SAITO-KUROKAWA LIFTS, $L$-VALUES FOR $GL_2$, AND CONGRUENCES BETWEEN SIEGEL MODULAR FORMS

by

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# TABLE OF CONTENTS

**DEDICATION** ............................................................................................................................ ii  
**ACKNOWLEDGEMENTS** ................................................................................................................ iii  

**CHAPTER**

I. Introduction ................................................................................................................................. 1  
1.1 Set-up of the Problem .................................................................................................................. 1  
1.2 Motivation and Outline of the Argument .................................................................................... 2  
1.3 Overview of the Argument ......................................................................................................... 4  

II. Modular Forms ............................................................................................................................ 9  
2.1 Characters .................................................................................................................................. 9  
2.2 Integral Weight Modular Forms .................................................................................................. 11  
2.3 Half-Integral Weight Modular Forms ........................................................................................ 17  
2.4 Jacobi Forms ............................................................................................................................. 21  
2.5 Siegel Modular Forms ................................................................................................................ 23  

III. The Saito-Kurokawa Correspondence ....................................................................................... 35  
3.1 Integer Weight Forms to Half-Integer Weight Forms ............................................................... 36  
3.2 Half-Integer Weight Forms to Jacobi Forms .......................................................................... 37  
3.3 Jacobi Forms to Siegel Forms .................................................................................................. 38  
3.4 CAP Forms ............................................................................................................................... 40  

IV. Eisenstein Series .......................................................................................................................... 42  
4.1 Basic Definitions ......................................................................................................................... 42  
4.2 The Fourier Coefficients of $E(Z, s, \chi)$ ................................................................................. 45  
4.3 Pullbacks and an Inner Product Relation .................................................................................. 50  

V. Simplifying Shimura’s Inner Product Relation .......................................................................... 53  
5.1 Relating $\langle F_f, F_f \rangle$ to $\langle \phi_f, \phi_f \rangle$ and $L(k, f)$ ......................................................... 53  
5.2 Relating $\langle \phi_f, \phi_f \rangle$ to $\langle g_f, g_f \rangle$ ........................................................................... 57  
5.3 Relating $\langle g_f, g_f \rangle$ to $\langle f, f \rangle$ .................................................................................. 59  
5.4 Decomposing the Standard $L$-function ..................................................................................... 60  

VI. Periods and a Certain Hecke Operator ....................................................................................... 63  
6.1 Eichler-Shimura Theory ............................................................................................................. 64  
6.2 Periods Associated to Newforms ............................................................................................. 66
6.3 A Certain Hecke Operator: Ordinary Case ......................... 68
6.4 A Certain Hecke Algebra: $M \geq 4$ ............................... 71

VII. The Congruence .................................................. 75
7.1 Congruent to a Modular Form ................................... 75
7.2 Congruent to a Cuspidal Eigenform ......................... 84

VIII. Selmer Groups and Galois Representations ..................... 88
8.1 Galois Representations: Definitions and Basic Facts ............. 88
8.2 Selmer Groups ..................................................... 92
8.3 Galois Arguments .................................................. 99
8.4 Numerical Example ............................................... 112

BIBLIOGRAPHY ......................................................... 117
CHAPTER I

Introduction

1.1 Set-up of the Problem

In mathematics some of the most beautiful and striking results occur when two seemingly unrelated objects come together. This trend is particularly apparent in number theory where one has many theorems and conjectures relating special values of $L$-functions, analytic objects, to the sizes of certain class groups, or more generally, Selmer groups, arithmetic objects. For example, Dirichlet’s class number formula provides a precise relationship between the class number of a number field $K$ and the residue at $s = 1$ of the $L$-function attached to the field $K$. One also has the famous conjecture of Birch and Swinnerton-Dyer that predicts that the order of vanishing at $s = 1$ of the $L$-function attached to an elliptic curve $E$ is related to the rank of the elliptic curve. The conjecture that motivates the work in this thesis is that of Bloch and Kato (a generalization of the Birch and Swinnerton-Dyer conjecture). The version of their conjecture we are interested in states that the special values of an $L$-function attached to an eigenform $f$ should measure the sizes of the Selmer groups attached to $f$. In particular, we will prove that if $f$ is an eigenform of weight $2k - 2$ defined over some $p$-adic ring $\mathcal{O}$ with uniformizer $\varpi$ and $\varpi \mid L_{\text{alg}}(k, f)$, then $\varpi$ divides the order of the Selmer group associated to $f$. There are in fact many
additional conditions that need to be satisfied, but for simplicity we omit them here (see Theorem VIII.24 in Chapter VIII for a precise statement).

1.2 Motivation and Outline of the Argument

We roughly follow the method developed by Ribet in his proof of the converse of Herbrand’s theorem ([57]) and then extended by Wiles ([87]) to prove the Main Conjecture of Iwasawa theory for totally real fields. We briefly review this method as it gives a good general framework for understanding the structure of the argument to be presented.

Let $\chi$ be a primitive Dirichlet character of conductor $N$. Let $p$ be an odd prime with $\gcd(p, N) = 1$ and $k \geq 1$ an integer so that $\chi(-1) = (-1)^k$. Denote the $p$-adic cyclotomic character by $\varepsilon$ and its reduction modulo $p$ by $\omega$. Associated to $\chi$ is an Eisenstein series $E_{\chi,k}(z)$ given by

$$E_{\chi,k}(z) = \sum_{n=0}^{\infty} c(n)q^n$$

where $c(\ell) = 1 + \chi(\ell)\ell^{k-1}$ for $\ell \nmid N$ and $c(0) = \frac{L(1-k, \chi)}{2}$. If $\varpi | L(1 - k, \chi)$ for $\varpi$ a uniformizer of $\mathbb{Z}_p[\chi]$, this Eisenstein series “looks like” a cusp form modulo $\varpi$. More precisely, for some $M$ with $N \mid M$ there exists a $p$-adic ring $\mathcal{O} \supseteq \mathbb{Z}_p[\chi]$ with prime $p$ and a cusp form $f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(M), \chi, \mathcal{O})$ so that $p \nmid a_f(p)$ and

$$a_f(n) \equiv c(n) \pmod{p}$$

for every $n$ with $\gcd(n, M) = 1$. In other words,

$$a_f(\ell) \equiv 1 + \chi(\ell)\ell^{k-1} \pmod{p}$$

for every prime $\ell \nmid M$. Using this congruence one is able to show that $f$ has a
non-split residual Galois representation of the form

\[ \bar{\rho}_{f,p} \simeq \begin{pmatrix} 1 & * \\ 0 & \chi \omega^{k-1} \end{pmatrix} \]

(so \(* \neq 0\), in particular). From this, under reasonable hypotheses on \(\chi\) (\(\chi\) has order prime to \(p\) suffices), one can show that \(*\) gives a non-zero cohomology class in \(H^1_{ur}(\mathbb{Q}, \chi^{-1} \omega^{1-k})\), which in turn gives a nontrivial piece of the class group of the splitting field of \(\chi^{-1} \omega^{1-k}\).

The general outline that one should take away from this is that one starts with a 1-dimensional object, a character, and associates to it an Eisenstein series. One then finds a 2-dimensional object, a cusp form, that is congruent to the Eisenstein series modulo a suitable prime dividing a certain \(L\)-value. From this congruence one can make deductions about the Galois representation associated to the cusp form to construct a cohomology class.

In our argument we will replace the character with a weight \(2k - 2\) newform \(f\), a 2-dimensional object. Associated to this newform via the Saito-Kurokawa correspondence is a Siegel cusp form \(F_f\) which “looks like” an Eisenstein series. We then show that if \(\varpi \mid L_{alg}(k,f)\) then there is a Siegel eigenform \(G\) that is not a Saito-Kurokawa lift so that \(F_f \equiv G(\mod \varpi)\). Attached to \(G\) is a 4-dimensional Galois representation. The congruence with \(F_f\) allows us to make deductions about this Galois representation, producing a non-trivial element of \(p\)-power order in our Selmer group.

Chapter II consists of a review of some basic definitions and theorems about the various types of modular forms that will be encountered in this thesis. The reader who is familiar with modular forms can skip this chapter and refer back to it if necessary. The Saito-Kurokawa correspondence is outlined in Chapter III with
references directing the reader to papers where the correspondence is demonstrated. In Chapter IV we discuss an Eisenstein series as constructed by Shimura. We show that with a suitable normalization this Eisenstein series is holomorphic and has $p$-integral Fourier coefficients. A theorem of Shimura that gives the inner product of this Eisenstein series with $F_f$ is also stated here. In Chapter V we prove a formula relating $\langle F_f, F_f \rangle$ and $\langle f, f \rangle$ as well as decompose the standard $L$-function of $F_f$ into a product of $L$-functions including the $L$-function of $f$. We discuss periods attached to modular forms as well as construct a related special Hecke operator in Chapter VI. These are both important in constructing our congruence with an eigenform $G$.

All of the previous results are combined in Chapter VII to produce the congruence between $F_f$ and $G$ alluded to above. Finally, Chapter VIII consists of the Galois representation arguments we use to produce the nontrivial element of the Selmer group.

1.3 Overview of the Argument

Let $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))$ be a newform with Fourier coefficients in some $p$-adic ring $\mathcal{O}$. The Saito-Kurokawa correspondence is a Hecke-equivariant isomorphism between $S_{2k-2}^{\text{new}}(\Gamma_0(M))$ and $S_k^{+,\text{new}}(\Gamma_0^4(M))$, the Maass space of Siegel newforms of weight $k$ on $\Gamma_0^4(M) \subset \text{Sp}_4(\mathbb{Z})$. The isomorphism is established by first showing there is an isomorphism between $S_{2k-2}^{\text{new}}(\Gamma_0(M))$ and Kohnen’s $+\text{-space}$ of half-integer weight modular forms $S_{k-1/2}^{+,\text{new}}(\Gamma_0(4M))$. Therefore, associated to $f$ we have a half-integer weight form $g_f$. Next one shows that $S_{k-1/2}^{+,\text{new}}(\Gamma_0(4M))$ is isomorphic to the space of Jacobi forms $J_{k,1}^{\text{cusp},\text{new}}(\Gamma_0(M))$, giving us a Jacobi form $\phi_f$ associated to $f$. Finally, one proves $J_{k,1}^{\text{cusp},\text{new}}(\Gamma_0(M))$ is isomorphic to $S_k^{\text{cusp},\text{new}}(\Gamma_0^4(M))$, giving us our Siegel modular form $F_f$ associated to $f$. One has the decomposition of the Spinor $L$-
function associated to $F_f$:

$$L^*_{\text{spin}}(s, F_f) = \zeta(s - k + 1)\zeta(s - k + 2)L(s, f),$$

see Theorem III.8. We show that the correspondence can be normalized so that $F_f$ has Fourier coefficients in $O$ (Corollary III.9).

Let $\chi$ be a Dirichlet character of conductor $N$ with $M \mid N$. Fix $n > 1$ and let $\mathfrak{h}^n$ be Siegel upper half-space, i.e.,

$$\mathfrak{h}^n = \{ Z \in M_n(\mathbb{C}) : tZ = Z, \text{Im}(Z) > 0 \}.$$

We study an Eisenstein series $E(Z, s, \chi)$ defined on $\mathfrak{h}^n \times \mathbb{C}$. We study this Eisenstein series normalized by multiplying by

$$\pi^{-\frac{n(n+2)}{4}} L_N(2s, \chi) \prod_{j=1}^{\lfloor n/2 \rfloor} L_N(4s - 2j, \chi^2)$$

and then evaluated at $s = (n + 1 - k)/2$. It is a theorem of Shimura that with such a normalization this Eisenstein series is a Siegel modular form of weight $k$ and level $N$ as long as $k > \max\{3, n + 1\}$. We prove that the Fourier coefficients of the normalized Eisenstein series all lie in $\mathbb{Z}_p[\chi, i^{nk}]$ for any prime $p$ such that $p > n$ and $p \nmid N$.

Our next step is to specialize to the case where $n = 4$. A great deal of work has been done by Garrett ([26], [27]) and Shimura ([72], [73]) concerning the notion of the pullback of the Eisenstein series to $\mathfrak{h}^2 \times \mathfrak{h}^2$. Pulling our Eisenstein series back gives us a Siegel modular form of weight $k$ and level $N$ in each of the variables $Z$ and $W$ in $\mathfrak{h}^2 \times \mathfrak{h}^2$. We apply an inner product relation of Shimura ([72], Equation 6.17) to deduce the formula

$$\langle \mathcal{E}(Z, W), F_f(W) \rangle = A \cdot L_N(5 - k, \lambda_{F_f}, \chi)F_f(Z)$$
where $\mathcal{E}$ is our Eisenstein series appropriately normalized, $A$ is a non-zero constant depending on $k$ and $N$, and $L_N(2s, \lambda_{F_f}, \chi)$ is the standard $L$-function associated to $F_f$ (see Theorem IV.9).

In the case where $M \neq N$, we take the trace of the Eisenstein series $\mathcal{E}$ to lower the level to $M$. Since $\mathcal{E}$ is now a Siegel modular form of weight $k$ and level $M$ in each of the variables $Z$ and $W$, we can write

$$\mathcal{E}(Z, W) = \sum_{i,j} c_{i,j}(F_i(Z) \otimes F_j(W)) \quad (1.2)$$

for $\{F_i\}$ a basis of $\mathcal{M}_k(\Gamma_0^1(M))$ with $F_0 = F_f$ and $F_i$ orthogonal to $F_f$ for $i > 0$. Using this expansion with Equation 1.1 and solving for $c_{0,0}$ we obtain the equation

$$c_{0,0} = \frac{AL_N(5 - k, \lambda_{F_f}, \chi)}{\langle F_f, F_f \rangle} \quad (1.3)$$

At this point we turn our attention to studying $c_{0,0}$, as this will ultimately produce the congruence we desire.

There are two main steps in studying $c_{0,0}$: factoring the standard $L$-function and writing $\langle F_f, F_f \rangle$ in terms of $\langle f, f \rangle$. It is a fairly easy result to show that

$$L_N(2s, \lambda_{F_f}, \chi) = L_N(2s - 2, \chi)L_N(2s + k - 3, f, \chi)L_N(2s + k - 4, f, \chi).$$

To express $\langle F_f, F_f \rangle$ in terms of $\langle f, f \rangle$ takes a bit more work. We use the explicit nature of the Saito-Kurokawa lift to first relate $\langle F_f, F_f \rangle$ to $\langle \phi_f, \phi_f \rangle$ by generalizing a result of Kohnen and Skoruppa ([41]). This step is where the factor $L(k, f)$ arises. Next we relate $\langle \phi_f, \phi_f \rangle$ to $\langle g_f, g_f \rangle$. Finally, we use a result of Kohnen ([39]) to relate $\langle g_f, g_f \rangle$ to $\langle f, f \rangle$. Combining the results we obtain

$$\langle F_f, F_f \rangle = B\frac{L(k, f)}{L(k - 1, f, \chi_D)}\langle f, f \rangle$$

where $\mathcal{E}$ is our Eisenstein series appropriately normalized, $A$ is a non-zero constant depending on $k$ and $N$, and $L_N(2s, \lambda_{F_f}, \chi)$ is the standard $L$-function associated to $F_f$ (see Theorem IV.9).
where $D$ is a discriminant with $(-1)^{k-1}D > 0$, $\gcd(N, D) = 1$, $c_{\eta_f}(|D|) \neq 0$ and $B$ is a non-zero constant. Combining these results with Equation 1.3 we obtain

\begin{equation}
(1.4) \quad c_{0,0} = C \frac{L(k-1, f, \chi_D)L_N(3-k, \chi)L_N(1, f, \chi)L_N(2, f, \chi)}{L(k, f)\langle f, f \rangle}
\end{equation}

where $C$ is a non-zero constant. Note that all of these non-normalized values are transcendental, but the ratio is algebraic.

In order to study the $p$-divisibility of $c_{0,0}$, we need to normalize the $L$-values and the transcendental factor $\langle f, f \rangle$. We accomplish this by conjecturing the existence of a Hecke operator $t$ with eigenvalue $u \frac{\langle f, f \rangle}{\Omega_f^+\Omega_f^-}$ when acting on $f$ for $u$ a unit in $O$ and such that $t$ kills all other forms in a basis of $S_{2k-2}^{\text{new}}(\Gamma_0(M), O)$. We prove this conjecture in the case that $f$ is ordinary at $p$ or the level of $f$ is greater than or equal to 4. (It appears that the restriction that $f$ is ordinary for levels less then 4 is purely a technical one that can be overcome with further work.) Using that the Saito-Kurokawa correspondence is Hecke equivariant, we have a Hecke operator $t_S$ on Siegel modular forms that has the same eigenvalue when acting on $F_f$ as $t$ does when acting on $f$ and is such that it kills all other Saito-Kurokawa lifts linearly independent from $F_f$. Applying $t_S$ to Equation 1.2 we obtain

\begin{equation}
((1 \otimes t_S)\mathcal{E})(Z, W) = \frac{\alpha}{\varpi^m}(F_f(Z) \otimes F_f(W)) + \sum_{i,j} c_{i,j}(F_i(Z) \otimes (t_SF_j)(W))
\end{equation}

for some $\varpi$-unit $\alpha$. Recalling that $\mathcal{E}$ and $F_f$ both have $\varpi$-integral Fourier coefficients, this gives us a congruence $F_f \equiv G(\mod \varpi^m)$ for a Siegel modular form $G$. Arguing with the Siegel operator $\Phi$ and Galois representations we show that we have a congruence between the eigenvalues of $F_f$ and those of a cuspidal Siegel eigenform that is not a Saito-Kurokawa lift. See Theorem VII.9 for the precise statement as well as the hypotheses.

Associated to our eigenform $G$ is a 4-dimensional $p$-adic Galois representation
(\rho_G, V_G). The congruence established between \(F_f\) and \(G\) gives us that the residual representation \(\overline{\rho}_G\) has semi-simplification \(\omega^{k-2} \oplus \overline{\rho}_f \oplus \omega^{k-1}\) where \(\omega\) is the reduction of the \(p\)-adic cyclotomic character \(\varepsilon\). Using this we show by brute force that there is a \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-stable lattice \(T_G\) so that \(\overline{\rho}_G\) has the form

\[
\overline{\rho}_G = \begin{pmatrix}
\omega^{k-2} & *_1 & *_2 \\
*_3 & \overline{\rho}_{f,p} & *_4 \\
& & \omega^{k-1}
\end{pmatrix}
\]

where either \(*_1\) or \(*_3\) is zero.

Next we show that the lattice can be chosen so that no conjugate of \(\overline{\rho}_G\) has \(*_2\) and \(*_4\) both equal to zero. We accomplish this by modifying an argument of Ribet ([57]) and using the fact that if \(\rho_G\) has a 1-dimensional sub-quotient, then it must be a CAP form. At this point there are 6 possibilities for the shape of \(\overline{\rho}_G\) and we split these into 2 cases. The first case corresponds to the situation in which the \(*_4\) is zero so \(*_2\) gives us a non-zero cohomology class. We are able to rule this case out by showing it would give a nontrivial quotient of the \(\omega^{-1}\)-isotypical piece of the \(p\)-part of the class group of \(\mathbb{Q}(\mu_p)\) (no such non-trivial piece can exist).

The second case corresponds to the situation when \(*_4\) gives a non-zero cohomology class in \(H^1(\mathbb{Q}, W_1)\) where \(W_1\) is the \(\omega\)-torsion in \(W_f = V_f/T_f\). We then show through a series of lemmas that this cohomology class gives a non-zero \(\omega\)-torsion element of the Selmer group. This provides the divisibility that we seek.

We conclude with a numerical example showing that for weight 54 and level 1 there is a newform \(f\) so that a prime over 516223 divides \(L_{\text{alg}}(28, f)\) as well as the order of the Selmer group attached to \(f\).
CHAPTER II

Modular Forms

In this chapter we present a quick overview of the information concerning modular forms necessary to understand the main results in this thesis. As this thesis deals primarily with various types of modular forms (e.g. integral and half-integral weight modular forms, Jacobi forms, and Siegel modular forms), relations between them, and objects associated to them (e.g. $L$-functions), it is essential to start with basic definitions and facts. The reader who is familiar with various types of modular forms can skip this chapter. Proofs are not included, as they are standard and given in the references listed. We start out with an elementary section on characters. Next we give an introduction to elliptic modular forms on $\text{SL}_2(\mathbb{Z})$ before moving on to half-integral weight modular forms, Jacobi forms, and then concluding with a discussion of Siegel modular forms. The descriptions of half-integer weight and Jacobi modular forms are brief, as they do not play a prominent role.

