a smooth function, which means $\forall p \in C, \ \exists$ an open set $p \in V_p \subset M$ and a smooth function $f_p : C \to \mathbb{R}$ s.t. $f_p|_{C \cap V_p} = f|_{C \cap V_p}$.

a. Show that there is a smooth function $\hat{f} : M \to \mathbb{R}$, such that $\hat{f}|_C = f$ and $\text{supp}(\hat{f}) \subset O$.

b. If the set $C$ is not assumed to be closed, then does the statement of part a) still hold? If yes, give the proof; and if not, give a counterexample.