1. Let $K$ be a compact topological space, and let $X_1 \subseteq X_2 \subseteq \ldots$ be a sequence of inclusions of T1 spaces (each $X_n$ has the induced topology from $X_{n+1}$). On $\bigcup X_n$, consider the union topology, which means that a set $U$ is open in $\bigcup X_n$ if and only if for all $n$, $U \cap X_n$ is open in $X_n$. Let $f : K \to \bigcup X_n$ be a continuous map. Prove that there exists an $n \in \mathbb{N}$ such that $f(K) \subseteq X_n$.

2. Prove that if $p(y)$ is a non-constant polynomial with complex coefficients with no roots of multiplicity $\geq 2$, then the set $\{(x, y) \in \mathbb{C}^2 \mid x^2 = p(y)\}$ is a smooth submanifold of $\mathbb{C}^2$.

3. Let a space $T$ be obtained from $S^1 \times [0, 1]$, where $S^1 = \{z \in \mathbb{C} \mid |z|^2 = 1\}$ by identifying $(z, 0) \sim (ze^{2\pi i/3}, 0)$, $(z, 1) \sim (ze^{2\pi i/7}, 1)$, with the quotient topology. Compute $\pi_1(T)$ in terms of generators and defining relations.

4. Recall the following definitions:
   - A map $f : X \to Y$ between topological spaces is proper if for all compact $K \subseteq Y$, the set $f^{-1}(K)$ is compact.
   - A Hausdorff space $Y$ is locally compact if every point in $Y$ has a neighborhood which is compact.

   Let $X$ and $Y$ be Hausdorff spaces, and suppose that $Y$ is locally compact. Prove that if $f : X \to Y$ is a proper local homeomorphism, then it is a covering map.

5. Show that the space $SO(2)$ of $2 \times 2$ real orthogonal matrices of determinant 1 is a smooth submanifold of $\mathbb{R}^4$ of dimension 1. Recall that a matrix $A$ is orthogonal if $AA^t = I$, where $I$ is the identity matrix.
1. Let a space $Z$ be obtained from the simplex
$$\Delta_3 = \{(x, y, z, t) \mid x, y, z, t \geq 0, x + y + z + t = 1\}$$
by identifying
$$(x, y, z, 0) \sim (x, y, 0, z)$$
and
$$(0, x, y, z) \sim (x, 0, y, z)$$
whenever $x + y + z = 1, x, y, z \geq 0$, with the quotient topology. Compute the Euler characteristic of $Z$.

2. Let $X$ be a path-connected topological space. Show that $X^N$, in the product topology, is also path-connected.

3. Let a space $Y$ be obtained from $\mathbb{R}^3$ by removing the circle
$$\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1/4\}.$$
Prove that the embedding of the unit sphere
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$
into $Y$ is not homotopic to a constant map.

4. Let $S^2$ be defined as in the previous problem, and let
$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - y = 0\}.$$
Prove or disprove: $S^2 \cap X$ is a smooth submanifold of $\mathbb{R}^3$.

5. Fix a point $a$ in the 1-sphere $S^1$. Consider the embedding $S^1 \subset S^1 \times S^1$ given by
$$z \mapsto (z, a).$$
Compute the relative homology $H_k(S^1 \times S^1, Q)$ where $Q$ is the image of the embedding.