2.1 Characters

Let $N$ be a positive integer. A Dirichlet character modulo $N$ is a group homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$. We can extend $\chi$ to a homomorphism on all of $\mathbb{Z}$ by declaring that $\chi(n) = 0$ if $\gcd(n, N) > 1$ and $\chi(n) = \chi(n \mod N)$ if $\gcd(n, N) = 1$. We say that $\chi'$ modulo $N'$ is induced by the character $\chi$ if $N \mid N'$ and $\chi'(n) = \chi(n)$
whenever $\gcd(n, N') = 1$. A character which is not induced by a character of strictly smaller level is said to be \textit{primitive}.

For any Dirichlet character $\chi$ we can define an associated $L$-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{Re}(s) > 1.$$ 

Since $\chi$ is multiplicative, we also have an Euler product expansion for $L(s, \chi)$ given by

$$L(s, \chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}.$$ 

If $\chi$ is not the trivial character $L(s, \chi)$ has an analytic continuation to the entire complex plane. If $\chi$ is a primitive character modulo $N$ and $a \in \{0, 1\}$ satisfies $\chi(-1) = (-1)^a$, then $L(s, \chi)$ satisfies the functional equation

$$ \left(\frac{N}{\pi}\right)^{\frac{2}{\pi}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi) = c_\chi \left(\frac{N}{\pi}\right)^{\frac{1-s}{2}} \Gamma\left(\frac{1-s+a}{2}\right) L(1-s, \overline{\chi})$$

where $c_\chi = i^{-a} \tau(\chi) N^{-1/2}$ and $\tau(\chi)$ is the usual Gauss sum associated to $\chi$ and the choice of $i = \sqrt{-1} \in \mathbb{C}$. For a positive integer $M$, write $L_M(s, \chi)$ to denote the Euler product $\prod_{p \nmid M} (1 - \chi(p)p^{-s})^{-1}$ and $L_p(s, \chi)$ to denote the $p^{th}$ Euler factor of $L(s, \chi)$.

In addition to Dirichlet characters we will also need Hecke characters. Let $A = A_\mathbb{Q}$ be the adeles of $\mathbb{Q}$. A \textit{Hecke character} $\psi$ is a continuous homomorphism $\psi : A^\times \to \mathbb{C}^\times$ such that $\psi(\mathbb{Q}^\times) = \{1\}$. For any such Hecke character, denote the restriction of $\psi$ to $\mathbb{Q}_\ell^\times$ by $\psi_\ell$. The continuity of $\psi$ ensures that it is trivial on a subgroup of the form

$$H = \prod_{p \nmid S} (1 + p^{k_p} \mathbb{Z}_p) \prod_{p \in S} \mathbb{Z}_p^\times$$

for a finite set of primes $S$. It is easy to see that there is an isomorphism

$$\hat{\mathbb{Z}}^\times / H \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

where $N = \prod_{p \in S} p^{k_p}$. Given a Hecke character, this allows us to associate a Dirichlet character to it. Conversely, suppose we are given a primitive Dirichlet character $\chi$
modulo $N$. We associate a Hecke character $\psi$ to $\chi$ as follows. Let $N = \prod p^{k_p}$ as above. The Chinese Remainder Theorem allows us to write
\[ (\mathbb{Z}/N\mathbb{Z})^\times \cong \prod (\mathbb{Z}/p^{k_p}\mathbb{Z})^\times. \]
Therefore we can reduce to the case that $N = p^k$, $p$ a prime. Define
\[ \psi_\infty(x) = \begin{cases} 1 & \chi(-1) = 1 \\ \text{sgn}(x) & \chi(-1) = -1 \end{cases} \]
(This just says that we want $\psi$ to be odd or even according to whether $\chi$ is odd or even.) For $\ell \neq p$, set $\psi_\ell(\ell^m u) = \chi(\ell)^m$ for $u \in \mathbb{Z}_p^\times$, $m \in \mathbb{Z}$. Set $\psi_p(u) = \chi^{-1}(b)$ for $b \equiv u \pmod{p^k}$ and $u \in \mathbb{Z}_p$. It is easy to see that $\psi = \prod_{\ell \leq \infty} \psi_\ell$ determines a finite-order Hecke character. This association allows us to move back and forth between Dirichlet characters and Hecke characters. This will be convenient in what follows as some of the constructions will be given in terms of Hecke characters and some in terms of Dirichlet characters.

### 2.2 Integral Weight Modular Forms

This section collects some of the necessary results on elliptic modular forms. There are many good books on this subject the reader can consult for proofs and further details. Two standard references are [55] and [70]. We will refer to elliptic modular forms just as modular forms when it is clear from the context.

For a commutative ring $R$ we let $M_n(R)$ denote the set of $n \times n$ matrices with entries in $R$. We use $\text{GL}_n(R)$ to denote the subset of $M_n(R)$ with unit determinant and $\text{SL}_n(R)$ the subgroup of $\text{GL}_n(R)$ with determinant equal to 1. Let $I_n$ denote the identity element of $\text{GL}_n(R)$. We will mainly be interested in $\text{SL}_2(\mathbb{Z})$ and its congruence subgroups. Let $N$ be a positive integer. The most interesting subgroups
of SL₂(ℤ) for us will be
\[
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 (\text{mod } N), b \equiv c \equiv 0 (\text{mod } N) \right\},
\]
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 (\text{mod } N) \right\},
\]
and
\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv 1 (\text{mod } N) \right\}.
\]
We say a subgroup of SL₂(ℤ) is a congruence subgroup of level N if it contains Γ(N).

In general we will denote our matrices by \( \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \) where the subscripts will be dropped when they are clear from context.

Let \( \mathfrak{h}^1 \) denote the complex upper half-plane, i.e., \( \mathfrak{h}^1 = \{ z = x + iy \in \mathbb{C} \mid y > 0 \} \).

There is an action of SL₂(ℝ) on \( \mathfrak{h}^1 \cup \{ \infty \} \cup \mathbb{Q} \) given by linear fractional transformations. Explicitly, for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \) and \( z \in \mathfrak{h}^1 \cup \mathbb{Q} \) we have \( \gamma z = \frac{az + b}{cz + d} \) and \( \gamma \infty = a/c \). Therefore we have an action of any subgroup of SL₂(ℤ) on \( \mathfrak{h}^1 \cup \{ \infty \} \cup \mathbb{Q} \). We refer to \( \mathbb{Q} \cup \{ \infty \} \) as the cusps. A congruence subgroup will permute these cusps.

For example, one can show that SL₂(ℤ) permutes the cusps transitively.

Let \( f \) be a function defined on \( \mathfrak{h}^1 \cup \mathbb{Q} \cup \{ \infty \} \) and taking values in \( \mathbb{C} \cup \{ \infty \} \). Let \( k \) be a positive integer. We can define an action of GL⁺₂(ℚ) on \( f \) where GL⁺₂(ℚ) is the subgroup of GL₂(ℚ) consisting of matrices with positive determinant. This action is given by
\[
(f|_k \gamma)(z) = (\det \gamma)^{k/2}(cz + d)^{-k} f(\gamma z)
\]
for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{Q}) \). We drop the weight from the notation when it is clear from the context.
Suppose that \( f \) is meromorphic on \( \mathbb{H}^1 \) and \( f|_\gamma = f \) for all \( \gamma \in \Gamma \). If we assume \( \Gamma \) is a congruence subgroup then there exists a minimal positive integer \( h \) so that \( \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \). Thus \( f(z + h) = f(z) \) and so \( f \) has a Fourier expansion of the form

\[
    f(z) = \sum_{n=-\infty}^{\infty} a_f(n) q_h^n
\]

where \( q_h = e^{2\pi iz/h} \). Note that for \( \Gamma = \Gamma_1(N) \) or \( \Gamma = \Gamma_0(N) \) we have \( h = 1 \).

Let \( \gamma \in \text{SL}_2(\mathbb{Z}) \). If \( f \) is invariant under \( \Gamma \supset \Gamma(N) \), \( f|_\gamma \) is invariant under the action of \( \gamma^{-1}\Gamma\gamma \). Since \( \Gamma(N) \) is normal in \( \text{SL}_2(\mathbb{Z}) \), we have that \( \Gamma(N) \subset \gamma^{-1}\Gamma\gamma \) as well. Therefore, \( f|_\gamma \) is also invariant under translation by \( N \) so it too has a Fourier expansion in powers of \( q_h \). Let \( \alpha \) be a cusp. Since \( \text{SL}_2(\mathbb{Z}) \) is transitive on the cusps, there exists \( \gamma \in \text{SL}_2(\mathbb{Z}) \) such that \( \alpha = \gamma \infty \). If \( f|_\gamma \) has a Fourier expansion with \( a_f(n) = 0 \) for all \( n < 0 \) we say that \( f \) is holomorphic at \( \alpha \).

**Definition II.1.** Let \( k \) be a positive integer and let \( \Gamma \) be a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Let \( f \) be a holomorphic function on \( \mathbb{H}^1 \cup \mathbb{Q} \cup \{\infty\} \) such that \( f|_\gamma = f \) for every \( \gamma \in \Gamma \). We say that \( f \) is a modular form of weight \( k \) for the congruence subgroup \( \Gamma \) and denote the space of such forms by \( M_k(\Gamma) \). If, in addition we have that \( a_f(0) = 0 \) in the Fourier expansion of \( f \) at each cusp then we call \( f \) a cusp form.

We denote the space of cusp forms by \( S_k(\Gamma) \). If the Fourier coefficients of \( f \) all lie in a ring \( \mathcal{O} \), we write \( f \in M_k(\Gamma, \mathcal{O}) \) if \( f \) is a modular form and similarly for \( f \) a cusp form.

Our next step is to recall Hecke operators for a congruence subgroup \( \Gamma \). Let \( G = \text{GL}_2^+(\mathbb{Q}) \) and

\[
    \Gamma' = \{ \alpha \in G : \alpha \Gamma \alpha^{-1} \sim \Gamma \}
\]

where \( \sim \) denotes that the groups are commensurable, i.e., their intersection has finite
index in each of the groups. Let $\alpha \in \Gamma'$. There exists a finite set of $\alpha_j \in \Gamma'$ such that

\[ \Gamma \alpha \Gamma = \bigsqcup_j \Gamma \alpha_j. \]

Define an action of $\Gamma \alpha \Gamma$ on a modular form $f \in M_k(\Gamma)$ by

\[ f|_{[\Gamma \alpha \Gamma]} = \det(\alpha)^{k/2 - 1} \sum_j f|_{\alpha_j}. \]

The Hecke algebra $T(\Gamma, \mathbb{Z})$ is the algebra consisting of finite sums $\sum n_i [\Gamma \alpha \Gamma]$ where $n_i \in \mathbb{Z}$ and $\alpha \in \Gamma'$. We will write $T_{\mathbb{Z}}$ for this Hecke algebra when $\Gamma$ is clear from the context. For a positive integer $n$ we define the $n^{th}$ Hecke operator $T(n)$ by

\[ T(n) = \sum_{\substack{\alpha \in \Gamma' \\ \det(\alpha) = n}} [\Gamma \alpha \Gamma] \]

If $\Gamma$ is a congruence subgroup of minimal level $N$ and $n \mid N$ then we denote the $n^{th}$ Hecke operator by $U(n)$ to emphasize that $n$ divides the level.

To make the notion of $\Gamma'$ a little more concrete we compute $\text{SL}_2(\mathbb{Z})'$ and $\Gamma_0(N)'$:

\[ \text{SL}_2(\mathbb{Z})' = M_2(\mathbb{Z}) \cap \text{GL}_2^+ (\mathbb{Q}) \]

and

\[ \Gamma_0(N)' = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : N | c, \gcd(a, N) = 1, \det(\alpha) > 0 \right\}. \]

Though the definition of Hecke operators that we have given is the definition that generalizes most easily, it is not necessarily the easiest definition for getting a feel for what the Hecke operators do to a modular form. One can work out explicitly the action of the Hecke operators by computing the $\alpha_j$’s. For example, for $\Gamma = \text{SL}_2(\mathbb{Z})$ the action of the $n^{th}$ Hecke operator is given by

\[ T(n) f(z) = \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \leq b < d} f \left( \frac{az + b}{d} \right). \]
If \( m \) and \( n \) are relatively prime, then \( T(mn) = T(m)T(n) \) and similarly for \( U(m) \) and \( U(n) \) and \( T(m) \) and \( U(n) \). We say that \( f \) is an eigenform of \( T(n) \) if there exists \( \lambda_f(n) \in \mathbb{C} \) such that \( T(n)f = \lambda_f(n)f \), and similarly for \( U(n) \).

Let \( f, g \in M_k(\Gamma) \). If at least one of \( f \) and \( g \) is a cusp form, we can define the Petersson product of \( f \) and \( g \) by

\[
\langle f, g \rangle = \frac{1}{[\text{SL}_2(\mathbb{Z}): \Gamma]} \int_{\Gamma \setminus \mathbb{H}} f(z)\overline{g(z)}y^{k-2}dxdy.
\]

where \( \text{SL}_2(\mathbb{Z}) \) means \( \text{SL}_2(\mathbb{Z})/ \pm I_2 \) and \( \Gamma \) is the image of \( \Gamma \) in \( \overline{\text{SL}_2(\mathbb{Z})} \). With this normalization the Petersson product is independent of the congruence group \( \Gamma \). It is also true that the Hecke operators are self-adjoint with respect to the Petersson product:

\[
\langle T(n)f, g \rangle = \langle f, T(n)g \rangle, \quad \gcd(n, N) = 1.
\]

Now we come to the definition of newforms. Let \( N \) be a positive integer and

\[
S_{k}^\text{old}(\Gamma_0(N)) = \sum_{0 \leq r \leq |N|} U(d)S_k(\Gamma_0(r)), \quad r \neq N.
\]

Set \( S_{k}^\text{new}(\Gamma_0(N)) \) to be the orthogonal complement of \( S_{k}^\text{old}(\Gamma_0(N)) \) in \( S_k(\Gamma_0(N)) \) with respect to the Petersson product. One can think of the elements in \( S_{k}^\text{new}(\Gamma_0(N)) \) as the analogues of primitive Dirichlet characters as they do not “come from” a modular form of lower level. We call \( f \in S_{k}^\text{new}(\Gamma_0(N)) \) a newform if \( a_f(1) = 1 \) and \( f \) is a Hecke eigenform for all Hecke operators. We have the following important theorem:

**Theorem II.2.** (Atkin-Lehner) There is an orthogonal basis \( f_1, \ldots, f_r \) of newforms for \( S_{k}^\text{new}(\Gamma_0(N)) \) with \( T(n)f_j = \lambda_{f_j}(n)f_j \) for all \( n \) and \( f_j(z) = \sum_{n=1}^{\infty} \lambda_{f_j}(n)q^n \).

This is in contrast to the case of \( S_k(\Gamma_0(N)) \) where in general we have only an eigenbasis with eigenforms for \( T(n) \) with \( (n, N) = 1 \).
Next we turn our attention to $L$-functions. Let $f \in S_k(\Gamma_0(N))$. Define the $L$-function associated to $f$ by

$$L(s, f) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$$

where the $a_f(n)$’s are the Fourier coefficients of $f$. It is well known that $L(s, f)$ has analytic continuation to the entire complex plane and satisfies a functional equation.

Let $\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and define $g = f|\omega_N$. Write

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, f).$$

This is known as the completed $L$-function of $f$. The functional equation for $L(s, f)$ is given by

$$\Lambda(s, f) = i^{k}\Lambda(k - s, g).$$

Suppose that $f \in S_{k}^{\text{new}}(\Gamma_0(N))$ is a newform. The $L$-function $L(s, f)$ has an Euler product expansion

$$L(s, f) = \prod_{p\nmid N} (1 - \lambda_f(p)p^{-s} + p^{k-1-2s})^{-1} \prod_{p\mid N} (1 - \lambda_f(p)p^{-s})^{-1}.$$ 

For a positive integer $M$ we will write $L_M(s, f)$ to mean the $L$-function so that the Euler product is only over primes not dividing $M$ and $L^{(p)}(s, f)$ to denote the $p^{th}$ Euler factor.

We will also need to consider the twisted $L$-functions $L(s, f, \psi)$ for $\psi$ a Dirichlet character. Define these twisted $L$-functions by

$$L(s, f, \psi) = \sum_{n=1}^{\infty} \psi(n)a_f(n)n^{-s}.$$ 

The functions $L(s, f, \psi)$ have analytic continuations, functional equations, and Euler product expansions similar to those of the $L(s, f)$’s.
2.3 Half-Integral Weight Modular Forms

In this section we continue to review facts about modular forms, this time turning our attention to the theory of half-integral weight modular forms as laid out by Shimura ([62]) and refined by Kohnen ([37], [38], [39]). We follow Koblitz ([36]) in our presentation of the basic material. The main interest for us in half-integral weight modular forms will be the role they play in the Saito-Kurokawa correspondence, so we limit ourselves to the facts that are necessary in this regard.

Let $\sqrt{z}$ be the branch of the square root function taking its argument in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For $m \in \mathbb{N}$, define $z^{m/2} = (\sqrt{z})^m$.

Following ([62], Page 442), we define the “quadratic residue symbol” $(\frac{a}{b})$ characterized by the following properties.

1. $(\frac{a}{b})$ if $\gcd(a, b) \neq 1$.
2. If $b$ is an odd prime, $(\frac{a}{b})$ coincides with the ordinary quadratic residue symbol, i.e., it is one less then the number of solutions of $x^2 \equiv a \pmod{b}$.
3. If $b > 0$, the map $a \mapsto (\frac{a}{b})$ defines a character modulo $b$.
4. If $a \neq 0$, the map $b \mapsto (\frac{a}{b})$ defines a character modulo a divisor of $4a$, whose conductor is the conductor of $\mathbb{Q}(\sqrt{a})$ over $\mathbb{Q}$. We denote this character by $\chi_a$.
5. $(\frac{a^2}{b^2}) = 1$ or $-1$ according as $a > 0$ or $a < 0$.
6. $(\frac{a}{b}) = 0$.

Note that this definition differs from the traditional one which has the property that $(\frac{a}{b}) = (\frac{a}{|b|})$. Let $k$ be a positive integer and $g$ a complex valued function on the complex upper half-plane. Define a slash operator in this case by

$$(g|k^{-1/2}\gamma)(z) = j(\gamma, z)^{-1}g(\gamma z)$$
where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $j(\gamma, z) = \left( \frac{a}{d} \right) \left( \frac{-d}{a} \right)^{-k-3/2} (cz + d)^{k-1/2}$. We use the more cumbersome $k - 1/2$ instead of $k/2$ as this will end up being the weight of the half-integral weight modular forms we are interested in. The Legendre symbols appear here to account for the fact that if we naively defined the slash operator as in the case for integral weight modular forms we would be off by a sign.

We would like to define this slash operator for all of $GL_2^+(\mathbb{Q})$ as in the case of integral weight modular forms so that we can define Hecke operators. The problem is that there is no preferred branch of the square root function for all elements in $GL_2^+(\mathbb{Q})$. The way around this difficulty is to work with a larger group then $GL_2^+(\mathbb{Q})$. We need this larger group $\mathfrak{G}$ to contain four “copies” of each element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, one for each branch of the square root of $(cz + d)$, as well as one for each branch of the square root of $-(cz + d)$. Therefore we need $\mathfrak{G}$ to be a four-sheeted cover of $GL_2^+(\mathbb{Q})$. This is accomplished by setting $\mathfrak{G}_{k-1/2}$ to be the group of pairs $(\gamma, \phi(z))$ where $\gamma \in GL_2^+(\mathbb{Q})$ and $\phi$ is a complex-valued holomorphic function on $\mathfrak{h}^1$ such that

$$|\phi(z)| = (\det \gamma)^{k/2+1/4} |cz + d|^{k-1/2}$$

with the group law defined by $(\gamma, \phi(z)) \cdot (\alpha, \psi(z)) = (\gamma \alpha, \phi(\alpha z) \psi(z))$. The covering map $pr : \mathfrak{G}_{k-1/2} \to GL_2^+(\mathbb{Q})$ is projection onto the first factor. The group $\mathfrak{G}_{k-1/2}$ is what is referred to as a metaplectic group.

