(1) Suppose that \( L \subseteq \mathbb{Z}^2 \) is the subgroup generated by \((5, 4)\) and \((2, 7)\). Show that there is a unique subgroup \( M \subseteq \mathbb{Z}^2 \) of index 9 that contains \( L \). Give generators of \( M \).

(2) Suppose that \( A \) is an invertible square matrix with complex entries. Show that if \( A^2 \) is diagonalizable, then so is \( A \).

(3) Suppose that \( R \) is the subring of the polynomial ring \( \mathbb{Z}[x] \) consisting of all polynomials \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \) for which the coefficients \( a_1, a_2, \ldots, a_n \) are even (but \( a_0 \) does not have to be even).

(a) Show that \( R \) contains a maximal ideal that is not finitely generated.

(b) What is the ring \( R/(3) \)? Is it finite?

(4) Consider the vector space \( V = \mathbb{F}_p^4 \) where \( \mathbb{F}_p \) is the field with \( p \)-elements.

(a) How many 2-dimensional subspaces does \( V \) have?

(b) Suppose that a subgroup \( G \subseteq \text{GL}_4(\mathbb{F}_p) \) is a \( p \)-group. Show that there exists a 2-dimensional subspace \( W \) of \( V \) such that \( g \cdot W \subseteq W \) for all \( g \in G \).

(5) Let \( K \) be the splitting field of \( X^4 - 2 \) over \( \mathbb{Q} \).

(a) What is the Galois group of \( K \) over \( \mathbb{Q} \)?

(b) Find all subfields \( L \) of \( K \) such that \([L : \mathbb{Q}] = 4\). (Here \([L : \mathbb{Q}] \) is the degree of the field extension \( L/\mathbb{Q} \).)
(1) Suppose that $M$ is a field containing $\mathbb{F}_p$ and $K$ and $L$ are subfields of $M$. Assume that the number of elements of $K$, $L$ and $M$ are with $p^6$, $p^{10}$ and $p^{60}$ respectively. How many elements do the fields $KL$ and $K \cap L$ have?

(2) Let $A \in \text{Mat}_{n,n}(K)$ be an $n \times n$ matrix with entries in the field $K$ and suppose that the characteristic polynomial of $A$ is irreducible over $K$.
(a) Let $L \subseteq \text{Mat}_{n,n}(K)$ be the $K$-span of $I, A, A^2, \ldots, A^{n-1}$. Show that $L$ is a subring of $\text{Mat}_{n,n}(K)$, and show that the ring $L$ is a field.
(b) For a nonzero vector $v \in K^n$, prove that $v, Av, \ldots, A^{n-1}v$ form a basis of $K^n$.

(3) Calculate the following groups:
(a) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(2)$.
(b) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(9)$.
(c) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(4) Let $V$ be a vector space of dimension 3.
(a) Show that there exists a linear map $\varphi : \Lambda^2 V \otimes V \rightarrow \Lambda^2 V \otimes V$ such that
$$\varphi((a \wedge b) \otimes c) = (a \wedge c) \otimes b - (b \wedge c) \otimes a.$$ 
(b) Determine the eigenvalues of $\varphi$ and their multiplicities.

(5) Let $G$ be a group of even order. Show that there exists an element in $G$ of order 2 whose conjugacy class has an odd number of elements.