1. Consider two disjoint squares $ABCD, EFGH$ in $\mathbb{R}^2$. Identify their sides as follows:

- $AD$ with $HG$,
- $DC$ with $EH$,
- $AB$ with $BC$,
- $EF$ with $FG$.

All identifications of sides are bijective linear, with the endpoints identified in the order given. Is the quotient space of the identification a compact surface (i.e. a compact topological 2-manifold)? If so, classify it.

2. For which $n \in \mathbb{N}$ does there exist a CW structure on $\mathbb{C}P^{2n}$ with no cell in dimension $n$? Prove your answer.

3. Let $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

Let $X$ be the quotient of $S^1 \times [0,1]$ by the smallest equivalence relation $\sim$ satisfying

$$(x, 0) \sim (e^{2\pi i / 3}x, 0),$$

$$(y, 1) \sim (e^{2\pi i / 6}y, 1)$$

for $x, y \in S^1$. Calculate $\pi_1(X)$.

4. Let

$$X_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

$$X_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \text{ and } z = 0\},$$

$$X_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0 \text{ and } 0 \leq z \leq 1\}.$$

Let $X = X_1 \cup X_2 \cup X_3$, with the induced topology from $\mathbb{R}^3$. Describe the universal covering space $\tilde{X}$ of $X$. 
5. Let

\[ S = \{ (z,t) \in \mathbb{C}^2 \mid |z|^2 + |t|^2 = 1 \}, \]
\[ T = \{ (z,t) \in S \mid t^3 \in [0,\infty) \subset \mathbb{R} \}. \]

Let \( X \) (resp. \( Y \)) be the quotient of \( S \) (resp. \( T \)) by the equivalence relation identifying

\[(z_1,t_1) \sim (z_2,t_2)\]

when \( z_1 = z_2\lambda, \ t_1 = t_2\lambda \) for some \( \lambda \in \mathbb{C} \) with \( \lambda^3 = 1 \).

(a) Calculate the homology of \( Y \).

(b) Calculate the homology of \( X \).
1. Let $S_1, S_2$ be two smooth embedded submanifolds of manifold $M$.
   1) Write down the definition for $S_1, S_2$ to be transversal.
   2) Show that if $S_1, S_2 \subset M$ are transversal, then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

2. Let $M$ be a smooth orientable manifold and let $\Psi : M \to \mathbb{R}$ be a smooth map. Show that if $0$ is a regular value of $\Psi$, then $\Psi^{-1}(0) \subset M$ is also a smooth orientable manifold.

3. Prove the following statement if it’s true, or disprove using a counterexample: Let $M$ and $N$ be two smooth manifolds, if the tangent bundles $TM$ and $TN$ are diffeomorphic, then $M$ and $N$ are diffeomorphic.

4. Consider the form $\omega = (x^2 + 2x + 3y + 4z)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $\iota : S^2 \to \mathbb{R}^3$ be the inclusion map.
   1) Evaluate the integral $\int_{S^2} \omega$.
   2) Construct a closed form $\theta$ on $\mathbb{R}^3$ s.t. $\iota^* \theta = \iota^* \omega$, or prove that such a form $\theta$ does not exist.

5. Prove the following statements about Lie groups:
   1) $SL_2(\mathbb{R}) = \{ A \in M_{2\times2}(\mathbb{R}) : \det A = 1 \}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.
   2) $SL_2(\mathbb{C}) = \{ A \in M_{2\times2}(\mathbb{C}) : \det A = 1 \}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$. 