The group $\mathfrak{G}_{k-1/2}$ acts on functions $g : \mathfrak{h}^1 \to \mathbb{C}$ by

$$(g | (\gamma, \phi))(z) = \phi(z)^{-1} g(\gamma z).$$
Let $\Gamma \subset \Gamma_0(4)$ be a congruence subgroup. Define
\[ \tilde{\Gamma} = \{ \tilde{\gamma} = (\gamma, j(\gamma, z)) : \gamma \in \Gamma \}. \]
Suppose that $\Gamma$ is a congruence subgroup of level $4N$. In this case the map $\gamma \mapsto (\gamma, j(\gamma, z))$ is a left inverse of the map $pr$, so we can define an action of any congruence subgroup of $\Gamma_0(4)$ on the set of functions $g : \mathfrak{h}^1 \to \mathbb{C}$ through the action of $\mathfrak{S}_{k-1/2}$. 
Suppose that $g$ is a meromorphic function on $\mathfrak{h}^1$ so that $(g|_{k-1/2}) = g$ for all $\gamma \in \Gamma$ where we understand that the action is through $\mathfrak{S}_{k-1/2}$ but omit the $\sim$'s. As in the case of integer weight modular forms we get a Fourier expansion of $g$ at each cusp and define $g$ to be holomorphic at the cusp if the Fourier coefficients $c_g(n)$ vanish for $n < 0$.

**Definition II.3.** Let $k$ be a positive integer and let $\Gamma \subseteq \Gamma_0(4)$ be a congruence subgroup. Let $g$ be a holomorphic function on $\mathfrak{h}^1$ such that $g|_{k-1/2} = g$ for all $\gamma \in \Gamma$. We say that $g$ is a *modular form of weight $k - 1/2$ for $\Gamma$* if $g$ is holomorphic at all the cusps. Denote the space of weight $k - 1/2$ modular forms by $M_{k-1/2}(\Gamma)$. If $g$ vanishes at all the cusps (i.e., $c_g(0) = 0$) we say that $g$ is a cusp form of weight $k - 1/2$ for $\Gamma$. Denote the space of weight $k - 1/2$ cusp forms by $S_{k-1/2}(\Gamma)$. If the Fourier coefficients of $g$ all lie in a ring $\mathcal{O}$, we write $g \in M_{k-1/2}(\Gamma, \mathcal{O})$ if $g$ is a modular form. Similarly if $g$ is a cusp form.

We now come to the definition of Hecke operators on half-integral weight modular forms. We will not make heavy use of these; we just need to know they exist and behave as we would expect. Therefore our discussion is brief. One should consult
([62], Section 1) for further details. Let $m$ be a positive integer; set

$$T_{k-1/2}(m^2) = m^{k-5/2} \left[ \left( 1, 0 \right), m^{k-1/2} \right] \Gamma_0(4N).$$

We define an action of $T_{k-1/2}(m^2)$ on $M_{k-1/2}(\Gamma)$ as in the integral weight case. We could define $T(m)$ similarly, but it turns out that unless $m$ is a perfect square the action of $T(m)$ is multiplication by 0. (See ([36], Page 204, Proposition 12) for a proof of this fact.) We write $U(m^2)$ in place of $T(m^2)$ when $m \mid 4N$.

Let $g \in S_{k-1/2}(\Gamma_0(4N))$ have a Fourier expansion given by $g(z) = \sum_{n=1}^{\infty} c_g(n)q^n$. Then for a prime $p$ such that $p \nmid 4N$ the action of Hecke operators on the Fourier expansion is given by

$$T(p^2)g(z) = \sum_{n=1}^{\infty} \left( c_g(p^2n) + \left( \frac{(-1)^{k-1}n}{p} \right) p^{k-2}c_g(n) + p^{2k-3}c_g(n/p^2) \right) q^n.$$ If $p \mid 4N$ the action of $U(p^2)$ is given by replacing the $n^{th}$ Fourier coefficient by the $p^2 n^{th}$ Fourier coefficient.

Let $f, g \in S_{k-1/2}(\Gamma)$. The Petersson product of $f$ and $g$ is defined to be

$$\langle f, g \rangle = \frac{1}{[\Gamma_0(4) : \Gamma]} \int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)} y^{k-5/2} \, dx \, dy.$$

We will be interested in a subspace $S^{+}_{k-1/2}(\Gamma_0(4N))$ of $S_{k-1/2}(\Gamma_0(4N))$ known as Kohnen’s $+$-space. We will see in the next chapter that there is a Hecke-equivariant isomorphism between $S^{+,new}_{k-1/2}(\Gamma_0(4N))$ and $S^{new}_{2k-2}(\Gamma_0(N))$, which is the reason for our interest in this particular subspace. It consists of all cusp forms $g$ so that the Fourier expansion of $g$ is given by

$$g(z) = \sum_{n \geq 1} c_g(n)q^n.$$

$(-1)^{k-1}n \equiv 0, 1 (\text{mod } 4)$
For our purposes it is most natural to define the space of newforms of half-integer weight in terms of the relation between half-integer weight modular forms and that of integral weight modular forms so we postpone this definition until the next chapter.

2.4 Jacobi Forms

In this section we will outline the basics of Jacobi forms. The standard reference for these facts is Eichler and Zagier’s book [21]. One can also find facts on Jacobi forms in [49].

Let $\phi$ be a complex-valued function defined on $\mathfrak{h}^1 \times \mathbb{C}$. Let $k$ and $m$ be positive integers. We will actually only be concerned with the case of $m = 1$, but we include general $m$ in the definitions here for the sake of completeness. Define an action of $\text{SL}_2(\mathbb{Z})$ and $\mathbb{Z}^2$ on $\mathfrak{h}^1 \times \mathbb{C}$ by

$$\gamma \cdot (\tau, z) = \left( \frac{a_\gamma \tau + b_\gamma}{c_\gamma \tau + d_\gamma}, \frac{z}{c_\gamma \tau + d_\gamma} \right)$$

and

$$(\lambda, \mu) \cdot (\tau, z) = (\tau, z + \lambda \tau + \mu).$$

One can show that these define an action of the semi-direct product

$$\Gamma^J_1 = \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 = \{(M, X) \in \text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2 \mid (M, X)(M', X') = (MM', XM' + X')\}$$

on $\mathfrak{h}^1 \times \mathbb{C}$. The group $\Gamma^J_1$ is the full Jacobi group. For a congruence subgroup $\Gamma$, let $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$.

Define an action of $\Gamma^J_1$ on $\phi$ by

$$(\phi|\gamma)(\tau, z) = (c_\gamma \tau + d_\gamma)^{-k} e_m \left( -\frac{c_\gamma^2 z^2}{c_\gamma \tau + d_\gamma} \right) \phi \left( \frac{a_\gamma \tau + b_\gamma}{c_\gamma \tau + d_\gamma}, \frac{z}{c_\gamma \tau + d_\gamma} \right)$$

and

$$(\phi | (\lambda, \mu))(\tau, z) = e_m (\lambda^2 \tau + 2\lambda z) \phi(\tau, z + \lambda \tau + \mu)$$

for $\gamma \in \text{SL}_2(\mathbb{Z})$, $(\lambda, \mu) \in \mathbb{Z}^2$, and $e_m(z) = e^{2\pi i m z}$. 
Definition II.4. Let $k, m \in \mathbb{N}$ and $\Gamma \subset \text{SL}_2(\mathbb{Z})$ be a congruence subgroup. Let $\phi : \mathfrak{h}^1 \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying

1. $\phi | \gamma = \phi$ for every $\gamma \in \Gamma$.
2. $\phi | (\lambda, \mu) = \phi$ for every $(\lambda, \mu) \in \mathbb{Z}^2$.
3. For each $\gamma \in \text{SL}_2(\mathbb{Z})$, $\phi | \gamma$ has a Fourier expansion of the form $\sum_{n,r} c_\phi(n,r) q^n \zeta^r$ where $q = e(\tau), \zeta = e(z)$, with $c_\phi(n,r) = 0$ unless $n \geq \frac{r^2}{4m}$.

We say $\phi$ is a Jacobi form of weight $k$ and index $m$. If, in addition we have that $c_\phi(n,r) = 0$ whenever $n = \frac{r^2}{4m}$ we say that $\phi$ is a cusp form. Denote the space of Jacobi forms on $\Gamma$ of weight $k$ and index $m$ by $J_{k,m}(\Gamma)$ and the cusp forms by $J^\text{cusp}_{k,m}(\Gamma)$. As in the case of integer weight and half-integral weight modular forms, if the Fourier coefficients of $\phi$ lie in a ring $\mathcal{O}$, we write $\phi \in J_{k,m}(\Gamma, \mathcal{O})$.

It is also often convenient to write the Fourier expansion of a Jacobi form in terms of a discriminant $D = r^2 - 4nm$:

$$\phi(\tau, z) = \sum_{\substack{D \leq 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4m}}} c_\phi(D, r) e\left(\frac{r^2 - D}{4m} \tau + rz\right).$$

We will switch back and forth between the notation depending upon which is more convenient in our situation and which is the standard notation in the literature for each situation.

Let $\phi, \psi \in J_{k,m}(\Gamma)$ with at least one a cusp form. Define the Petersson product of $\phi$ and $\psi$ to be

$$\langle \phi, \psi \rangle = \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma]} \int_{\mathfrak{h}^1 \times \mathbb{C}} \phi(\tau, z) \overline{\psi(\tau, z)} v^{k-3} e^{-4\pi my^2/v} \, dx \, dy \, du \, dv$$

where $\tau = u + iv, z = x + iy$.

Let $\phi \in J_{k,m}(\Gamma_0(N)) = J_{k,m}(N)$ with $c_\phi(D, r)$ its $(D, r)$-th Fourier coefficient as above. The following are formulas for the linear operators $V_n, n \in \mathbb{N}$ and Hecke
operators $T_J(p), p \nmid mN$, $U_J(p), p \mid mN$:

$$V_n\phi = \sum_{D \leq 0, r \in \mathbb{Z}}^{D \equiv r^2 \mod 4mn} \left( \sum_{d \mid \gcd(r^2 - D, n, r)} d^k c_\phi \left( \frac{D}{d^2}, \frac{r}{d} \right) \right) e \left( \frac{r^2 - D}{4mn} \tau + rz \right),$$

$$T_J(p)\phi = \sum_{D \leq 0, r \in \mathbb{Z}}^{D \equiv r^2 \mod 4m} c_\phi^*(D, r) e \left( \frac{r^2 - D}{4mn} \tau + rz \right) \quad \text{for} \ p \nmid mN$$

where

$$c_\phi^*(D, r) = c_\phi(p^2 D, pr) + p^{k-2} \left( \frac{D}{p} \right) c_\phi(D, r) + p^{2k-3} c_\phi \left( \frac{D}{p^2}, \frac{r}{p} \right),$$

and

$$U_J(p)\phi = \sum_{D \leq 0, r \in \mathbb{Z}}^{D \equiv r^2 \mod 4m} c_\phi(p^2 D, pr) e \left( \frac{r^2 - D}{4mn} \tau + rz \right) \quad \text{for} \ p \mid mN.$$

One can check that the Hecke operators preserve the space of cusp forms. The operator $V_n$ is an index changing operator and maps $J_k(mN_0(N))$ to $J_k(mnN_0(N))$.

As in the case of half-integer weight modular forms it is most natural to define the space of Jacobi newforms in terms of the relation between Jacobi forms and integer weight newforms. Therefore we postpone the definition until the next chapter.

### 2.5 Siegel Modular Forms

Good references for Siegel modular forms are more difficult to find then for elliptic modular forms. Klingen’s book ([35]) contains some of the basic material, but does not address Hecke operators or $L$-functions. One can consult many papers listed in the references for a quick summary of facts on Siegel modular forms. For instance, Andrianov ([1], [5]) and Shimura ([71]) both give overviews of a lot of the necessary information on Siegel modular forms.

Let $n$ be a positive integer greater then 1. Recall that the real positive symplectic group $G^{2n} = \text{GSp}_{2n}^+(\mathbb{R})$ is defined by

$$\text{GSp}_{2n}^+(\mathbb{R}) = \{ \gamma \in M_{2n}(\mathbb{R}) : {}^t \gamma t_n \gamma = \mu(\gamma) t_n, \ \mu(\gamma) > 0 \}$$
where $\iota = \iota_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and the subscript $n$ denotes an $n \times n$ matrix. The $n$ will be dropped from the notation when it is clear from context. Let

$$\Sigma = \Sigma^{2n} = C^{2n} \cap M_{2n}(\mathbb{Z}),$$

$$\Sigma_m^{2n} = \{ \gamma \in \Sigma^{2n} : \mu(\gamma) = m \},$$

and

$$S^n(R) = \{ \gamma \in M_n(R) : \iota^t \gamma = \gamma \}$$

for any commutative ring $R$.

The Siegel modular group is

$$\text{Sp}_{2n}(\mathbb{Z}) = \{ \gamma \in \Sigma : \mu(\gamma) = 1 \}.$$

For a positive integer $N$, we define the congruence subgroups $\Gamma^{2n}(N)$, $\Gamma_0^{2n}(N)$, and $\Gamma_1^{2n}(N)$ analogously to the SL$_2(\mathbb{Z})$ case:

$$\Gamma^{2n}(N) = \left\{ \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \text{Sp}_{2n}(\mathbb{Z}) \mid A_n \equiv D_n \equiv 1_n \pmod{N}, B_n \equiv C_n \equiv 0_n \pmod{N} \right\},$$

$$\Gamma_0^{2n}(N) = \left\{ \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \text{Sp}_{2n}(\mathbb{Z}) \mid C_n \equiv 0_n \pmod{N} \right\},$$

and

$$\Gamma_1^{2n}(N) = \left\{ \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in \Gamma_0^{2n}(N) \mid A_n \equiv 1_n \pmod{N} \right\}$$

with the convention that the congruences regarding the matrices are with respect to their entries. We will drop the subscripts on the matrix entries when they are clear from the context. If $\Gamma \subset \text{Sp}_{2n}(\mathbb{Z})$ is a subgroup of finite index that contains $\Gamma^{2n}(N)$ for some $N$ we say that $\Gamma$ is a congruence subgroup of level $N$. 
The Siegel upper half-space:

$$\mathfrak{h}^n = \{ Z \in \mathbb{S}^n(\mathbb{C}) \mid \text{Im}(Z) > 0 \}$$

provides a generalization of complex upper half-space on which the real positive symplectic group acts. Note that each $Z \in \mathfrak{h}^n$ can be written as $Z = X + iY$ with $X, Y \in \mathbb{S}^n(\mathbb{R})$ and $Y > 0$ where we write $Y > 0$ to mean $Y$ is positive definite. Whenever we write $Z = X + iY$ we will mean with such $X$ and $Y$. The real positive symplectic group acts on Siegel upper half-space via

$$G^{2n} \times \mathfrak{h}^n \to \mathfrak{h}^n, \quad (\gamma, Z) \mapsto \gamma Z = (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}.$$ 

Let $k$ be a positive integer, $\gamma \in G^{2n}$, and $F$ a complex-valued function on $\mathfrak{h}^n$. Define a slash operator of $\gamma$ on $F$ by

$$(F|_{k}\gamma)(Z) = \det(C_\gamma Z + D_\gamma)^{-k}F(\gamma Z).$$

For a subgroup $K$ of $G^{2n}$, if $F|_{k}\gamma = F$ for every $\gamma \in K$, then we say that $F$ is $K$-automorphic of weight $k$. We will drop the weight from the slash operator when it is clear from the context.

Suppose $K$ is a subgroup of $G^{2n}$ that is commensurable with $\text{Sp}_{2n}(\mathbb{Z})$. Then $K$ necessarily contains a subgroup of $\text{Sp}_{2n}(\mathbb{Z})$ of finite index; so there exists integers $r \in \mathbb{N}$ such that every matrix of the form

$$B = \begin{pmatrix} I_n & rb \\ 0_n & I_n \end{pmatrix}$$

belongs to $K$ where $b \in \mathbb{S}^{2n}(\mathbb{Z})$. Thus we have that

$$F|(B) = F(Z + rb) = F(Z)$$
for every $b \in \mathbb{S}^{2n}(\mathbb{Z})$. Therefore we have that $F$ is periodic of period $r$ in each of the complex variables $Z_{i,j}$ where the $Z_{i,j}$'s are the entries of $Z$. This gives a Fourier expansion, which can be shown to be of the form

$$F(Z) = \sum_{T \in P_n} A_F(r^{-1}T)e^{2\pi i \text{Tr}(r^{-1}TZ)}$$

where $\text{Tr}(A)$ is the trace of the matrix $A$ and

$$P_n = \{T \in \mathbb{S}^n(\mathbb{Q}) : t_{i,i}, 2t_{i,j} \in \mathbb{Z}(i \neq j)\}.$$

Note that for $\Gamma = \Gamma_{0}^{2n}(N), r = 1$.

**Definition II.5.** Let $K$ be a subgroup of $G^{2n}$ commensurable with $\text{Sp}_{2n}(\mathbb{Z})$. A function $F : \mathfrak{h}^n \to \mathbb{C}$ is said to be a *Siegel modular form* for $K$ of weight $k$ if $F|\gamma = F$ for every $\gamma \in K$ and $F$ is holomorphic. Let $\mathcal{M}_k(K)$ denote the space of all Siegel modular forms for $K$ of weight $k$. If the Fourier coefficients of $F$ all lie in a ring $\mathcal{O}$, we write $F \in \mathcal{M}_k(\Gamma, \mathcal{O})$ if $F$ is a modular form and likewise for $F$ a cusp form.

Note that by the Koecher principle that we have $A_F(T) = 0$ unless $T > 0$. Also note that for $\gamma \in \Sigma$ we have that $F|\gamma$ is still a Siegel modular form.

**Definition II.6.** Let $F \in \mathcal{M}_k(K)$. For $\gamma \in \Sigma$, write

$$F|\gamma(Z) = \sum_{T \in P_n} A_{F|\gamma}(T)e^{2\pi i \text{Tr}(TZ)}.$$

We say $F$ is a *Siegel cusp form* (or just cusp form) if $A_{F|\gamma}(T) = 0$ unless $T$ is positive definite for all $\gamma \in \Sigma$. The vector space of Siegel cusp forms of weight $k$ for $K$ is denoted by $\mathcal{S}_k(K)$.

Our next step is to recall the Petersson product in the Siegel modular form case. We begin by recalling that the there is a $G^{2n}$ invariant measure on $\mathfrak{h}^n$ given by

$$d\mu Z = \det(Y)^{-(n+1)} \prod_{\alpha \leq \beta} dx_{\alpha,\beta} \prod_{\alpha \leq \beta} dy_{\alpha,\beta}$$
where \( Z = X + iY = (x_{\alpha,\beta}) + i(y_{\alpha,\beta}) \) and \( dx_{\alpha,\beta}, dy_{\alpha,\beta} \) are the usual Euclidean measures on \( \mathbb{R} \).

Let \( D(K) \) be a fundamental domain of \( K \) on \( \mathfrak{h}^n \), i.e., a closed subset of \( \mathfrak{h}^n \) which contains a representative of each \( K \)-orbit on \( \mathfrak{h}^n \), and has no interior points belonging to the same \( K \)-orbit. For \( F, G \) two Siegel modular forms with at least one of them a cusp form for \( K \) of weight \( k \), define the Petersson product of \( F \) and \( G \) by

\[
\langle F, G \rangle = \frac{1}{[\text{Sp}_{2n}(\mathbb{Z}) : K_1]} \int_{D(K_1)} F(Z) \overline{G(Z)} \det(Y)^k d\mu_Z
\]

where \( K_1 \) is a subgroup of \( K \cap \text{Sp}_{2n}(\mathbb{Z}) \) of finite index, \( K_1 = K_1 \cup (-I_{2n})K_1/(\pm I_{2n}) \) and \( \overline{\text{Sp}_{2n}(\mathbb{Z})} = \text{Sp}_{2n}(\mathbb{Z})/(\pm I_{2n}) \). The integral converges absolutely and the Petersson product is independent of the choice of \( K_1 \). We also have that for \( \gamma \in G^{2n} \) with rational entries,

\[
\langle F|\gamma, G|\gamma \rangle = \mu(\gamma)^{-nk}\langle F, G \rangle.
\]

We now come to the definitions of the Hecke algebra and Hecke operators for the symplectic case. Let \( g \in \Sigma \) and let \( K \) be a subgroup of \( G^{2n} \) that is commensurable with \( \text{Sp}_{2n}(\mathbb{Z}) \). The double coset \( KgK \) is a finite union of left cosets modulo \( K \) ([5], Lemma 1.4). Write \( KgK = \bigsqcup Kg_i \). For a ring \( R \), the Hecke algebra is then given by the \( R \)-algebra consisting of formal finite linear combinations \( t = \sum a_i (Kg_i) \) that are invariant under the natural right action of \( K \). We denote the Hecke algebra by \( \mathbb{T}_{S,R}(K) \), where we drop the \( K \) when it is understood from the context. The elements \( KgK = \sum_i Kg_i \) form a free \( R \)-basis of \( \mathbb{T}_{S,R}(K) \). Elements in \( \mathbb{T}_{S,R}(K) \) are called Hecke operators.

