September 2015, Qualifying Review Algebra, Morning

Please justify all your answers. We write $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}$ for the complex numbers, the field with $p$ elements, the rational numbers, the real numbers and the integers respectively.

Problem 1. Suppose that a finite group $G$ acts 2-transitively on a set $X$ of cardinality at least 2. Prove the following counting formula:

$$|G_x \cap G_y| \cdot (|X|^2 - |X|) = |G|$$

where $x$ and $y$ are any two distinct elements of $X$. ($2$-transitive means: given elements $x \neq y$ and $x' \neq y'$ of $X$ there exists $g \in G$ such that $gx = x'$ and $gy = y'$.)

Problem 2. Suppose that $R$ is a commutative ring with $1 \neq 0$ such that every proper principal ideal is prime. Show that $R$ is a field. (A proper ideal of $R$ is an ideal $I$ with $I \neq R$. A principal ideal is an ideal that can be generated by one element.)

Problem 3. Show that every group of order $255 = 3 \cdot 5 \cdot 17$ is cyclic.

Problem 4. Let $\alpha$ be a complex number with $\alpha^2 = \sqrt{3} - \sqrt{5}$.

(a) What is the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$?
(b) What is the minimal polynomial of $\alpha$ over $\mathbb{Q}$?
(c) Show that the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is not Galois.

Problem 5. Let $V$ be an $n$-dimensional real vector space.

(a) Show that there exists a unique linear map

$$\varphi: \bigwedge^2(V^*) \otimes \bigwedge^2(V) \to V^* \otimes V$$

satisfying

$$\varphi((f_1 \wedge f_2) \otimes (v_1 \wedge v_2)) = (f_1(v_1)f_2 - f_2(v_1)f_1) \otimes v_2 - (f_1(v_2)f_2 - f_2(v_2)f_1) \otimes v_1.$$  

(b) Determine the rank of $\varphi$, as a function of $n$. 

Problem 1. Let \( f: G \rightarrow H \) be a surjection of finite groups and let \( K \) be a \( p \)-Sylow subgroup of \( G \). Show that \( f(K) \) is a \( p \)-Sylow of \( H \).

Problem 2. Let \( R = \mathbb{C}[x, y] \), and define \( R \)-modules by \( M = R/(x + y)R \), and \( N = R/(x^2 - y^2 - 1)R \). Show that \( M \otimes_R N = 0 \).

Problem 3. Compute the rational canonical form of the following matrix.
\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
The empty blocks consist solely of zeros.

Problem 4. Suppose that \( p(x) \in \mathbb{Q}[x] \) is an polynomial of degree 5 whose roots are \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). Show that, if \( \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)/\mathbb{Q} \) is a Galois extension with Galois group \( S_5 \), then we have
\( \mathbb{Q}(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4) \cap \mathbb{Q}(2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = \mathbb{Q} \).

Problem 5. Let \( V \) and \( W \) be finite dimensional complex vector spaces of dimension \( m \) and \( n \) respectively, and let \( A \) and \( B \) be linear maps \( V \rightarrow W \), with \( A \) surjective. Show that \( A + tB \) is surjective for all but at most \( n \) values of \( t \in \mathbb{C} \).