(1) (5pt) Show that
\[ \omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \]
restricts to a nonzero deRham cohomology class of
\[ S^1 = \{(x, y) \in \mathbb{R}^2; x^2 + x^2 = 1\}. \]

(2) (5pt) Any non-constant smooth function of a compact connected manifold of dimension has at least two critical points.

(3) (10pt) For each \( n \geq 1 \), there is a diffeomorphism
\[ (TS^n) \times \mathbb{R} \cong S^n \times \mathbb{R}^{n+1}. \]

(4)(15pt) Assuming that every \( n \)-dimensional compact manifold \( M^n \) can be embedded into some \( \mathbb{R}^N \), prove that we can choose \( N = 2n + 1 \). (Hint: Given a nonzero vector \( v \neq 0 \) in \( \mathbb{R}^N \), one can define parallel projection \( \phi_v \) from \( \mathbb{R}^N \) to the orthogonal complement of \( v \). If \( N > 2n + 1 \), we can choose some \( v \neq 0 \) such that the \( \phi_v|_{M^n} \) is an embedding).

(5)(15pt) (a) Show that the space of orthogonal matrices
\[ O(n) = \{A \in Mat_{n \times n}(\mathbb{R}); AA^t = \text{Id}\} \]
is a smooth manifold.
(b) Verify that the tangent space at identity matrix
\[ o(n) = T_{Id}U(n) = \{A \in Mat_{n \times n}(\mathbb{R}); A + A^t = 0\}. \]
(c) Show that the tangent bundle \( TO(n) \) can be trivialized, i.e.
\[ TO(n) \cong O(n) \times o(n). \]
1. Solution

(1) $\omega$ restricts to a one form on $S^1$. For dimension reason, it is automatically closed. It is enough to show that it is not exact. By Stokes theorem, it is enough to show

$$\int_{S^1} \omega \neq 0.$$ 

Using the parametrization $x = \cos \theta, y = \sin \theta$, we have

$$\int_{S^1} \omega = \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = 2\pi \neq 0.$$

(2) Suppose that $f : M \to \mathbb{R}$ is a non-constant smooth function. Since $M$ is compact, $f$ has both maximal and minimal and they are different. Moreover, dim $M \geq 1$ due to the connectedness. We claim that both maximum and minimum are critical points. We will prove it for maximum and the proof for minimum is the same.

Suppose that $x_0 \in M$ is a maximum of $f$ and $v \in T_{x_0}M$. Let $x(t) \in M$ such that $x(0) = x_0, \dot{x}(0) = v$. Then, $g(t) = f(x(t))$ is a one-variable function such that $t = 0$ is a maximum. By Cal I,

$$0 = \dot{g}(0) = df_{x_0}(v).$$

Hence, $x_0$ is a critical point.

(3) Suppose $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere. We observe that at any $x \in S^n$, $\mathbb{R}^{n+1}$ can be decomposed as the direct sum of $T_xS^n$ and normal bundle $N_x(S^n)$. Namely

$$TS^n \oplus N(S^n) \cong S^n \times \mathbb{R}^{n+1}.$$ 

We claim that $N(S^n)$ is trivial. For any $x \in S^n$, $x$ can also be viewed as a unit vector $v_x$ perpendicular to $S^n$. Hence,

$$x \to v_x$$

defines a nowhere vanishing section of $N(S^n)$. Furthermore, $N(S^n)$ is of the rank 1 and hence is trivial.

(4) Suppose that $M^n \subset \mathbb{R}^N$ is a submanifold. If $N \leq 2n + 1$, $\mathbb{R}^N \subset \mathbb{R}^{2n+1}$ and there is nothing to prove. Suppose that $N \geq 2n + 2$. By the hint, we would like to find a nonzero vector $v \in \mathbb{R}^N$ such that the parallel projection $\phi_v|_{M^n}$ is an embedding, i.e., one-to-one and immersion. Note that $\phi_v$ only depends on $v$ up to a scalar. Therefore, we can assume that $v \in S^{N-1}$. Note that

$\phi_v|_{M^n}$ is one-to-one iff $v \neq \frac{x-y}{||x-y||}$ for any $x, y \in M^n$ and $x \neq y$.

$\phi_v|_{M^n}$ is an immersion iff $v \notin T_xM^n$ for any $x$.

Define maps

$$T_1 : M^n \times M^n - \Delta = \{(x, x) \in M^n \times M^n\} \to S^{N-1},$$

by

$$(x, y) \to \frac{x-y}{||x-y||}$$

and

$$T_2 : TM^n - \{\text{zero section}\} \to S^{N-1}$$
by $v \rightarrow ||v||$. By Sard's theorem, there exists a $v \in S^{N-1}$ which is a regular value for both $T_1, T_2$. Note that

$$\dim M^n \times M^n - \Delta = \dim TM^n - \{\text{zero section}\} = 2n < \dim S^{N-1}.$$ 

Hence, $v \notin T_1(M^n \times M^n - \Delta) \cup T_2(TM^n - \{\text{zero section}\})$. By the previous argument, $\phi_v|_{M^n}$ is an embedding.

(5a) Define a map from $M_{n \times n}(\mathbb{R})$ to the space of symmetric matrices $Sym_{n \times n}$ by 

$$\Phi(A) = AA^t.$$ 

$Sym_{n \times n}$ is a vector space and hence a smooth manifold. It is enough to show that $Id$ is a regular value of $\Phi$. Suppose that $\Phi(A) = Id$. Note that the tangent map 

$$d\Phi_A(X) = XA^t + AX^t.$$ 

Any tangent vector $V \in T_Id Sym_{n \times n}$ can be written as $V = Y + Y^t$ for an upper triangular matrix $Y$. Let $X = YA$, i.e., $Y = XA^t$. Then, $Y^t = AX^t$. Namely, $d\Phi_A(X) = V$.

(5b) Suppose that $X \in 0(n)$. Then 

$$e^{tX}e^{tX^t} = Id.$$ 

. By differentiating with respect to $t$, we have 

$$X + X^t = 0.$$ 

(5c) For any $A \in O(n)$, the matrix multiplication by $A$ defines a smooth family of diffeomorphism 

$$\phi^A : O(n) \rightarrow O(n)$$ 

such that $\phi^A(Id) = A$. The tangent map 

$$d\phi^A_{Id} : o(n) \rightarrow T_A(O(n))$$ 

defines a trivialization of tangent bundle $O(n) \times o(n) \cong TO(n)$. 