Let \( F \) be a \( K \)-invariant complex-valued function on \( \mathfrak{h}^n \) of weight \( k \). The space of such functions is invariant under the action of Hecke operators given by

\[
t : F \mapsto tF = \sum_i a_i \mu(g_i)^{nk-(\langle n \rangle)} F|g_i
\]
where \( t = \sum_i a_i K g_i \) and \( \langle n \rangle = n(n + 1)/2 \). These are what are referred to as the normalized Hecke operators by Andrianov ([5]) (because of the \( \mu(g_i) \) factor). These are the operators used in most applications.

**Proposition II.7.** ([5], Proposition 1.8) The spaces \( \mathcal{M}_k(K) \) and \( \mathcal{S}_k(K) \) are invariant under all Hecke operators.

We now specialize to the case of \( n = 2 \) and \( K = \Gamma_0^1(N) \) as this is the case we will be interested in and there are more explicit results for this case. We refer the reader to [1] and [51] for more details as well as proofs of the results. Let

\[
\widetilde{\Sigma}^4 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : C \equiv 0 \pmod{N}, A \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \right\}.
\]

Let \( p \) be a prime and \( m \) a positive integer and let \( R_p(m) \) denote the set of matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) where \( \{a, b\} \) runs through a complete set of representatives modulo the equivalence relation \( \{a, b\} \sim \{a', b'\} \) if there exists \( r \in \mathbb{Z}, \gcd(r, p) = 1 \) such that \( ra = a' \) and \( rb = b' \) modulo \( p^m \). Set

\[
V(p^m) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Sigma^4_{p^m} : D \in \begin{pmatrix} p^a & 0 \\ 0 & p^{a+b} \end{pmatrix} R_p(b), a + b \leq m, a \geq 0, b \geq 0, A = p^m D^{-1}, B(\text{mod } D) \right\}.
\]

Let \( \widetilde{\Sigma}^4_{p^m} = \Sigma^4_{p^m} \cap \widetilde{\Sigma}^4 \).

**Lemma II.8.** ([51], Lemma 1.7) With the notation as above, the following assertions hold.

1. If \( p \) is a prime which does not divide \( N \), then

\[
\widetilde{\Sigma}^4_{p^m} = \prod_{\alpha \in V(p^m)} \Gamma_0^1(N) \alpha,
\]
2. If \( p \) is a prime factor of \( N \), then

\[
\tilde{\Sigma}_p^4 = \prod_s \Gamma_0^4(N) \begin{pmatrix} I_2 & S \\ 0_2 & p^m I_2 \end{pmatrix} = \Gamma_0^4(N) \begin{pmatrix} I_2 & 0_2 \\ 0_2 & p^m I_2 \end{pmatrix} \Gamma_0^4(N),
\]

where \( S \) runs through all integral symmetric matrices modulo \( p^m \).

For each positive integer \( m \), define the \( m \)th Hecke operator \( T_S(m) \) by

\[
T_S(m) = \sum_{\alpha} \Gamma_0^4(N) \alpha \Gamma_0^4(N)
\]

where the sum is taken over all \( \alpha \in \Gamma_0^4(N) \backslash \tilde{\Sigma}_p^4 / \Gamma_0^4(N) \).

**Lemma II.9.** ([51], Lemma 1.9) For positive integers \( m \) and \( n \) satisfying \( \gcd(m,n) = 1 \), we have the identity

\[
T_S(m)T_S(n) = T_S(mn).
\]

From this Lemma it is easy to see that it suffices to understand \( T_S(p^m) \) for the primes \( p \) in order to understand all the Hecke operators \( T_S(m) \). We use the work of Andrianov ([1], Proposition 2.1.2) and Matsuda ([51]) to determine the action of the Hecke operators \( T_S(p^m) \) on the Fourier coefficients of a Siegel modular form \( F \).

Let \( a \) be a function on \( M_2(\mathbb{Q}) \) that is zero away from \( P_2 \) and satisfies the relation \( a(UTU') = a(T) \) for all \( U \in \mathrm{SL}_2(\mathbb{Z}) \). For a prime number \( p \), define operators

\[
(\Delta(p^m)a)(T) = a(p^mT)
\]

and

\[
(\Pi(p^m)a)(T) = \sum_{U,U' \in R_p(m)} a \begin{pmatrix} p^{-m} & 0 \\ 0 & 1 \end{pmatrix} U T U' \begin{pmatrix} 1 & 0 \\ 0 & p^m \end{pmatrix}.
\]

Let \( F \in M_k(\Gamma_0^4(N)) \) have a Fourier expansion given by

\[
F(Z) = \sum_{T \in P_2} a_F(T) e^{2\pi i \text{Tr}(TZ)}
\]
and write
\[ T_S(m)F(Z) = \sum_{T \in P_2} a_F(m; T)e^{2\pi i \text{Tr}(TZ)}. \]

**Lemma II.10.** ([51], Lemma 2.1) The notation being as above, the following equalities hold.

1. If \( p \nmid N \),
   \[ a_F(p^m; T) = \begin{cases} 
   \sum_{b+c+d=m} p^{(2k-2)c+(2k-3)d}(\Delta(p^{-d})\Pi(p^c)\Delta(p^d)a_F)(T) & p^{-d}T \in P_2 \\
   0 & p^{-d}T \notin P_2,
   \end{cases} \]

2. If \( p \mid N \), \( a_F(p^m; T) = a_F(p^mT) \).

We define Hecke eigenforms as in the case of integer weight forms. We say that \( F \) is a **Hecke eigenform**, or just an **eigenform**, if there exists \( \lambda_F(m) \in \mathbb{C} \) such that \( T_S(m)F = \lambda_F(m)F \) for all \( m \geq 1 \). However, in the case of Siegel modular forms there is no nice relationship between the eigenvalues of a eigenform \( F \) and the Fourier coefficients of \( F \).

In addition to the Hecke operators we will need the Siegel operator \( \Phi \). The Siegel operator is a linear operator that takes modular forms on \( \text{Sp}_{2n}(\mathbb{Z}) \) to modular forms on \( \text{Sp}_{2n-2}(\mathbb{Z}) \). The only case we will be interested in is the case taking \( \mathcal{M}_k(\Gamma) \) to \( M_k(\Gamma') \) for congruence subgroups \( \Gamma \subseteq \text{Sp}_4(\mathbb{Z}) \) and \( \Gamma' \subseteq \text{SL}_2(\mathbb{Z}) \). The Siegel operator is defined by

\[
\Phi(F(\tau)) = \lim_{\lambda \to \infty} F \left( \begin{array}{cc} \tau & 0 \\ 0 & i\lambda \end{array} \right)
\]

where \( \tau \in \mathfrak{h}^1 \) for \( F \in \mathcal{M}_k(\text{Sp}_4(\mathbb{Z})) \). We can also express the action of the Siegel operator in terms of the Fourier coefficients of \( F \). Let \( Z = \left( \begin{array}{cc} \tau & z \\ z & \tau' \end{array} \right) \), then we have

\[
\Phi(F(\tau)) = \sum_{n \geq 0} a_F \left( \begin{array}{cc} n & 0 \\ 0 & 0 \end{array} \right) e^{2\pi int}. \quad (2.2)
\]
See ([4], Equation 3.50). Note that Equation 2.2 shows that if $F$ has Fourier coefficients in some ring $\mathcal{O}$, then so does $\Phi(F)$. We can characterize Siegel cusp forms in terms of the Siegel operator.

**Theorem II.11.** ([4], Theorem 3.13) Let $F \in \mathcal{M}_k(\Gamma)$ for $\Gamma$ a congruence subgroup of $\text{Sp}_4(\mathbb{Z})$. If $\Phi(F|\gamma) = 0$ for every $\gamma \in \text{Sp}_4(\mathbb{Z})$ then $F$ is a cusp form, and conversely.

We will also be interested in the Siegel operator’s interaction with the Hecke operators.

**Theorem II.12.** ([25], Satz 4.4) Let $\Gamma$ be a congruence subgroup of $\text{Sp}_4(\mathbb{Z})$ of level $N$ and $F \in \mathcal{M}_k(\Gamma)$. For a prime number $p \nmid N$, we have

$$\Phi(T_S(p)F) = (1 - p^{2-k})T(p)\Phi(F).$$

**Theorem II.13.** ([25], Satz 4.5) If $F$ is an eigenform of the operator $T_S(p)$, then $\Phi(F)$ is an eigenform of $T(p)$.

Let $\Gamma \subset \text{Sp}_4(\mathbb{Z})$ be a congruence subgroup. In this thesis we will be particularly interested in a certain subspace of $\mathcal{M}_k(\Gamma)$ defined and studied by Maass ([45], [46]). Appropriately, this is called the Maass space and denoted by $\mathcal{M}_k^\ast(\Gamma)$. Before we give the definition, note that since $n = 2$ we can write the Fourier expansion of $F$ as

$$F(\tau, z, \tau') = \sum_{\substack{n, r, m \in \mathbb{Z} \\ n, m, 4nm - r^2 \geq 0}} A(n, r, m) e(n\tau + rz + m\tau')$$

where $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathfrak{h}^1$, $z \in \mathbb{C}$, $\text{Im}(z)^2 < \text{Im}(\tau)\text{Im}(\tau')$ and

$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$. We say that $F$ is in the space Maass forms if the Fourier co-
Coefficients of $F$ satisfy the relation

$$A(n, r, m) = \sum_{d \mid \gcd(n, r, m)} d^{k-1} A \left( \frac{nm}{d^2}, \frac{r}{d}, 1 \right)$$

for every $n, r, m \in \mathbb{Z}$ with $n, m, 4nm - r^2 \geq 0$. The space of Maass cusp forms is given by $S_k^*(\Gamma) := S_k(\Gamma) \cap M_k^*(\Gamma)$. Note that the spaces of Maass forms and Maass cusp forms are invariant under the action of Hecke operators. This can be seen by looking at the explicit formulas for the action of the Hecke operators as given above. In the case of level 1 the Maass space can be characterized by the positivity of eigenvalues as well. In particular, we have the following theorem of Breulmann.

**Theorem II.14.** ([8]) Let $F \in S_k(\text{Sp}_4(\mathbb{Z}))$ be an eigenform of all Hecke operators $T_S(n)$ with eigenvalues $\lambda_F(n)$. Then $F \in S_k^*(\text{Sp}_4(\mathbb{Z}))$ if and only if $\lambda_F(n) > 0$ for every $n \in \mathbb{N}$.

Let $F$ be a non-zero Hecke eigenform of weight $k$ and level $N$ with eigenvalues $\lambda_F(m)$. Associated to $F$ are two different $L$-functions: the spinor zeta function and the standard zeta function. We start with the spinor zeta function. This is defined by

$$L_{\text{spin}}(s, F) = \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \lambda_F(m)m^{-s}.$$ 

It has an Euler product expansion given by

$$L_{\text{spin}}(s, F) = \prod_p L_{\text{spin}}^{(p)}(s, F)^{-1} \quad (\text{Re}(s) \gg 0)$$

where

$$L_{\text{spin}}^{(p)}(s, F) = 1 - \lambda_F(p)p^{-s} + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})p^{-2s} - \lambda_F(p)p^{2k-3-3s} + p^{4k-6-4s}$$

for $p \nmid N$ and

$$L_{\text{spin}}^{(p)}(s, F) = 1 - \lambda_F(p)p^{-s}$$
for $p \mid N$ ([6], Equation 0.15). For $F$ of full level, Andrianov shows ([1], Theorem 3.1.1) that $L_{\text{spin}}(s, F)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with at most simple poles at $s = k - 2, k$. He also shows that it satisfies a functional equation

$$L^*_{\text{spin}}(s, F) = (-1)^k L^*_{\text{spin}}(2k - 2 - s, F)$$

where

$$L^*_{\text{spin}}(s, F) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) L_{\text{spin}}(s, F).$$

The corresponding results for level $N > 1$ are given in ([6], Theorem 1). We will also be interested in the modified Spinor zeta function $L^*_{\text{spin}}(s, F)$. This is defined by

$$(2.4) \quad L^*_{\text{spin}}(s, F) = \left( \prod_{p \mid N} (1 - p^{k-1-s})^{-1} (1 - p^{k-2-s})^{-1} \right) L_{\text{spin}}(s, F).$$

In order to define the standard zeta function we return to the case of general $n$. Suppose that $F \in S_k(\Gamma_0^n(N))$ is a Hecke eigenform with eigenvalues $\lambda_F(m)$. Define the standard zeta function associated to $F$ by

$$L(s, \lambda_F) = \prod_p W_p(p^{-s})^{-1}$$

where

$$W_p(t) = \begin{cases} 
\prod_{i=1}^n (1 - p^n \alpha_{p,i} t) & \text{if } p \mid N \\
(1 - p^nt) \prod_{i=1}^n (1 - p^n \alpha_{p,i} t)(1 - p^n \alpha_{p,i}^{-1} t) & \text{if } p \nmid N 
\end{cases}$$

and $\alpha_{p,i}$ are Satake parameters. The Satake parameters are defined in terms of $\lambda_F$; we will calculate them precisely in the context we need them later. Given a Hecke character $\phi$ define the standard zeta function with character by

$$(2.5) \quad L(s, \lambda_F, \phi) = \prod_p W_p(\phi(p)p^{-s})^{-1}.$$ 

Similarly, we define $L_N(s, \lambda_F, \phi)$ to be the restricted Euler product given by

$$(2.6) \quad L_N(s, \lambda_F, \phi) = \prod_{p \mid N} W_p(\phi(p)p^{-s})^{-1}.$$
The standard zeta function is absolutely and uniformly convergent for
\( \text{Re}(s) > n + 1 + \delta, \ (\delta > 0) \) and is holomorphic in its domain. Put
\[
\Gamma_n(s) = \pi^{n(n-1)/4} \prod_{j=1}^{n-1} \Gamma(s - j/2)
\]
and
\[
g^n(s, h) = \begin{cases} 
\Gamma_n \left( s + \frac{h-n}{2} \right) \Gamma \left( s + \frac{h}{2} - \left[ \frac{h-n}{2} \right] \right) & \text{if } h \geq n \\
\Gamma_{2k-2-n} \left( s + \frac{h-n}{2} \right) \Gamma \left( s - \frac{h}{2} \right) \prod_{a=h+2}^{n} \Gamma(2s - a) & \text{if } \frac{(n-2)}{2} \leq h < n
\end{cases}
\]
Set
\[
\Gamma_{k,\epsilon}^n(s) = \Gamma_n \left( s + \frac{k + \epsilon - n - 1}{2} \right) g^n(s, k - \epsilon)
\]
where \( \phi_\infty(x) = \text{sgn}(x_\infty)^{k+\epsilon} \) and \( 0 \leq \epsilon \leq 1 \). We have the following theorem of Shimura:

**Theorem II.15.** ([71], Theorem 6.1) The function \( Z(s, F, \phi) = \Gamma_{k,\epsilon}^n(s/2)L(s, \lambda_F, \phi) \) has meromorphic continuation to the entire complex plane with only finitely many poles, each of them simple. More precisely, for \( \phi^2 \neq 1 \) we have \( Z(s, F, \phi) \) is entire. If \( \phi^2 = 1 \) and \( N \neq 1 \) and \( k > n/2 \), then the only possible pole is at \( s = (n+2)/4 \) and this only occurs if \( 2k - n \in 4\mathbb{Z} \). If \( \phi^2 = 1 \) and \( N \neq 1 \) with \( k \leq n/2 \), then the possible poles are in the set \( \{ j/2 : j \in \mathbb{Z}, \left[ \frac{n+3}{2} \right] \leq j \leq n + 1 - k \} \). If \( \phi^2 = 1 \) and \( N = 1 \), then each pole is in \( \{ j/2 : j \in \mathbb{Z}, \left[ \frac{n+3}{2} \right] \leq j \leq n + 1 - k \} \) or \( \{ j/2 : j \in \mathbb{Z}, 0 \leq j \leq \lfloor n/2 \rfloor \} \) where we do not need to include \( j = 0 \) if \( \phi \neq 1 \).
CHAPTER III

The Saito-Kurokawa Correspondence

The Saito-Kurokawa correspondence was first conjectured by Saito and Kurokawa in 1977; see [43]. They based their conjecture on numerical evidence which suggested that given a Hecke eigenform \( f \in S_{2k-2}(SL_2(\mathbb{Z})) \) there exists a Siegel eigenform \( F_f \) of degree 2 and weight \( k \) such that

\[
L_{\text{spin}}(s, F_f) = \zeta(s - k + 1) \zeta(s - k + 2) L(s, f).
\]

This conjecture was proven in a series of papers by Maass [46], Andrianov [2], and Zagier [89]. In 1993 Manickham, Ramakrishnan, and Vasudevan [48] extended this result to include the case of odd square-free level. Manickham and Ramakrishnan were later able to improve their previous result to include all positive levels ([49], [50]). One can view the Saito-Kurokawa correspondence from an automorphic representation viewpoint as well, which is done in [54] as well as the recent work of Schmidt ([58], [59]). This is perhaps the most natural way to view the correspondence, but for our purposes this viewpoint will not be necessary and we stick to the more classical approach with its explicit formulas.

In this chapter we will present a brief overview of the Saito-Kurokawa correspondence, citing references containing more details for the interested reader. Throughout this chapter \( M \) and \( k \) are fixed positive integers.
3.1 Integer Weight Forms to Half-Integer Weight Forms

The correspondence of integer weight elliptic modular forms of weight $2k - 2$ to half-integer weight modular forms of weight $k - 1/2$ was first observed by Shimura: see [62]. However, determining the level of the half-integer weight form was complicated. His results were made more precise by Kohnen ([38], [40]). In this section we will describe Kohnen’s results as they establish the first step in the Saito-Kurokawa correspondence.

Let $D$ be a fundamental discriminant with $(-1)^{k-1}D > 0$. There exists a Shimura lifting $\zeta_D$ that maps $S_{k-1/2}^+(\Gamma_0(4M))$ to $M_{2k-2}(\Gamma_0(M))$ and a Shitani lifting $\zeta_D^*$ mapping $S_{2k-2}(\Gamma_0(M))$ to $S_{k-1/2}^+(\Gamma_0(4M))$. These maps are adjoint on cusp forms with respect to the Petersson products. Explicitly, for

$$g(z) = \sum_{n \geq 1} c_g(n)q^n \in S_{k-1/2}^+(\Gamma_0(M))$$

one has

$$\zeta_D g(z) = \sum_{n \geq 1} \left( \sum_{d | n \text{ gcd}(d,M) = 1} \left( \frac{D}{d} \right) d^{k-2} c_g(|D|n^2/d^2) \right) q^n$$

and for $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))$ a newform one has

$$\zeta_D^* f(z) = (-1)^{(k-1)/2} 2^{k-1} \sum_{m \geq 1} r_{k-1,M,D}(f; |D|m) q^m$$

where $r_{k-1,M,D}(f; |D|m)$ is a certain integral. For the definitions of these integrals see ([40], Section 1).

Using these liftings, one has the following theorem:

**Theorem III.1.** ([38], [40], [47]) For $D$ a fundamental discriminant with
\((-1)^{k-1}D > 0 \text{ and } \gcd(D, M) = 1\), the Shimura and Shintani liftings give a Hecke-equivariant isomorphism between \(S_{k-1/2}^{+}(\Gamma_0(4M))\) and \(S_{2k-2}^{\text{new}}(\Gamma_0(M))\).

Let \(\mathcal{O}\) be a ring so that an embedding of \(\mathcal{O}\) into \(\mathbb{C}\) exists. Choose such an embedding. We identify \(\mathcal{O}\) with its image in \(\mathbb{C}\) via this embedding. Assume that \(\mathcal{O}\) contains the Fourier coefficients of \(f\). The Shintani lifting \(g_f := \zeta_D^*f\) is determined only up to a constant multiple. By phrasing the Shintani lifting completely in the language of cohomology, Stevens was able to prove the following result.

**Theorem III.2. ( [79] Proposition 2.3.1)** Let \(f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))\) be a newform. If the Fourier coefficients of \(f\) are in \(\mathcal{O}\), then there exists a corresponding Shintani lifting \(g_f\) of \(f\) with Fourier coefficients in \(\mathcal{O}\) as well.

**Remark III.3.** Throughout this thesis we fix our \(g_f\) to have Fourier coefficients in \(\mathcal{O}\) as in Theorem III.2. If, in addition, \(\mathcal{O}\) is a discrete valuation ring, we fix our \(g_f\) to have Fourier coefficients in \(\mathcal{O}\) with some Fourier coefficient in \(\mathcal{O}^\times\).

### 3.2 Half-Integer Weight Forms to Jacobi Forms

In this section we use the results of ([49], [50]) which gives the correspondence for arbitrary levels \(M\). The case of level 1 is presented nicely in [21]. For odd square-free level one can consult [48]. The arguments there are similar to the ones used in ([49], [50]), just a little less complicated.

