1. Fix $1 < p < \infty$. Let $f \in L^p(E)$, where $E$ is a measurable subset of $\mathbb{R}^d$. Assume that
\[ \int_E f(x)g(x) \, dx = 0 \]
for all compactly supported continuous functions $g : \mathbb{R}^d \to \mathbb{R}$. Is $f(x) = 0$ for almost every $x$ in $E$? If your answer is positive, prove it. Otherwise, given a counterexample.

2. Let $(a, b)$ be an interval on $\mathbb{R}$. Let $f \in L^1(a, b)$. Assume that
\[ \int_a^b f(x)g'(x) \, dx = 0 \]
for all $C^1$ functions $g$ with support compactly contained in $(a, b)$. Prove that there is a constant $c$ such that $f(x) = c$ for almost every $x$ in $(a, b)$.

3. Let $g_k, k = 1, 2, \ldots$, be a sequence of functions that are absolutely continuous on the interval $[a, b]$. Suppose that there is a $c \in [a, b]$, such that the series $\sum_{k=1}^{\infty} g_k(c)$ is convergent, and
\[ \sum_{k=1}^{\infty} \int_a^b |g_k'(x)| \, dx < \infty. \]
(a) Show that $\sum_{k=1}^{\infty} g_k(x)$ is convergent for all $x \in [a, b]$.
(b) Let $f(x) = \sum_{k=1}^{\infty} g_k(x)$. Show that $f$ is absolutely continuous on $[a, b]$ and $f'(x) = \sum_{k=1}^{\infty} g_k'(x)$ for almost every $x \in [a, b]$.

4. Let $f_1(s, t), f_2(s, t)$, and $f_3(s, t)$ be nonnegative measurable functions on $\mathbb{R}^2$. Set
\[ I_k = \int_{\mathbb{R}^2} (f_k(s, t))^2 \, ds \, dt, \quad k = 1, 2, 3. \]
Prove that
\[ \int_{\mathbb{R}^3} f_1(x_2, x_3)f_2(x_1, x_3)f_3(x_1, x_2) \, dx_1 \, dx_2 \, dx_3 \leq (I_1I_2I_3)^{1/2}. \]

5. Let $E$ be a measurable subset of $\mathbb{R}$ such that $m(E) < \infty$. Let $f \in L^\infty(E)$ with $\|f\|_\infty > 0$. Show that
\[ \lim_{n \to \infty} \frac{\|f\|_{n+1}}{\|f\|_n} = \|f\|_\infty. \]
Here $\|f\|_n := \|f\|_{L^n(E)}$, $\|f\|_{n+1} := \|f\|_{L^{n+1}(E)}$. 

Analysis QR exam, September 11, 2015
Morning Session, 9:00–12:00
1. Prove that if \( f(z) \) is a non-constant, entire function and there is a constant \( c \in \mathbb{C} \) such that \( f(z) = c \) has infinitely many distinct solutions, then \( f(z) \) has an essential singularity at \( z = \infty \) in the sense that \( g(z) = f(\frac{1}{z}) \) has an essential singularity at \( z = 0 \).

2. Let \( w \) be a complex number such that \( |w| < 1 \). Evaluate the integral
\[
\int_0^{2\pi} \log|1 - we^{i\theta}|d\theta.
\]

3. Let \( S_6 \) and \( S_7 \) be the open squares centered at the origin of side length 6 and 7, respectively. Let \( \Gamma_6 \) and \( \Gamma_7 \) be the boundary of \( S_6 \) and \( S_7 \), respectively, and let \( \Omega \) be the region between \( \Gamma_6 \) and \( \Gamma_7 \).

   (a) Let \( f \) be a function that is holomorphic in an open neighborhood of \( \overline{\Omega} \). Prove that there are functions \( f_+ \) and \( f_- \) where \( f_+ \) is holomorphic in \( S_7 \), \( f_- \) is holomorphic in \( \mathbb{C} \setminus \overline{S}_6 \), and
\[
f(z) = f_+(z) + f_-(z), \quad \text{for all } z \in \Omega.
\]

   (b) How many different ways are there of writing \( f \) as the sum of \( f_+ \) and \( f_- \) as in part (a)? Find all possible ways.

4. Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( f \) be a holomorphic function on \( D \) such that its image \( f(D) \subset D \). Is it possible that \( f \) have two distinct fixed points in \( D \)? (A fixed point of \( f \) is a point \( z_0 \in D \) such that \( f(z_0) = z_0 \).) Find an example showing that it is possible or prove that it is not possible.

5. Prove that there are no polynomials \( p(z) \) satisfying
\[
|p(z) - \frac{1}{z}| < \frac{1}{2}, \quad 1 < |z| < 2.
\]