Let \(P_D^+\) denote the \(D\)-th Poincare series in \(S_{k-1/2}^{+}(\Gamma_0(4M))\) for \(D\) a negative fundamental discriminant. It is characterized by the formula

\[
\langle g, P_D^+ \rangle = \frac{\Gamma(k - 3/2)}{(4\pi |D|)^{k-3/2}} c_g(|D|).
\]

Let

\[
S_{k-1/2}(4M) = \begin{cases} 
S_{k-1/2}(\Gamma_0(4M)) & \text{if } 2 \mid M \\
S_{k-1/2}^+(\Gamma_0(4M)) & \text{if } 2 \nmid M
\end{cases}
\]
as in [49]. Let \( \mathcal{P} \) denote the subspace in \( S_{k-1/2}(\Gamma_0(4M)) \) (if \( 2 \mid M \)) or in \( S_{k-1/2}^+(\Gamma_0(4M)) \) (if \( 2 \nmid M \)) generated by the Poincare series \( P|_D \) (if \( 2 \mid M \)) or \( P^+_|_D \) (if \( 2 \nmid M \)), with \( D \) running over all discriminants \( D \equiv 0, 1 \pmod{4} \) such that \((-1)^{k-1}D > 0\). Define

\[
S_{k-1/2}^{\text{new}}(4M) = \mathcal{P} \cap \bigoplus_{f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))} S_{k-1/2}^{\text{new}}(4M, f)
\]

where

\[
S_{k-1/2}^{\text{new}}(4M, f) = \{ g \in S_{k-1/2}(4M) : T(p^2)g = a_f(p)g \text{ for } p \nmid M \},
\]

and

\[
J_{k,1}^{\text{cusp, new}}(M) = \bigoplus_{f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))} J_{k,1}^{\text{cusp, new}}(M, f)
\]

where

\[
J_{k,1}^{\text{cusp, new}}(M, f) = \{ \phi \in J_{k,1}^{\text{cusp}}(M) : T_f(p)\phi = a_f(p)\phi \text{ for } p \nmid M \}.
\]

**Theorem III.4.** ([49], Theorem 5.4) The map defined by

\[
\sum_{\substack{D \in \mathbb{Z}, r \in \mathbb{Z} \\mid \\text{gcd}(D, r) = 1 \\text{ and } D \equiv r^2 \pmod{4} \\text{ and } D = 0, 1 \pmod{4} \\text{ or } D \equiv 0, 1 \pmod{4}}} c(D,r)e \left( \frac{r^2 - D}{4} \tau + rz \right) \mapsto \sum_{\substack{D \in \mathbb{Z}, r \in \mathbb{Z} \\mid \\text{gcd}(D, r) = 1 \\text{ and } D \equiv 0, 1 \pmod{4} \\text{ or } D \equiv 0, 1 \pmod{4}}} c(D)e(|D|\tau),
\]

is a canonical isomorphism between \( J_{k,1}^{\text{cusp, new}}(M) \) and \( S_{k-1/2}^{\text{new}}(4M) \), which commutes with the action of Hecke operators. It also preserves the Hilbert space structure. (Note that in this case, the Fourier coefficients \( c(D,r) \) of a Jacobi newform depend only on \( D \) and not \( r \), and so we write \( c(D) \) on the right hand side instead of \( c(D,r) \).)

**Corollary III.5.** If \( g \in S_{k-1/2}^{\text{new}}(4M, \mathcal{O}) \) for a ring \( \mathcal{O} \), then the corresponding Jacobi form \( \phi \) also has Fourier coefficients in \( \mathcal{O} \) and vice versa.

### 3.3 Jacobi Forms to Siegel Forms

Let \( F \in \mathcal{M}_k^+((\Gamma_0^4(M))) \) have Fourier expansion given by

\[
F(\tau, z, \tau') = \sum_{n, r, m \in \mathbb{Z}, n, m, 4nm - r^2 \geq 0} A(n, r, m) e(n\tau + rz + m\tau')
\]
as in Equation 2.3. One also has that $F$ admits a Fourier-Jacobi expansion

\[(3.1) \quad F(\tau, z, \tau') = \sum_{m \geq 0} \phi_m(\tau, z) e(m\tau')\]

where the $\phi_m$ are Jacobi forms of weight $k$, index $m$, and level $M$. See ([21], Theorem 6.1) for a proof of this fact.

**Theorem III.6.** ([50], Theorem 5.1) The association $F \mapsto \phi_1$ gives an isomorphism between $M^*_k(\Gamma^4_0(M))$ and $J_{k,1}(\Gamma^J_0(M))$. This isomorphism commutes with the action of Hecke operators.

The inverse map to the map $F \mapsto \phi_1$ is given as follows. Let $\phi(\tau, z) \in J_{k,1}(\Gamma^J_0(M))$. Define

\[F(\tau, z, \tau') = \sum_{m \geq 0} V_m \phi(\tau, z) e(m\tau')\]

where $V_m$ is the linear operator defined in Section 2.4. Then $F$ is a Maass form. For the details consult ([21], §6) and [48].

**Corollary III.7.** Let $\phi \in J_{k,1}(\Gamma^J_0(M), \mathcal{O})$ where $\mathcal{O}$ is some ring. If $F$ is the Siegel modular form associated to $\phi$ in Theorem III.6 then $F$ has Fourier coefficients in $\mathcal{O}$.

**Proof.** Using that $F$ is in the Maass space and the definition of $V_m$ we obtain

\[A(n, r, m) = \sum_{d | \gcd(n, r, m)} d^{k-1} c \left( \frac{4nm - r^2}{d^2} : \frac{d, r}{d} \right)\]

where $c(D, r)$ are the Fourier coefficients of $\phi$. The rest is clear. \qed

As in [49], we define

\[S^{*, \text{new}}_k(\Gamma^4_0(M), f) = \left\{ F(Z) = \sum_{m \geq 0} V_m \phi(\tau, z) e(m\tau') : \phi \in J^{\text{cusp, new}}_{k,1}(\Gamma^J_0(M), f) \right\}\]

and

\[S^{*, \text{new}}_k(\Gamma^4_0(M)) = \bigoplus_{f \in \mathcal{S}^{\text{new}}_{2k-2}(\Gamma_0(M))} S^{*, \text{new}}_k(\Gamma^4_0(M), f).\]
Therefore, we have a correspondence taking $f$ to $F_f$ via first associating a half-
integer weight form $g_f$ to $f$, normalized as in Remark III.3. We then associate a
Jacobi form $\phi_f$ to $g_f$. Finally, associated to $\phi_f$ is the Siegel form $F_f$. We summarize
this in the following theorem.

**Theorem III.8.** ([50], Theorem 5.2) The space $S^{*\text{new}}_k(\Gamma_0^4(M))$ has a basis of eigen-
forms with respect to all Hecke operators. Each form $F$ belonging to $S^{*\text{new}}_k(\Gamma_0^4(M), f)$
corresponds to $f \in S^{\text{new}}_{2k-2}(\Gamma_0(M))$ via the Saito-Kurokawa correspondence outlined
above with

$$L^*_{\text{spin}}(s, F) = \zeta(s-k+1)\zeta(s-k+2)L(s, f).$$

**Corollary III.9.** If $f \in S^{\text{new}}_{2k-2}(\Gamma_0(M), \mathcal{O})$ is a newform, then the Siegel modular
form $F_f$ corresponding to $f$ under the Saito-Kurokawa correspondence has Fourier
coefficients in $\mathcal{O}$ as well. If $\mathcal{O}$ is a discrete valuation ring, $F_f$ has a Fourier coefficient
in $\mathcal{O}^\times$.

### 3.4 CAP Forms

So far in this chapter we have seen that there are a class of Siegel modular forms
that are associated to elliptic modular forms via the Saito-Kurokawa correspondence.
In fact, this is a special case of a more general phenomenon. The more general
forms are referred to as CAP forms, where CAP stands for “Cuspidal Associated
to Parabolic”. One can consult [54] or [77] for the general definition and properties
of CAP forms in the language of automorphic forms. There are three different
parabolics for the algebraic group $\text{Sp}_4$ over $\mathbb{Q}$. The one we are interested in is the
Siegel parabolic. The reason for this is because of the shape of the associated Galois
representation. If a form $G$ is CAP associated to the Klingen parabolic, then the
Galois representation will be reducible of the form $\rho_G = \sigma_1 \oplus \sigma_2$ where the $\sigma_i$ are
2-dimensional. For the other parabolic we would get a completely reducible Galois representation (i.e., a sum of characters). It will be clear when we get to Chapter VIII why we want to consider the CAP case when the parabolic is the Siegel parabolic: this is the case where we get Galois representations of the form \( \rho_G = \chi_1 \oplus \sigma_1 \oplus \chi_2 \), where the \( \chi_i \) are characters and \( \sigma_1 \) is 2-dimensional. Therefore, we make the following definition.

**Definition III.10.** A Siegel cusp form \( F \) of weight \( k \) and level \( N \) that is an eigenform for all Hecke operators \( T_S(\ell) \) with \( \ell \nmid N \) is a CAP form if the eigenvalues of \( F \) are given by

\[
\lambda_F(\ell) = \lambda_f(\ell) + \ell^{k-1} + \ell^{k-2} \quad (\ell \nmid N)
\]

for some elliptic cusp form \( f \) with eigenvalues \( \lambda_f(\ell) \).
CHAPTER IV

Eisenstein Series

In this chapter we study an Eisenstein series $E(Z, s, \chi)$ as defined by Shimura ([68], [71], [73]). We will with the basic definitions and then move to a study of the Fourier coefficients of the Eisenstein series. We show that under a suitable normalization and choosing a certain value of $s$ we have that $E(Z, s, \chi)$ is a holomorphic Siegel modular form with Fourier coefficients that are $p$-integral for a prime $p$ of our choosing. We next move to studying an inner product relation of Shimura that calculates the inner product of $E(Z, s, \chi)$ with a Siegel cusp form $F$ in terms of $F$ and a standard $L$-function associated to $F$.

Throughout this chapter we fix an embedding of $\mathbb{Q}_p$ into $\mathbb{C}$ for a prime $p$ to be chosen.

4.1 Basic Definitions

For more general information about Eisenstein series of the type introduced in this section the interested reader can consult ([68], [73]).

Denote the adeles over $\mathbb{Q}$ by $\mathbb{A}$. Given an algebraic group $G$ over $\mathbb{Q}$, we denote the adelic group of $G$ by $G(\mathbb{A})$. For example, we write $\text{Sp}_{2n}(\mathbb{A})$ for the adelic symplectic group. Given a place $v$ of $\mathbb{Q}$, we denote the $v$th component of $G(\mathbb{A})$ by $G_v$ or $G(\mathbb{Q}_v)$. 
We let $\mathfrak{f}$ denote the finite set of places. Set $|x|_\mathbb{Q} = \prod_{v \in \{\infty\} \cup \mathfrak{f}} |x_v|_v$ where $| \cdot |_\infty$ is the normal absolute value on $\mathbb{R}$ and $|\ell|_\ell = \ell^{-1}$ for $\ell$ a finite prime.

For a square invertible matrix, write $x^* = ^t\overline{x}$ and $\hat{x} = (x^*)^{-1}$ where $\overline{x}$ denotes the complex conjugate of $x$.

Before we can define the Eisenstein series we need to define some subgroups of $\text{Sp}_{2n}(\mathbb{A})$ and $\text{Sp}_{2n}(\mathbb{Q})$. Let $a$ and $b$ be non-zero ideals in $\mathbb{Z}$. Set

$$D[a, b] = \text{Sp}_{2n}(\mathbb{R}) \prod_{\ell \in \mathfrak{f}} D_\ell[a, b]$$

where

$$D_\ell[a, b] = \{ x \in \text{Sp}_{2n}(\mathbb{Q}_\ell) : a_x \in M_n(\mathbb{Z}_\ell), b_x \in M_n(a_\ell), c_x \in M_n(b_\ell), d \in M_n(\mathbb{Z}_\ell) \}.$$

Define a maximal compact subgroup $C_v$ of $\text{Sp}_{2n}(\mathbb{Q}_v)$ by

$$C_v = \begin{cases} 
\{ \alpha \in \text{Sp}_{2n}(\mathbb{R}) : \alpha(i) = i \} & v = \infty, \\
\text{Sp}_{2n}(\mathbb{Q}_v) \cap \text{GL}_{2n}(\mathbb{Z}_v) & v \in \mathfrak{f},
\end{cases}$$

and set $C = \prod_{v \in \mathfrak{f} \cup \{\infty\}} C_v$. Let $P$ be the parabolic of $\text{Sp}_{2n}(\mathbb{Q})$ defined by

$$P = \{ x \in \text{Sp}_{2n}(\mathbb{Q}) : c_x = 0 \}.$$

Recall

$$S^n(\mathbb{R}) = \{ x \in M_n(\mathbb{R}) : ^t x = x \}$$

and that we write every element $Z \in \mathfrak{h}^n$ as $Z = X + iY$ with $X, Y \in S^n(\mathbb{R})$ and $Y > 0$.

Let $\lambda = \frac{n+1}{2}$, $N$ a positive integer, and $k$ a positive integer such that $k > \max\{3, 2\lambda\}$. In order to define the Eisenstein series we need a Hecke character $\chi$ of $\mathbb{A}^\times$ satisfying

$$\chi_\infty(x) = \text{sgn}(x)^k,$$

$$\chi_\ell(a) = 1 \quad \text{if } \ell \in \mathfrak{f}, a \in \mathbb{Z}_\ell^\times, \text{ and } N \mid (a - 1).$$
Set $D = D[1, N]$ and define functions $\mu$ and $\varepsilon$ on $\text{Sp}_{2n}(\mathbb{A})$ by

\[
\mu(x) = \begin{cases} 0 & \text{if } x \notin P(\mathbb{A})D \\ \chi(\det(d_p))^{-1}\chi(N)(\det(d_w))^{-1}\det(d_p)^{-k} & \text{if } x = pw \in P(\mathbb{A})D \end{cases}
\]

and

\[
\varepsilon(x_\infty) = |j(x_\infty, i)|^2 \\
\varepsilon(x_f) = \det(d_p)^{-2} \text{ for } x = pw
\]

where $\chi(N) = \prod_{\ell \mid N} \chi_\ell$ and $j(x_\infty, Z) = \det(c_{x_\infty}Z + d_{x_\infty})$ and it is understood that $i$ here denotes the $n \times n$ identity matrix multiplied by the complex number $i$.

We now have all the ingredients necessary to define the Eisenstein series we are interested in. For $x \in \text{Sp}_{2n}(\mathbb{A})$ and $s \in \mathbb{C}$, define

\[
E(x, s) = E(x, s; \chi, D) = \sum_{\alpha \in A} \mu(\alpha x)\varepsilon(\alpha x)^{-s}, \quad A = P \setminus \text{Sp}_{2n}(\mathbb{Q}).
\]

This gives us an Eisenstein series defined on $\text{Sp}_{2n}(\mathbb{A}) \times \mathbb{C}$, but we will ultimately be interested in an Eisenstein series $E(Z, s)$ defined on $\mathfrak{h}^n \times \mathbb{C}$. The Eisenstein series $E(Z, s)$ converges locally uniformly in $\mathfrak{h}^n$ for $\Re(s) > \lambda$. We associate the Eisenstein series $E(Z, s)$ to $E(x, s)$ as follows.

More generally, let $F_0$ be a function on $\text{Sp}_{2n}(\mathbb{A})$ such that

\[
F_0(\alpha x w) = F_0(x)J(w, i)^{-1} \quad \text{for } \alpha \in \text{Sp}_{2n}(\mathbb{Q}) \text{ and } w \in C'
\]

where $C'$ is an open subgroup of $C$ and $J(x, z)$ is defined by

\[
J(x, z) = J_{k,s}(x, z) = j(x, z)^kJ(x, z)^{s}.
\]

Our Eisenstein series is such a function. Let $\Gamma' = \text{Sp}_{2n}(\mathbb{Q}) \cap C'\text{Sp}_{2n}(\mathbb{R})$ and define a function $F$ on $\mathfrak{h}^n$ by

\[
F(x(i)) = F_0(x)J(x, i) \quad \text{for } x \in C'\text{Sp}_{2n}(\mathbb{R}).
\]
Using the strong approximation theorem \((\text{Sp}_{2n}(\mathbb{A}) = \text{Sp}_{2n}(\mathbb{Q}) C' \text{Sp}_{2n}(\mathbb{R}))\) we have that this is well-defined and satisfies

\[
F(\gamma Z) = F(Z)J(\gamma, Z) \quad \text{for } \gamma \in \Gamma' \text{ and } Z \in \mathfrak{h}^n.
\]

Therefore, we have an associated Eisenstein series \(E(Z, s)\) defined on \(\mathfrak{h}^n \times \mathbb{C}\). Conversely, given a function \(F\) satisfying Equation 4.5, we can define a function \(F_0\) satisfying Equation 4.3 and Equation 4.4 by

\[
F_0(\alpha x) = F(x(i))J(x, i)^{-1} \quad \text{for } \alpha \in \text{Sp}_{2n}(\mathbb{Q}) \text{ and } x \in C' \text{Sp}_{2n}(\mathbb{R}).
\]

We will also make use of the fact that if \(G = F|_{\gamma^{-1}}\) for \(\gamma \in \text{Sp}_{2n}(\mathbb{Z})\) with \(F\) a Siegel modular form, then \(G_0(x) = F_0(x\gamma_f)\) and vice versa. See ([68], Page 424) for more information on this.

### 4.2 The Fourier Coefficients of \(E(Z, s, \chi)\)

We will now focus our attention on the Fourier coefficients of \(E(x, s)\) and in turn \(E(Z, s)\). It turns out that it is actually easier to study the Fourier coefficients of a simple translation of \(E(x, s)\) given by

\[
E^*(x, s) = E(x\tau^{-1}, s; \chi, D)
\]

where we recall \(\tau = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}\). Using the discussion above, we get a corresponding form \(E^*(Z, s)\). For \(z \in \mathbb{C}\), put \(e(z) = e^{2\pi i z}\) and define characters \(e_v : \mathbb{Q}_v \to S^1\) by \(e_\infty(x) = e(x)\) and \(e_\ell(x) = e(-y)\) with \(y \in \mathbb{Q}\) such that \(x - y \in \mathbb{Z}_\ell\). Set \(e_\Lambda(x) = \prod_{v \in \infty \cup \mathfrak{f}} e_v(x_v)\). Let \(L = S^n(\mathbb{Q}) \cap M_n(\mathbb{Z})\) and set \(L' = \{ s \in S^n(\mathbb{Q}) : \text{Tr}(sL) \subseteq \mathbb{Z} \}\) and \(M = N^{-1}L'\). The Eisenstein series \(E^*(x, s)\) has a Fourier expansion

\[
E^* \left( \begin{pmatrix} q & \sigma \hat{q} \\ 0 & \hat{q} \end{pmatrix}, s \right) = \sum_{h \in \mathcal{M}} c(h, q, s)e_\Lambda(\text{Tr}(h\sigma))
\]
where \( q \in \text{GL}_{2n}(\mathbb{A}) \), \( \sigma \in \mathbb{S}^n(\mathbb{A}) \), and \( c(h, q, s) \in \mathbb{C} \) ([73], Eq. 18.6.6). We also have a corresponding Fourier expansion

\[
E^*(Z, s) = \sum_{h \in M} a(h, Y, s) e(\text{Tr}(hX))
\]

for \( Z = X + iY \in \mathfrak{h}^n \).

**Remark IV.1.** ([68], Page 460) The Fourier coefficients of \( E^*(Z, s) \) are nonvanishing only when \( h \) is totally positive definite due to the fact that we have restricted our \( k \) to be larger then 3.

Shimura has calculated the values of \( c(h, q, s) \). Moreover, he has expressed the values of \( a(h, Y, s) \) in terms of \( c(h, q, s) \).

**Proposition IV.2.** ([73], Prop. 18.14) For \( N \neq 1 \),

\[
c(h, q, s) = \chi(\det(q)^{-1} N^{-n\lambda} | \det(qq^*)_\mathbb{R} | \lambda^{-s} | \det(qq^*_\infty)|^s \cdot \alpha_N(q^* h q, 2s, \chi) \xi(qq^*, h; s + k/2, s - k/2)
\]

where \( \xi \) is defined by

\[
\xi(Y, h; s, t) = \int_{S_{\infty}} e(-\text{Tr}(hX)) \det(X + iY)^{-s} \det(X - iY)^{-t} dX
\]

with \( 0 < Y \in \mathbb{S}^n(\mathbb{R}) \), \( h \in \mathbb{S}^n(\mathbb{R}) \), and \( s, t \in \mathbb{C} \) and \( \alpha_N \) is a Whittaker integral, or what Shimura refers to as a Siegel series. One can consult [73] for the definition of \( \alpha_N \); it will not be needed here.

**Lemma IV.3.** ([73], Lemma 18.7) Let \( Z = X + iY \in \mathfrak{h}^n \) and let \( q \in \text{GL}_n(\mathbb{A}) \) such that \( q_\infty = Y^{1/2} \) and \( q_\mathbb{R} = 1 \). Then \( a(h, Y, s) = \det(Y)^{-k/2} c(h, q, s) \).

Using the fact that \( q_\mathbb{R} = 1 \) and \( q_\infty = Y^{1/2} > 0 \), we have that

\[
\chi(\det(q))^{-1} = \chi(\det(q_\infty))^{-1} = 1.
\]

Thus we have:
Corollary IV.4. Given the set-up in Lemma IV.3, we have
\[ a(h, Y, s) = \det(Y)^{-k/2} N^{-n \lambda} \det(Y)^s \alpha_N(q^* hq; 2s, \chi) \xi(Y, h, s + k/2, s - k/2). \]

For a Dirichlet character \( \psi \), set \( \Lambda_N(s, \psi) = L_N(2s, \psi) \prod_{j=1}^{[n/2]} L_N(4s - 2j, \psi^2) \). We normalize \( E^*(Z, s) \) by multiplying it by \( \pi^{-n(n+2)/4} \Lambda_N(s, \chi) \) and call this normalized Eisenstein series \( D_E^*(Z, s) = D_E^*(Z, s; k, \chi, N) \).

Consider the Fourier expansion of \( D_E^*(Z, s) \) at \( s = \lambda - k/2 \):
\[ D_E^*(Z, \lambda - k/2) = \sum_{h \in M} \pi^{-n(n+2)/4} \Lambda_N(\lambda - k/2, \chi) a(h, Y, \lambda - k/2) e(\text{Tr}(hX)) = \sum_{h \in M} b(h, Y, \lambda - k/2) e(\text{Tr}(hX)). \]

Proposition IV.5. ([69], Proposition 4.1) The normalized Eisenstein series \( D_E^*(Z, \lambda - k/2) \) is in \( \mathcal{M}_k(\mathbb{Q}^{ab}) \) where \( \mathbb{Q}^{ab} \) is the maximal abelian extension of \( \mathbb{Q} \).

We show that the coefficients of \( D_E^*(Z, \lambda - k/2) \) actually lie in a finite extension of \( \mathbb{Z}_p \) for a suitably chosen prime \( p \). Using ([66], Equations 4.34K, 4.35IV) we have that
\[ \xi(Y, h; \lambda, \lambda - k) = \frac{i^{nk} \pi^{n(n+2)/4} 2^{n(k-1)} \det(Y)^{k-\lambda}}{\mathcal{P}_n} e(i \text{ Tr}(hY)) \]
where
\[ \mathcal{P}_n = \prod_{j=0}^{[\lambda]} j! \prod_{j=0}^{[\lambda]-1} (2j+1)!! \]
and
\[ n!! = \begin{cases} n(n-2) \ldots 5 \cdot 3 \cdot 1 & n > 0, \text{ odd} \\ n(n-2) \ldots 6 \cdot 4 \cdot 2 & n > 0, \text{ even} \end{cases} \]

Using that \( h \) is totally positive definite we have:

Proposition IV.6. ([73], Prop. 19.2) Set \( \chi_h \) to be the Hecke character corresponding to \( \mathbb{Q}(\sqrt{-\det(h)})/\mathbb{Q} \). Then
\[ \alpha_N(h, s, \chi) = \Lambda_N^{-1}(s, \chi) \Lambda_{h, N}(s, \chi) \prod_{\ell \in \mathcal{C}} f_{h, Y, \ell}(\chi(\ell)|\ell|^{2s}) \]
where $C$ is a finite subset of $f$, the $f_{h,Y,\ell}$ are polynomials with a constant term of 1 and coefficients in $\mathbb{Z}$ independent of $\chi$, and

$$\Lambda_{h,N}(s, \chi) = \begin{cases} L_N(2s - n/2, \chi \chi_h) & n \in 2\mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$$

To ease the notation set $F_{h,Y}(s, \chi) = \prod_{\ell \in C} f_{h,Y,\ell}(\chi(\ell)|\ell|^s)$. Combining Equation 4.6, Corollary IV.4, and Proposition IV.6 we have

$$b(h, Y, \lambda - k/2) = \begin{cases} \frac{i^n 2^{n(k - 1)} L_N(2\lambda - k - n/2, \chi \chi_h) F_{Y,h}(2\lambda - k, \chi)}{N^{n+k} \mathcal{P}_n} e(i \text{ Tr}(hY)) & n \in 2\mathbb{Z} \\ \frac{i^n 2^{n(k - 1)} F_{Y,h}(2\lambda - k, \chi)}{N^{n+k} \mathcal{P}_n} e(i \text{ Tr}(hY)) & \text{otherwise} \end{cases}$$

In particular, we can write

$$D_{E^*}(Z, \lambda - k/2) = \begin{cases} \sum_{h \in M} \left( \frac{i^n 2^{n(k - 1)} L_N(2\lambda - k - n/2, \chi \chi_h) F_{Y,h}(2\lambda - k, \chi)}{N^{n+k} \mathcal{P}_n} \right) e(\text{ Tr}(hZ)) & n \in 2\mathbb{Z} \\ \sum_{h \in M} \left( \frac{i^n 2^{n(k - 1)} F_{Y,h}(2\lambda - k, \chi)}{N^{n+k} \mathcal{P}_n} \right) e(\text{ Tr}(hZ)) & \text{otherwise} \end{cases}$$

Let $p$ be an odd prime with $\gcd(p, N) = 1$ and $p > 2\lambda - 1$. We show that the $b(h, Y, \lambda - k/2)$ all lie in $\mathbb{Z}_p[\chi, i^n \chi]$ where $\mathbb{Z}_p[\chi]$ is the extension of $\mathbb{Z}_p$ generated by the values of $\chi$. It is clear that $i^n 2^{n(k - 1)} N^{-n\lambda} \in \mathbb{Z}_p[\chi, i^n \chi]$ by our choice of $p$. The fact that $p > 2\lambda - 1$ and $n \geq 1$ so that $2\lambda - 1 \geq \lambda$ shows that $\mathcal{P}_n$ is in $\mathbb{Z}_p$. The fact that we have chosen $k > 2\lambda$ gives us that $2\lambda - k < 0$. This in turn shows that $|p|^{2\lambda - k} = p^{k-2\lambda} \in \mathbb{Z}_p$. Using that the coefficients of $f_{h,Y,\ell}$ all lie in $\mathbb{Z}$ and this fact, we have that $F_{Y,h}(2\lambda - k, \chi) \in \mathbb{Z}_p[\chi, i^n \chi]$ for all $h$. Therefore it remains to show that $L_N(2\lambda - k - n/2, \chi \chi_h) \in \mathbb{Z}_p[\chi, i^n \chi]$.

We will in fact show that for any Dirichlet character $\psi$ of conductor $N$ and any positive integer $n$ that $L_N(1 - n, \psi) \in \mathbb{Z}_p[\psi]$.

Before we can state the necessary theorem we need to define the Teichmuller character. Observe that the equation $x^{p-1} - 1 = 0$ has distinct roots modulo $p$, so
reducing modulo $p$ gives an isomorphism $\mu_{p-1} \cong \mathbb{F}_p^\times$. Therefore, given $a \in \mathbb{Z}_p^\times$ with $p \nmid a$, there is a unique $(p - 1)^{st}$ root of unity $\omega(a) \in \mathbb{Z}_p$ such that $a \equiv \omega(a) \pmod{p}$.

The Teichmüller character is the character $\omega$ so defined. We are now in a position to state the following theorem defining a $p$-adic $L$-function and giving the values at $1 - n$ in terms of generalized Bernoulli numbers.

**Theorem IV.7.** ([85], Thm. 5.11) For a non-trivial Dirichlet character $\psi$ of conductor $N$ there exists a $p$-adic analytic function $L_p(s, \psi)$ on

$$\{s \in \mathbb{C}_p : |s| < (p - 1)p^{-1/(p-1)}\}$$

such that

$$L_p(1 - n, \psi) = (1 - \psi \omega^{-n}(p)p^{n-1}) \frac{B_n,\psi \omega^{-n}}{n}$$

for $n \geq 1$.

Using this theorem and the well-known fact that one has $L(1 - n, \psi) = -\frac{B_{n,\psi}}{n}$, we can write

$$L_N(1 - n, \psi) = -(1 - \psi(p)p^{n-1})^{-1} \prod_{\ell | N} (1 - \psi(\ell)\ell^{1-n}) L_p(1 - n, \psi \omega^n).$$

One can see that $(1 - \psi(p)p^{n-1})^{-1} \in \mathbb{Z}_p[\psi]$ by expanding it in a convergent geometric series. We use the fact that $\gcd(p, N) = 1$ to conclude that $\prod_{\ell | N} (1 - \psi(\ell)\ell^{1-n}) \in \mathbb{Z}_p[\psi]$.

To finish our proof that $L_N(1 - n, \psi) \in \mathbb{Z}_p[\psi]$ for all $n \in \mathbb{N}$, we note that $L_p(m, \psi)$ is a $p$-adic integer for all $m$ and $\psi$ with conductor $N$ such that $\gcd(p, N) = 1$ by ([85], Corl. 5.13). Therefore we have proven:
Theorem IV.8. Let \( n, N, \) and \( k \) be positive integers such that \( k > \max\{3, n + 1\} \). Let \( \chi \) be a Dirichlet character as in Equation 4.1. Let \( p \) be a prime such that \( p > n \) and \( (p, N) = 1 \). Then \( D_{E^*}(Z, (n+1)/2 - k/2) \) is in \( \mathcal{M}_k(\Gamma_0(N), \mathbb{Z}[\chi, i^{nk}]) \).

4.3 Pullbacks and an Inner Product Relation

In this section we will use the results in the previous section with \( n = 4 \) so that \( E(Z, s, \chi) \) is defined on \( \mathfrak{h}^4 \).

We turn our attention to studying the pullback of our Eisenstein series \( E(Z, s, \chi) \) via maps

\[
\mathfrak{h}^2 \times \mathfrak{h}^2 \hookrightarrow \mathfrak{h}^4
\]

\[
(Z, W) \mapsto \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} = \text{diag}[Z, W]
\]

and

\[
\text{Sp}_4(\mathbb{Z}) \times \text{Sp}_4(\mathbb{Z}) \hookrightarrow \text{Sp}_8(\mathbb{Z})
\]

\[
(\alpha, \beta) \mapsto \alpha \times \beta = \begin{pmatrix} a_\alpha & 0 & b_\alpha & 0 \\ 0 & a_\beta & 0 & b_\beta \\ c_\alpha & 0 & d_\alpha & 0 \\ 0 & c_\beta & 0 & d_\beta \end{pmatrix}.
\]

These pullbacks have been studied extensively by Shimura ([72], [73]) as well as by Garrett ([26], [27]). In particular, if one has a Siegel modular form \( G \) on \( \text{Sp}_8(\mathbb{Z}) \) of weight \( k \) and level \( N \), then its pullback to \( \text{Sp}_4(\mathbb{Z}) \times \text{Sp}_4(\mathbb{Z}) \) is a Siegel modular form in each of the variables \( Z \) and \( W \) of weight \( k \) and level \( N \). We will be interested primarily in the results found in [72] particularly the inner product relation found there.
Let \( \sigma_f \in \text{Sp}_8(\mathbb{Q}_f) \) be defined as \( \sigma_f = (\sigma_\ell) \) with
\[
\sigma_\ell = \begin{cases} 
I_8 & \text{if } \ell \nmid N \\
\begin{pmatrix}
I_4 & 0_4 \\
0_2 & I_2 \\
I_2 & 0_2 \\
I_2 & 0_2
\end{pmatrix} & \text{if } \ell \mid N.
\end{cases}
\]
The strong approximation gives an element \( \rho \in \text{Sp}_8(\mathbb{Z}) \cap D[1,N]\sigma_f \) such that
\( N_\ell \mid a(\sigma_\ell \rho^{-1})_\ell - I_4 \) for every \( \ell \mid N \). In particular, we have that \( E|\rho \) corresponds to \( E(x\sigma_f^{-1}) \).

Let \( F \in S_k(\Gamma_0^4(N), \mathbb{R}) \) be a Siegel eigenform with eigenvalues given by
\( T(m)F = \lambda_F(m)F \) for \( m \in \mathbb{N} \). We specialize a result of Shimura that gives the inner product of \( E|\rho \) with such an \( F \). Applying ([72], Equation 6.17) to our situation we get
\[
\langle D_{E|\rho}(\text{diag}[Z,W],(5-k)/2),(F|\iota)^c(W) \rangle = \pi^{-3} \mathcal{A}_{k,N} L_N(5-k,\lambda_F,\chi) F(Z)
\]
where \( \mathcal{A}_{k,N} = \frac{(-1)^k 2^{2k-3} v_\Gamma}{3 [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^4(N)]} \), \( v_\Gamma = \pm 1 \) depending on \( N \), \( L_N(5-k,\lambda_F,\chi) \) is the standard zeta function as defined in Equation 2.6, and \( (F|\iota)^c \) denotes taking the complex conjugates of the Fourier coefficients of \( F|\iota \) where \( F|\iota \) is now a Siegel form on
\[
\Gamma^{4,0}(N) = \left\{ \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in \text{Sp}_4(\mathbb{Z}) | B_2 \equiv 0(\text{mod } N) \right\}.
\]
We can use the \( q \)-expansion principle for Siegel modular forms ([10], Proposition 1.5) to conclude that \( F|\iota \) has real Fourier coefficients since we chose \( F \) to have real Fourier coefficients. Therefore the \( (F|\iota)^c(W) \) in Equation 4.9 becomes \( (F|\iota)(W) \). Thus we have
\[
\langle D_{E|\rho}(\text{diag}[Z,W],(5-k)/2),(F|\iota)(W) \rangle = \langle D_{E|\rho(1\times i_2^{-1})}(\text{diag}[Z,W],(5-k)/2),F(W) \rangle
\]
where we have used the fact that

$$\langle G_1|\gamma, G_2 \rangle = \langle G_2, G_2|\gamma^{-1} \rangle$$

for Siegel modular forms $G_1$ and $G_2$.

Our next step is to make sure that the Fourier coefficients of $E(Z,W)$ are still in some finite extension of $\mathbb{Z}_p$, where

$$E(Z,W) := D_{E|\rho(1 \times \iota_2^{-1})}(\text{diag}[Z,W], (5 - k)/2).$$

Recall from Theorem IV.8 that $D_{E^*}(Z, (5 - k)/2) \in \mathcal{M}_k(\Gamma_0^8(N), \mathbb{Z}_p[\chi])$. Therefore, applying the $q$-expansion principle ([10], Proposition 1.5) to $D_{E^*}(\text{diag}[Z,W], (5 - k)/2)$ slashed by $\iota_4^{-1}\rho(1 \times \iota_2^{-1})$, we get that $D_{E|\rho(1 \times \iota_2^{-1})}(\text{diag}[Z,W], (5 - k)/2)$ has Fourier coefficients in $\mathbb{Z}_p[\chi]$.

Summarizing, we have the following theorem.

**Theorem IV.9.** Let $N > 1$ and $k > 3$. For $F \in S_k(\Gamma_0^4(N), \mathbb{R})$ a Hecke eigenform with eigenvalues given by $T_S(n)F = \lambda_F(n)F$ and $p$ a prime with $p > 2$ and $\gcd(p,N) = 1$ we have

$$\langle \mathcal{E}(Z,W), F(W) \rangle = \pi^{-3} A_{k,N} L_N(5 - k, \lambda_F, \chi) F(Z)$$

with $\mathcal{E}(Z,W)$ having Fourier coefficients in $\mathbb{Z}_p[\chi]$. 

CHAPTER V

Simplifying Shimura’s Inner Product Relation

In this chapter we will work to simplify Equation 4.10. We do this by finding a relation between $\langle f, f \rangle$ and $\langle Ff, Ff \rangle$ and also decomposing the standard zeta function $L_N(2s, \lambda_F, \chi)$ into $L_N(2s - 2, \chi)L_N(2s + k - 3, f, \chi)L_N(2s + k - 4, f, \chi)$.

5.1 Relating $\langle Ff, Ff \rangle$ to $\langle \phi_f, \phi_f \rangle$ and $L(k, f)$

In this section we seek to generalize the following result of Kohnen and Skoruppa from level $M = 1$ to general level.

**Theorem V.1. ([41], Corollary to Theorem 2)** Let $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, $Ff \in S_k^*(\text{Sp}_4(\mathbb{Z}))$ the Saito-Kurokawa lift of $f$, and $\phi_f$ the Jacobi form associated via the Saito-Kurokawa correspondence. Then the formula

$$
\langle Ff, Ff \rangle = \frac{\langle \phi_f, \phi_f \rangle}{\pi^k c_k} L(k, f)
$$

holds, where $c_k = \frac{3 \cdot 2^{k+1}}{(k-1)!}$.

We follow Kohnen and Skoruppa’s arguments; generalizing some results where needed.

Let $F, G \in S_k^*(\Gamma_0^1(M))$ be eigenforms with Fourier-Jacobi expansions given by

$$
F(Z) = \sum_{M \geq 1} \phi_M(\tau, z)e(M\tau')
$$
and
\[ G(Z) = \sum_{M \geq 1} \psi_M(\tau, z)e(M\tau'). \]

Define a Dirichlet series attached to \( F \) and \( G \) by
\[ D_{F,G}(s) = \zeta(2s - 2k + 4) \sum_{M \geq 1} \langle \phi_M, \psi_M \rangle M^{-s}. \]

Set
\[ (5.1) \quad D_{F,G}^*(s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) \prod_{p|M} (1 - p^{-2(s-k+2)}) D_{F,G}(s). \]

Horie has shown in [32] that \( D_{F,G}^*(s) \) has meromorphic continuation to \( \mathbb{C} \). It is entire if \( \langle F, G \rangle = 0 \) and otherwise has a simple pole at \( s = k \). Calculating the residue of \( D_{F,G} \) at \( s = k \) will give us the desired generalization of Theorem V.1.

Define an Eisenstein series
\[ E_{s,M}(Z) = \sum_{\gamma \in C_{2,1}(M) \backslash \Gamma_0^4(M)} \left( \frac{\det(\text{Im}\gamma Z)}{\text{Im}(\gamma Z)_1} \right)^s. \]

and
\[ (5.2) \quad E_{s,M}^*(Z) = \pi^{-s} \Gamma(s) \zeta(2s) \prod_{p|M} (1 - p^{-2s}) E_{s,M}(Z) \]

where \( (\gamma Z)_1 \) denotes the upper left entry and
\[ C_{2,1}(M) = \left\{ \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^4(M) \left| \begin{array}{c} \lambda', \mu' = (\lambda, \mu) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \right. \right\}. \]

One has that \( E_{s,M}^*(Z) \) has meromorphic continuation to \( \mathbb{C} \) with possible simple poles at \( s = 0, 2 \) ([32]). Kohnen and Skoruppa have calculated that \( \text{res}_{s=2} E_{s,1}^*(Z) = 1 \) ([41], Page 545). Note that this is independent of \( Z \), so we have \( \text{res}_{s=2} E_{s,1}^*(MZ) = 1 \) for
all positive integers $M$. Equation 5.2 gives $\text{res}_{s=2} E_{s,1}(Z) = \frac{90}{\pi^2}$. As above, this residue is independent of $Z$ so we have $\text{res}_{s=2} E_{s,1}(MZ) = \frac{90}{\pi^2}$ for all positive integers $M$. Horie ([32], Page 51) gives the following formula, allowing one to calculate the residue of $E_{s,M}(Z)$ inductively in terms of $E_{s,d}(Z)$ for $d \mid M$:

$$E_{s,1}(MZ) = \frac{1}{M^s} \sum_{d \mid M} d^{2s} \prod_{p \mid d} (1 - p^{-2s}) E_{s,d}(Z).$$

In fact, for $M = p_1^{m_1} \ldots p_n^{m_n}$, we have

$$\text{res}_{s=2} E_{s,M}(Z) = \left( \frac{90}{\pi^2} \right) h(p_1, \ldots, p_n) \prod_{i=1}^{n} \left( \frac{1}{p_i^{2m_i - 2}} \right) \left( \frac{1}{p_i^4 - 1} \right)$$

where $h$ is a polynomial with coefficients in $\mathbb{Z}$ uniquely determined by $M$. For example, for $M = p^n$ for a prime $p$, we have

$$h(p) = p^2 - 1$$

and for $M = p_1 \ldots p_n$ a product of distinct primes, we have

$$h(p_1, \ldots, p_n) = \prod_{i=1}^{n} (p_i^2 - 1).$$

Appealing to Equation 5.2 one obtains for $M = p_1^{m_1} \ldots p_n^{m_n}$

$$\text{res}_{s=2} E_{s,M}^*(Z) = h(p_1, \ldots, p_n) \prod_{i=1}^{n} \left( 1 - p_i^{-4} \right) \left( \frac{1}{p_i^{2m_i - 2}} \right) \left( \frac{1}{p_i^4 - 1} \right).$$

Now we can turn our attention back to calculating the residue of $D_{F,G}(s)$ at $s = k$.

Horie ([32], Equation 4) shows that

$$\pi^{-k+2} [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)] \langle FE_{s-k+2,M}^*, G \rangle = M^s D_{F,G}^*(s).$$

Taking the residue of this equation at $s = k$ and solving for $\text{res}_{s=k} D_{F,G}^*(s)$ we obtain

$$\text{res}_{s=k} D_{F,G}^*(s) = \frac{\pi^{2-k} [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)]}{M^k} \text{res}_{s=2} E_{s,M}^*(Z) \langle F, G \rangle$$

$$= \frac{\pi^{2-k} [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)]}{M^k} h(p_1, \ldots, p_n) \prod_{i=1}^{n} \left( \frac{1}{p_i^{2m_i - 2}} \right) \left( \frac{1}{p_i^4 - 1} \right) \langle F, G \rangle.$$
On the one hand, taking the residue at \( s = k \) of Equation 5.1 we have

\[
\text{res}_{s=k} D^*_{F,G}(s) = (2\pi)^{-2k} (k-1)! \prod_{p|M} (1-p^{-4}) \text{res}_{s=k} D_{F,G}(s).
\]

Now we combine these two results and solve for \( \text{res}_{s=k} D_{F,G}(s) \) to obtain

\[
(5.5) \quad \text{res}_{s=k} D_{F,G}(s) = \frac{2^{2k} \pi^{k+2} [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)] h(p_1, \ldots, p_n)}{M^k (k-1)! \prod_{i=1}^{\infty} (p_i^{2m_i-2})(p_i^4 - 1)} \langle F, G \rangle.
\]

**Lemma V.2.** Let \( f \in S_{2k-2}(\Gamma_0(M)) \) be a newform, \( F_f \) the Saito-Kurokawa lift of \( f \), and \( \phi_f \) the corresponding Jacobi form obtained in the Saito-Kurokawa correspondence. Then we have

\[
\langle \phi_f, \phi_f \rangle = \frac{\pi^k 2^{2k+1} [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^4(M)] h(p_1, \ldots, p_n)}{M^k (k-1)! \prod_{i=1}^{\infty} (p_i^{2m_i-2})(p_i^4 - 1)} \langle F_f, F_f \rangle.
\]

**Proof.** Dabrowski ([12], Thm. 4.2) gives the formula

\[
(5.6) \quad D_{F_f,F_f}(s) = \langle \phi_f, \phi_f \rangle L^*_{\text{spin}}(s, F_f)
\]

for \( M \) odd and square-free, where we recall that

\[
L^*_{\text{spin}}(s, F_f) = \prod_{p|M} (1-p^{k-1-s})^{-1} (1-p^{k-2-s})^{-1} L_{\text{spin}}(s, F_f).
\]

However, the only reason that the level is restricted is because the Saito-Kurokawa correspondence was not known for arbitrary level at that time. In light of [50], we have Dabrowski’s formula for general level. In fact, this result was originally proven by Kohnen and Skoruppa for level 1: see ([41], Theorem 2). Equation 3.2 gives us

\[
L^*_{\text{spin}}(s, F_f) = \zeta(s-k+1)\zeta(s-k+2)L(s, f).
\]

Combining this with Equation 5.6 and taking residues at \( s = k \) gives

\[
\text{res}_{s=k} D_{F_f,F_f}(s) = \frac{\pi^2}{6} L(k, f) \langle \phi_f, \phi_f \rangle.
\]

Combining this with Equation 5.5 we obtain the result. \( \square \)
5.2 Relating $\langle \phi_f, \phi_f \rangle$ to $\langle g_f, g_f \rangle$

As in the previous section, let $\phi_f$ denote the Jacobi form associated to $f$ via the Saito-Kurokawa correspondence. Likewise, let $g_f$ denote the half-integral weight modular form associated to $f$ via the Saito-Kurokawa correspondence. In this section we will calculate a relationship between $\langle \phi_f, \phi_f \rangle$ and $\langle g_f, g_f \rangle$. Combining this with Lemma V.2 we will obtain a relationship between $\langle F_f, F_f \rangle$ and $\langle g_f, g_f \rangle$.

Let $g_f(z) = \sum_{n=1}^{\infty} c_g(n) q^n$ be the Fourier expansion of $g_f$. Consider the summation

$$\sum_{n=1}^{\infty} c_g(n)^2 n^{s+k-3/2}.$$ 

Applying the Rankin-Selberg method to this summation we have for sufficiently large $s$:

$$(4\pi)^{-s-k-1/2} \Gamma(s + k - 3/2) \sum_{n=1}^{\infty} \frac{c_g(n)^2}{n^{s+k-3/2}} = \int_{b^+ / \Gamma_0(4M)} |g_f(z)|^2 y^{s+k-5/2} dxdy$$

$$= \int_{b^+ / \Gamma_0(4M)} y^{k-1/2} |g_f(z)|^2 E_s^4(z) \frac{dxdy}{y^2}$$

with $E_s^4(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4M)} (\text{Im } (\gamma z))^s$ and $\Gamma_\infty$ the stabilizer of $\infty$. In other words,

$$\sum_{n=1}^{\infty} \frac{c_g(n)^2}{n^{s+k-3/2}} = (4\pi)^{s+k-1/2} \Gamma(s + k - 3/2) \int_{\Gamma_0(4M) \backslash b^+} E_s^4(z) g_f(z) g_f(z) y^{k-1/2} dxdy \frac{y^2}{y^2}.$$ 

Taking residues at $s = 1$ we obtain

$$\text{res}_{s=1} \left( \sum_{n=1}^{\infty} \frac{c_g(n)^2}{n^{s+k-3/2}} \right) = \frac{(4\pi)^{k-1/2} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]}{\Gamma(k - 1/2)} \langle g_f, g_f \rangle \text{res}_{s=1} E_s^4(z)$$

$$= \frac{3 \cdot 2^{k-1} (4\pi)^{k-1/2}}{\pi^{3/2}(2k - 3)!!} \langle g_f, g_f \rangle$$

where we have used that

$$\text{res}_{s=1} E_s^4(z) = \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]} \text{res}_{s=1} E_s(z)$$

$$= \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]} \left( \frac{3}{\pi} \right)$$
and \( E_s(z) \) is the Eisenstein series for \( \text{SL}_2(\mathbb{Z}) \). Solving the above residue calculation for \( \langle g_f, g_f \rangle \) we have

\[
\langle g_f, g_f \rangle = \frac{(2k - 3)!!}{3 \cdot 2^{3k-2} \pi^{k-2}} \text{res}_{s=1} \left( \sum_{n \geq 1} \frac{c_g(n)^2}{n^{s+k-3/2}} \right).
\]

We define two half-integral weight modular forms \( g_0 \) and \( g_1 \) by

\[
g_j(z) = \sum_{n \equiv j \pmod{4}} c_g(n) q^{n/4}
\]

for \( j = 0, 1 \) as in ([21], Page 64-65). Using that \( g_f \) is in Kohnen’s ‘+−-space’, we see that \( g_f(z) = g_0(4z) + g_1(4z) \). Applying the same process to \( g_0 \) and \( g_1 \) as we just did to \( g_f \) we obtain

\[
\langle g_j, g_j \rangle = \frac{(2k - 3)!!}{3 \cdot 2^{3k-2} \pi^{k-2}} \cdot 2^{2k-1} \text{res}_{s=1} \left( \sum_{n \equiv j \pmod{4}} \frac{c_g(n)^2}{n^{s+k-3/2}} \right).
\]

Thus we have

\[
\langle g_0, g_0 \rangle + \langle g_1, g_1 \rangle = 2^{2k-1} \langle g_f, g_f \rangle.
\]

We need a slight generalization of Theorem 5.3 in [21]. In [21], the formula given only deals with the case \( M = 1 \). However, the proof carries through verbatim to the general case.

**Theorem V.3.** ([21], Theorem 5.3) For \( \phi_f \) and \( g_j \) as defined above, one has

\[
\langle \phi_f, \phi_f \rangle = \frac{1}{2[\text{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M) \backslash \mathcal{H}} \sum_{j=0}^{1} g_j(z) \overline{g_j(z)} v^{-k-3/2} \frac{du dv}{v^2}.
\]

Combining Equation 5.9 and Equation 5.10 we have:

**Lemma V.4.** For \( \phi_f \) and \( g_f \) defined as above we have

\[
\langle \phi_f, \phi_f \rangle = \frac{2^{2k-2}}{[\Gamma_0(M) : \Gamma_0(4M)]} \langle g_f, g_f \rangle.
\]
5.3 Relating $\langle g_f, g_f \rangle$ to $\langle f, f \rangle$

The only remaining hurdle in establishing a relationship between $\langle F_f, F_f \rangle$ and $\langle f, f \rangle$ is to relate $\langle g_f, g_f \rangle$ to $\langle f, f \rangle$. This is much easier than the other steps as the work has already been done for us.

Let $\ell$ be a prime dividing $M$. Define the Atkin-Lehner involution on $S_{2k-2}^\text{new}(\Gamma_0(M))$ associated to $\ell$ by

$$f|W_\ell = \frac{1}{\sqrt{\ell}} \begin{pmatrix} \ell & \alpha \\ M & \ell \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{Z}$ and $\ell^2 \beta - M \alpha = \ell$. We can define $w_\ell \in \{\pm 1\}$ for every $\ell \mid M$ by

$$f|W_\ell = w_\ell f.$$

Recall that for a discriminant $D$, we defined $\chi_D(n) = \left(\frac{D}{n}\right)$ as in Chapter II, Section 2.3.

**Lemma V.5.** ([39], Corollary 1) Let $M$ be odd and let $D$ be a discriminant with $(-1)^{k-1}D > 0$ and suppose that for all primes $\ell \mid M$ we have $\left(\frac{D}{\ell}\right) = w_\ell$. Then

$$\frac{|c_g(|D|)|^2}{\langle g_f, g_f \rangle} = 2^{\nu(M)} \frac{(k-2)!}{\pi^{k-1}} |D|^{k-3/2} \frac{L(k-1, f, \chi_D)}{\langle f, f \rangle}$$

(5.11)

where $\nu(M)$ is the number of primes dividing $M$.

The condition on the discriminant in Lemma V.5 is not a major restriction. If for some prime $\ell \mid M$ we have $w_\ell = -\left(\frac{D}{\ell}\right)$, then $c_g(|D|) = 0$ ([39], Page 243). So as long as we choose $D$ so that $(M, D) = 1$ and $c_g(|D|) \neq 0$, then our condition will be satisfied.

We are now in a position to gather our results and state the relationship between $\langle f, f \rangle$ and $\langle F_f, F_f \rangle$. 
**Theorem V.6.** Let $M = p_1^{m_1} \ldots p_n^{m_n}$ be odd, $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))$, and $F_f \in S_{k}^{\text{new}}(\Gamma_0^1(M))$ the Siegel modular form associated to $f$ via the Saito-Kurokawa correspondence. Let $D$ be a discriminant with $(-1)^{k-1}D > 0$, $\gcd(M, D) = 1$, and $c_g(|D|) \neq 0$. Then one has

$$\langle F_f, F_f \rangle = B_{k,M} \frac{|c_g(|D|)|^2 L(k, f)}{\pi |D|^{k-3/2} L(k - 1, f, \chi_D)} \langle f, f \rangle$$

with

$$B_{k,M} = \frac{M^k (k - 1) \prod_{i=1}^{n} (p_i^{2m_i-2}(p_i^4 - 1))}{2^{\nu_2(M)+3} 3^h(p_1, \ldots, p_n)[Sp_4(\mathbb{Z}) : \Gamma_0^1(M)] [\Gamma_0(M) : \Gamma_0(4M)]}$$

where $h(p_1, \ldots, p_n)$ is defined as Equation 5.3.

Note that for $M = 1$, we do not need to add the condition $c_g(|D|) \neq 0$ as this will be automatic. This condition was added in Lemma V.5 to cover the case when $M > 1$.

### 5.4 Decomposing the Standard $L$-function

In Equation 4.10 at the end of Chapter IV, we had a standard $L$-function of the form $L_N(5 - k, \lambda_f, \chi)$. We now assume that the $F$ that occurs there is a Saito-Kurokawa lift $F_f$. This assumption allows us to decompose the standard $L$-function into a product of $L$-functions associated to $f$ and $\chi$. In particular, we have the following theorem.

**Theorem V.7.** For $F_f$ a Saito-Kurokawa lift of $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M))$ with eigenvalues $\lambda_{F_f}$ and $N > 1$ an integer so that $M \mid N$, we have

$$L_N(2s, \lambda_{F_f}, \chi) = L_N(2s - 2, \chi)L_N(2s + k - 3, f, \chi)L_N(2s + k - 4, f, \chi).$$

**Proof.** We start by recalling the definition of the standard $L$-function:

$$L_N(2s, \lambda_{F_f}, \chi) = \prod_{p \mid N} \left[ (1 - \chi(p)p^{2-2s}) \prod_{i=1}^{2}(1 - \chi(p)\alpha_{p,i}p^{2-2s})(1 - \chi(p)\alpha_{p,i}^{-1}p^{2-2s}) \right]^{-1}.$$
We need to relate the Satake parameters $\alpha_{p,i}$ to the eigenvalues of $f$ in order to decompose $L_N(2s, \lambda_{F_f}, \chi)$. In order to accomplish this, we use the following formula (see [56]):

$$L^{(p)}_{\text{spin}}(s, F_f) = (1 - \alpha_0 p^{-s})(1 - \alpha_0 \alpha_1 p^{-s})(1 - \alpha_0 \alpha_1 \alpha_2 p^{-s}).$$

Recall that by Equation 3.2 we have

$$L^{(p)}_{\text{spin}}(s, F_f) = (1 - p^{k-1-s})(1 - p^{k-s-2})(1 - a_f(p) p^{-s} + p^{2k-3-2s}).$$

Letting $x = p^{-s}$, we have the polynomial identity

$$(1-\alpha_0 x)(1-\alpha_0 \alpha_1 x)(1-\alpha_0 \alpha_1 \alpha_2 x) = (1-p^{k-1} x)(1-p^{k-2} x)(1-a_f(p) x + p^{2k-3} x^2).$$

Therefore we have that $\alpha_0$, $\alpha_0 \alpha_1$, $\alpha_0 \alpha_2$, and $\alpha_0 \alpha_1 \alpha_2$ are

$$\left\{ p^{k-1}, p^{k-2}, \frac{2p^{2k-3}}{a_f(p) \pm \sqrt{a_f(p)^2 - 4p^{2k-3}}} \right\}.$$

The values $\alpha_0 \alpha_1$ and $\alpha_0 \alpha_2$ are completely symmetrical so we set

$$\alpha_0 \alpha_1 = \frac{2p^{2k-3}}{a_f(p) + \sqrt{a_f(p)^2 - 4p^{2k-3}}}$$

and

$$\alpha_0 \alpha_2 = \frac{2p^{2k-3}}{a_f(p) - \sqrt{a_f(p)^2 - 4p^{2k-3}}}.$$

Since we have $\alpha_0^2 \alpha_1 \alpha_2 = p^{2k-3}$, $\alpha_0 = p^{k-1}$ or $p^{k-2}$ but is arbitrary up to this choice.

We fix $\alpha_0 = p^{k-1}$. Pick $\alpha_p$ and $\beta_p$ such that

$$\alpha_p + \beta_p = a_f(p)$$

and

$$\alpha_p \beta_p = p^{2k-3}.$$

Thus

$$\alpha_1 = \beta_p p^{1-k}$$

$$\alpha_2 = \alpha_p p^{1-k}.$$
Therefore we can write

\[(1 - \chi(p)\alpha_1 p^{2-2s})(1 - \chi(p)\alpha_2 p^{2-2s}) = 1 - \chi(p)a_f(p)p^{3-2s-k} + \chi(p)^2p^{3-4s}\]

and

\[(1 - \chi(p)\alpha_1^{-1} p^{2-2s})(1 - \chi(p)\alpha_2^{-1} p^{2-2s}) = 1 - \chi(p)a_f(p)p^{4-2s-k} + \chi(p)^2p^{5-4s}.\]

Substituting this back in for $L_N(2s, \lambda_{F_f}, \chi)$ we have the result. \qed
CHAPTER VI

Periods and a Certain Hecke Operator

Throughout this chapter we make the following assumptions. Let $k$ and $M$ be positive integers with $k \geq 2$. Let $p$ be a prime so that $\gcd(p, M) = 1$ and $p > 2k - 2$. Fix a system of compatible embeddings $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, $\mathbb{Q} \hookrightarrow \mathbb{C}$, and $\mathbb{Q}_p \hookrightarrow \mathbb{C}$. We let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer $\varpi$. Let $p$ be the prime of $\mathcal{O}$ lying over $p$.

We begin the chapter by recalling some of the important elements of Eichler-Shimura theory that we will require. We then recall the definition of periods $\Omega_f^\pm$ associated to a newform $f$. These periods will allow us to normalize special values of the $L$-function attached to $f$ in Chapter VII. They also appear in the following conjecture we assert.

**Conjecture VI.1.** Let $f = f_1, f_2, \ldots, f_r$ be a basis of eigenforms for $S_{2k-2}^{\text{new}}(\Gamma_0(M))$ with $M$ square-free and $f$ a newform. Enlarge $\mathcal{O}$ if necessary so that the basis is defined over $\mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{T}_\mathcal{O}$ associated to $f$ and assume that $\rho_\mathfrak{m}$ is irreducible. Then there exists a Hecke operator $t \in \mathbb{T}_\mathcal{O}$ so that

$$tf_i = \begin{cases} u_{f_i} \langle f, f_i \rangle \Omega_f^+ & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

for $u$ a unit in $\mathcal{O}$ and $\Omega_f^\pm$ complex periods we will construct in Section 6.2.
We spend the last two sections of this chapter proving the conjecture in the case that \( f \) is ordinary at \( p \) or the case that \( M \geq 4 \).

### 6.1 Eichler-Shimura Theory

We will need some results of Eichler-Shimura theory, so we briefly recall the setup. Our treatment follows [31] and [83]. One can see [70] for a complete account. Let \( R \) be a ring. We denote the symmetric polynomial algebra of degree \( n \) over \( R \) by \( L_n(R) \), i.e., \( L_n(R) \) is the set of homogeneous polynomials of degree \( n \) with coefficients in \( R \). One has an action of \( \text{GL}_2(\mathbb{Q}) \cap \text{M}_2(\mathbb{Z}) \) on \( L_n(R) \) given by

\[
\gamma \cdot P(x, y) = P((x, y) \det(\gamma)\gamma^{-1}).
\]

We briefly recall the definition of the first cohomology group as well as the parabolic cohomology group in some generality. Let \( G \) be a group and \( L \) a left \( G \)-module. We call a map \( u : G \to L \) a 1-cocyle if it satisfies

\[
u(ab) = u(a) + au(b)
\]

for all \( a, b \in G \). We let \( Z(G, L) \) denote the group of 1-cocyles of \( G \) taking values in \( L \). We call \( u \in Z(G, L) \) a 1-coboundary if it satisfies

\[
u(a) = (a - 1)x
\]

for some \( x \in L \) and all \( a \in G \). Denote the set of coboundaries in \( Z(G, L) \) by \( B(G, L) \). The first cohomology group of \( G \) and \( L \) is defined by

\[
H^1(G, L) = Z(G, L)/B(G, L).
\]

Fix a subset \( P \) of \( G \). Set

\[
Z_P(G, L) = \{ u \in Z(G, L) : u(a) \in (a - 1)L \text{ for all } a \in P \}.
\]
It is clear that one has $B(G, L) \subset Z_P(G, L)$. The first parabolic cohomology group relative to $P$ is defined by

$$H^1_P(G, L) = Z_P(G, L)/B(G, L).$$

Now consider the case when $G = \Gamma_1(M)$ and $L = L_{2k-4}(R)$. Fix a minimal subset $P$ of $\Gamma_1(M)$ so that every parabolic element of $\Gamma_1(M)$ is conjugate to a power of an element of $P$ in $\Gamma_1(M)$. Then $P$ corresponds bijectively to the set of all equivalence classes of cusps of $\Gamma_1(M)$. We have by ([70], 8.1.30) that $H^1_P(\Gamma_1(M), L_{2k-4}(R))$ does not depend on the choice of $P$. We will refer to $H^1_P(\Gamma_1(M), L_{2k-4}(R))$ as the first parabolic cohomology group of $\Gamma_1(M)$.

We can define an action of the double cosets $\Gamma_1(M) \alpha \Gamma_1(M)$ on $H^1_P(\Gamma_1(M), L_{2k-4}(\mathbb{C}))$ for $\alpha \in \Gamma_1(M)'$ where we recall from Chapter II that

$$\Gamma_1(M)' = \left\{ \alpha \in \text{GL}_2^+(\mathbb{Q}) : \alpha \Gamma_1(M) \alpha^{-1} \sim \Gamma_1(M) \right\}.$$

Write $\Gamma_1(M) \alpha \Gamma_1(M) = \bigsqcup_{i=1}^d \Gamma_1(M) \alpha_i$. Given $\gamma \in \Gamma_1(M)$, let $\alpha_i \gamma = \gamma_i \alpha_j$ for some $j$ and some $\gamma_i \in \Gamma_1(M)$. If $u \in H^1_P(\Gamma_1(M), L_{2k-4}(\mathbb{C}))$, we define $v = [\Gamma_1(M) \alpha \Gamma_1(M)] u$ by

$$v(\gamma) = \sum_{i=1}^d \iota_1^{-1} \iota_1 t \alpha_i \gamma_i u(\gamma_i)$$

where $\iota_1 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. It is straightforward to check that this gives a map

$$[\Gamma_1(M) \alpha \Gamma_1(M)] : H^1_P(\Gamma_1(M), L_{2k-4}(\mathbb{C})) \to H^1_P(\Gamma_1(M), L_{2k-4}(\mathbb{C})).$$

One can consult ([70], Section 8.3) for the details. This allows one to define an action of the Hecke algebra for $\Gamma_1(M)$ on $H^1_P(\Gamma_1(M), L_{2k-4}(\mathbb{C}))$.

Let $f$ be a weight $2k-2$ modular form for $\Gamma_1(M)$. Define a differential form with
values in $L_{2k-4}(\mathbb{C})$ by
\begin{equation}
\omega_f = f(z)(zx + y)^{2k-4}dz.
\end{equation}

Pick a basepoint $z_0 \in \mathfrak{h}^1$. The map
\begin{equation}
\delta(f)(\gamma) = \int_{z_0}^{\gamma z_0} \omega_f
\end{equation}
defines a 1-cocyle on $\Gamma_1(M)$ with values in $L_{2k-4}(\mathbb{C})$ independent of the choice of $z_0$. In fact, if $f$ happens to be a cuspform then the class associated to $\omega_f$ actually lies in the parabolic subgroup associated to $\Gamma_1(M)$. The map $\delta(f)$ induces the Eichler-Shimura isomorphism
\[ S_{2k-2}(\Gamma_1(M)) \oplus \overline{S}_{2k-2}(\Gamma_1(M)) \cong H^1_P(\Gamma_1(M), L_{2k-4}(\mathbb{C})) \]
where $\overline{S}_{2k-2}(\Gamma_1(M))$ is the space of complex conjugates of elements in $S_{2k-2}(\Gamma_1(M))$. Note that this is a Hecke-equivariant isomorphism. See ([70], Chapter 8) for details.

### 6.2 Periods Associated to Newforms

In this section we will define periods $\Omega_{f}^{\pm}$ associated to our modular form $f$ as in Conjecture VI.1. These periods allow us to normalize the $L$-values $L(m,f)$ for $0 < m < 2k - 2$ to be algebraic integers.

Let $f \in S^{\text{new}}_{2k-2}(\Gamma_1(M))$ be a newform with eigenvalues in $\mathcal{O}$. The congruence class of $f$ in $S_{2k-2}(\Gamma_1(M))$ determines a maximal ideal $m$ of $\mathbb{T}_{\mathcal{O}}$ and a residual representation
\[ \rho_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{T}_{\mathcal{O}}/m) \]
so that $\text{Tr}(\rho_m(\text{Frob}_\ell)) = T(\ell)$ for all primes $\gcd(\ell, M\mathcal{O}) = 1$ where $\mathbb{T}_{\mathcal{O}}/m$ is of characteristic $p$. This fact is essentially due to Deligne, see ([57], Proposition 5.1) for a detailed proof. We make the assumption that $\rho_m$ is irreducible. This allows us to
identify the localization at \( \mathfrak{m} \) of the parabolic cohomology group \( H^1_P \) with that of the regular cohomology group \( H^1 \) ([83], Page 401).

Associated to \( f \) is a surjective \( \mathcal{O} \)-algebra map \( \pi_f : \mathbb{T}_{\mathcal{O}, \mathfrak{m}} \to \mathcal{O} \) given by \( T_\ell \mapsto a_f(\ell) \).

We can view this as a map into \( \mathbb{C} \) as well via the embeddings \( \mathcal{O} \hookrightarrow K \hookrightarrow \mathbb{C} \) where the embedding of \( K \) into \( \mathbb{C} \) was fixed at the beginning of this chapter. Let \( \wp_f \) be the kernel of \( \pi_f \).

For \( R = \mathcal{O}, K, \) or \( \mathbb{C} \) the cohomology group \( H^1(\Gamma_1(M), L_{2k-4}(R)) \) decomposes as

\[
H^1(\Gamma_1(M), L_{2k-4}(R)) = H^1(\Gamma_1(M), L_{2k-4}(R))^+ \oplus H^1(\Gamma_1(M), L_{2k-4}(R))^-
\]

with respect to the action of the Atkin-Lehner involution. We have natural maps

\[
H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))^\pm \to H^1(\Gamma_1(M), L_{2k-4}(K))^\pm
\]

with kernels the torsion subgroups of \( H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))^\pm \). Our assumption that \( \rho_\mathfrak{m} \) is irreducible gives us that \( H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))_\mathfrak{m} \) is torsion-free, so we have injections

\[
H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))^\pm_\mathfrak{m} \to H^1(\Gamma_1(M), L_{2k-4}(K))^\pm_\mathfrak{m}.
\]

Combining these injections with the natural injections

\[
H^1(\Gamma_1(M), L_{2k-4}(K))^\pm_\mathfrak{m} \to H^1(\Gamma_1(M), L_{2k-4}(\mathbb{C}))^\pm_\mathfrak{m}
\]

give us injections

\[
H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))^\pm_\mathfrak{m} \to H^1(\Gamma_1(M), L_{2k-4}(\mathbb{C}))^\pm_\mathfrak{m}.
\]

Now consider the embeddings

\[
H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))^\pm_{\wp_f} \hookrightarrow H^1(\Gamma_1(M), L_{2k-4}(\mathbb{C}))^\pm_{\wp_f}
\]

where for an \( R \)-module \( M \) and an ideal \( I \) of \( R \) we write \( M[I] \) for the intersection over the elements \( r \in I \) of the kernels \( r : M \to M \). The fact that \( f \) is a newform
implies that \( H^1(\Gamma_1(M), L_{2k-4}(O)) \cong O \) and \( H^1(\Gamma_1(M), L_{2k-4}(C)) \cong \mathbb{C} \).

Let \( \delta_f^\pm \) be an \( O \)-basis of \( H^1(\Gamma_1(M), L_{2k-4}(O)) \). Let \( \delta(f)^\pm \) be a \( C \)-basis of \( H^1(\Gamma_1(M), L_{2k-4}(C)) \) where \( \delta(f)^\pm \) is defined by \( \delta(f) = \delta(f)^+ + \delta(f)^- \). The periods \( \Omega_f^\pm \) are then defined to be the complex numbers so that we have

\[
\delta(f)^\pm = \Omega_f^\pm \delta_f^\pm.
\]

These periods are determined up to a change of \( O \)-basis, i.e., up to a \( O \)-unit. We have the following theorem essentially due to Shimura.

**Theorem VI.2.** ([65], Theorem 1) Let \( f \in S_{2k-2}(\Gamma_1(M)) \) be a newform with Fourier coefficients in \( O \). There exist complex periods \( \Omega_f^\pm \) such that for each integer \( m \) with \( 0 < m < 2k-2 \) and every Dirichlet character \( \chi \) one has

\[
\frac{L(m, f, \chi)}{\tau(\chi)(2\pi i)^m} \in \begin{cases} 
\Omega_f^+ \mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^m \\
\Omega_f^- \mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^m^{-1},
\end{cases}
\]

where \( \tau(\chi) \) is the Gauss sum of \( \chi \) and \( \mathcal{O}_\chi \) is the extension of \( O \) generated by the values of \( \chi \).

### 6.3 A Certain Hecke Operator: Ordinary Case

In this section we will establish the validity of Conjecture VI.1 in the case that \( f \) is ordinary at \( p \). Throughout this section we assume that \( M \) is square-free. The reason for this assumption is to ensure that our Hecke algebra is reduced. We use results of Hida ([29]) to construct \( t \).

Let \( f \) be a newform in \( S^\text{new}_{2k-2}(\Gamma_1(M), O) \). Let \( \pi_f : T_{\mathcal{O}, m} \to \mathcal{O} \) be the surjective \( \mathcal{O} \)-algebra homomorphism given in Section 6.2. We again let \( \varphi_f \) denote the kernel of \( \pi_f \). Let \( f = f_1, f_2, \ldots, f_r \) be a basis of eigenforms for \( S_{2k-2}(\Gamma_0(M)) \). We enlarge \( \mathcal{O} \) here if necessary so that this basis is defined over \( \mathcal{O} \). As with \( f \), there are maps \( \pi_{f_i} \) for each \( i \) as well as kernels \( \varphi_{f_i} \).
The fact that $f$ is a newform allows us to write

$$\mathbb{T}_{\mathcal{O},m} \otimes_{\mathcal{O}} K = K \oplus D$$

with a $K$-algebra $D$ so that $\pi_f$ induces the projection of $\mathbb{T}_{\mathcal{O},m}$ onto $K$. In this direct sum, $K$ corresponds to the Hecke algebra acting on the eigenspace generated by $f$ and $D$ corresponds to the Hecke algebra acting on the space generated by the rest of the $f_i$'s. Let $\varrho$ be the projection map of $\mathbb{T}_{\mathcal{O},m}$ to $D$. Set $I_f$ to be the kernel of $\varrho$. Using that our Hecke algebra is reduced, it is clear from the definition that we have

$$I_f = \text{Ann}(\varrho_f) = \bigcap_{i=2}^r \varrho_{f_i}$$

where $\text{Ann}(\varrho_f)$ denotes the annihilator of the ideal $\varrho_f$. Since $\mathbb{T}_{\mathcal{O},m}$ is reduced, we have that $\varrho_f \cap I_f = 0$. Therefore we have that

$$\mathbb{T}_{\mathcal{O},m}/(\varrho_f \oplus I_f) \cong \mathbb{T}_{\mathcal{O},m}/(\varrho_f, I_f) \cong \mathcal{O}/\pi_f(I_f)$$

where we use here that

$$\pi_f : \mathbb{T}_{\mathcal{O},m}/\varrho_f \cong \mathcal{O}.$$ 

Since $\mathcal{O}$ is a principal ideal domain, there exists $a \in \mathcal{O}$ so that $\pi_f(I_f) = a\mathcal{O}$. Therefore, we have

$$\mathcal{O}/a\mathcal{O} \cong \mathbb{T}_{\mathcal{O},m}/(\varrho_f \oplus I_f).$$

For each prime $\ell$, choose $\alpha_f(\ell)$ and $\beta_f(\ell)$ so that $\alpha_f(\ell) + \beta_f(\ell) = a_f(\ell)$ and $\alpha_f(\ell)\beta_f(\ell) = \ell^{2k-3}$. Set

$$D(s, \pi_f) = \prod_{\ell} H^{(\ell)}(s, \pi_f)^{-1}$$

$$= \prod_{\ell} ((1 - \alpha_f(\ell)^2 \ell^{-s})(1 - \alpha_f(\ell)\beta_f(\ell)\ell^{-s})(1 - \beta_f(\ell)^2 \ell^{-s}))^{-1}.$$
Shimura has shown this Euler product converges if the real part of $s$ is sufficiently large and can be extended to a meromorphic function on the entire complex plane that is holomorphic except for possible simple poles at $s = 2k - 2$ and $2k - 3$ ([63], Theorem 1). The values of $D(2k - 2, \pi_f)/U(\pi_f)$ are in $\mathcal{O}$ ([29], Page 86) where

$$U(\pi_f) = \frac{(2\pi)^{2k-1} \Omega_f^+ \Omega_f^-}{(2k-3)! M \varphi(M)}.$$

Following Hida we define $\varepsilon \in K$ by

$$a = \frac{D(2k - 2, \pi_f)}{\varepsilon \cdot U(\pi_f)}$$

where $a$ is given by Equation 6.4.

**Theorem VI.3.** ([29], Theorem 2.5) Let $f \in S^{\text{new}}_{2k-2}(\Gamma_1(M), \mathcal{O})$ be a newform. Let $\mathfrak{p}$ be the prime of $\mathcal{O}$ over $p$. If $f$ is ordinary at $\mathfrak{p}$, then $\varepsilon$ is a unit in $\mathcal{O}$.

Combining ([31], Theorem 5.1) and ([70], 8.2.17) we have

$$D(2k - 2, \pi_f) = \frac{2^{4k-4} \pi^{2k-1}}{(2k-3)! M \varphi(M)} \langle f, f \rangle$$

where we have used that $\langle f, f \rangle$ is independent of the congruence subgroup we consider our modular forms on by the way we have defined the inner product. Inserting this expression for $D(2k - 2, \pi_f)$ into Equation 6.5 and simplifying we obtain

$$a = \frac{2^{2k-3}}{\varepsilon \cdot \Omega_f^+ \Omega_f^-} \langle f, f \rangle.$$

Combining Equations 6.3 and 6.4 we can write

$$T_{\mathcal{O}, m}/(\varphi_f \oplus \bigcap_{i=2}^{r} \varphi_{f_i}) \cong \mathcal{O}/a\mathcal{O}$$

where

$$a = \frac{2^{2k-3}}{\varepsilon \cdot \Omega_f^+ \Omega_f^-} \langle f, f \rangle.$$
Since $\mathbb{T}_{\mathcal{O},m}/\wp_f \cong \mathcal{O}$, there exists a $t \in I_f$ that maps to $a$ under the above isomorphism. Thus we have that

$$\begin{align*} tf_i &= \begin{cases} af & \text{if } i = 1 \\ 0 & \text{if } 2 \leq i \leq r. \end{cases} \end{align*}$$

This is the Hecke operator we seek. Using the fact that

$$\mathbb{T}_\mathcal{O} \cong \prod \mathbb{T}_{\mathcal{O},m}$$

where the product is over the maximal ideals of $\mathbb{T}_\mathcal{O}$, we can view $\mathbb{T}_{\mathcal{O},m}$ as a subring of $\mathbb{T}_\mathcal{O}$. Therefore we have the following theorem.

**Theorem VI.4.** Let $f = f_1, f_2, \ldots, f_r$ be a basis of eigenforms of $S_{2k-2}(\Gamma_0(M), \mathcal{O})$ with $k > 2$. Suppose that the residual representation $\rho_m$ associated to $f$ is irreducible and $f$ is ordinary at $\mathfrak{p}$. There exists a Hecke operator $t \in \mathbb{T}_\mathcal{O}$ such that $tf = af$ and $tf_i = 0$ for $i \geq 2$ where $a$ is as in Equation 6.7 with $\varepsilon$ a unit in $\mathcal{O}$.

### 6.4 A Certain Hecke Algebra: $M \geq 4$

In this section we show how we can avoid appealing to Theorem VI.3 by assuming $M \geq 4$. This allows us to construct the Hecke operator we seek without needing to impose the condition that $f$ be ordinary at $\mathfrak{p}$. We again assume that $M$ is square-free.

We continue with the notation of the last section, now imposing the condition that $M \geq 4$. In particular, we recall the isomorphism

$$\mathcal{O}/a\mathcal{O} \cong \mathbb{T}_{\mathcal{O},m}/(\wp_f \oplus I_f).$$

Let $L = H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))_m$ and $L^\pm = H^1(\Gamma_1(M), L_{2k-4}(\mathcal{O}))^\pm_m$. Since we are still under the assumption that $p > 2k - 2$ and $\gcd(p, M) = 1$, we can use ([83], Theorem 1.13) to conclude that the $\mathbb{T}_{\mathcal{O},m}$-modules $L^\pm$ are free of rank 1.
There is a skew-symmetric perfect pairing

\[ L^\pm \times L^\mp \to \mathcal{O} \]

adjoint with respect to the Hecke operators ([29], Equation 3.3). We write this pairing as \( (x, y) \mapsto A(x, y) \). Note that to give such a perfect pairing is equivalent to giving an isomorphism

\[ L^\pm \to \text{Hom}_\mathcal{O}(L^\mp, \mathcal{O}) \]

(the equivalence is given by \( x \mapsto A(x, \cdot) \).)

Since we are assuming that \( p > 2k - 2 \), \( \rho_m \) is irreducible and \( M \geq 4 \) is prime to \( p \), ([83] Theorem 1.13) gives isomorphisms

\[ \theta^\pm : L^\pm \cong S_{2k-2}(\Gamma_1(M), \mathcal{O})_m. \] (6.8)

Let \( \delta_f^\pm \) be the cohomology classes in \( L^\pm \) given by

\[ \Theta^\pm(\delta_f^\pm) = f \]

as in Equation 6.8. (Note that although we defined \( \delta_f^\pm \) in the previous section, this definition agrees with the previous one up to \( p \)-adic unit in \( \mathcal{O} \), which will not effect our calculations.) We clearly have that

\[ L^\pm[\varphi_f] = \mathcal{O}\delta_f^\pm, \]

i.e., \( (\delta_f^+, \delta_f^-) \) is a basis for \( L[\varphi_f] \).

**Lemma VI.5.** The ideal \( a\mathcal{O} \) is generated by \( A(\delta_f^+, \delta_f^-) \).

**Proof.** (See [13], Lemma 4.17) The modules \( L[\varphi_f] \) and \( L/L[I_f] \) are both free of rank 2 over \( \mathcal{O} \). We also have that the perfect pairing \( A(\cdot, \cdot) \) induces an isomorphism

\[ L/L[I_f] \cong \text{Hom}_\mathcal{O}(L[\varphi_f], \mathcal{O}). \]
The ideal \(a\mathcal{O}\) annihilates the \(\mathcal{O}\)-module \(L/(L[\wp_f] \oplus L[I_f])\). Since we know that \(L\) is free of rank 2 over \(\mathbb{T}_{\mathcal{O},m}\), we have that
\[
\#((\mathcal{O}/a\mathcal{O})^2 = \#(L/(L[\wp_f] \oplus L[I_f])).
\]
The cardinality of \(L/(L[\wp_f] \oplus L[I_f])\) is that of the cokernel of the map
\[
L[\wp_f] \hookrightarrow \text{Hom}_\mathcal{O}(L[\wp_f], \mathcal{O})
\]
arising from the pairing \(A(\cdot, \cdot)\). This cardinality is precisely
\[
\#(\mathcal{O}/\text{det}(A(x_i, x_j)))
\]
where the \(x_i\) and \(x_j\) run over a basis of \(L[\wp_f]\). However, we know this basis is precisely \(\delta_f^\pm\). Therefore using the skew-symmetry of the pairing \(A(\cdot, \cdot)\) we obtain the result.

It now remains to calculate \(A(\delta_f^+, \delta_f^-)\) in terms of \(\Omega_f^\pm\) and \(\langle f, f \rangle\). Note that by the definition of \(\Omega_f^\pm\) up to \(p\)-adic unit in \(\mathcal{O}\) we have
\[
A(\delta_f^+, \delta_f^-) = \frac{1}{\Omega_f^+ \Omega_f^-} A(\delta(f)^+, \delta(f)^-)
\]
where \(\delta(f)\) is defined as in Equation 6.2. Applying ([31], Theorem 5.1) we obtain (again up to \(p\)-adic unit in \(\mathcal{O}\))
\[
\frac{1}{\Omega_f^+ \Omega_f^-} A(\delta(f)^+, \delta(f)^-) = \frac{(2k - 3)! M \varphi(M)}{(-2\pi i)^{2k-1} \Omega_f^+ \Omega_f^-} D(2k - 2, \pi_f).
\]
We combine this with the following equation of Shimura ([64], Equation 2.5)
\[
D(2k - 2, \pi_f) = \frac{4^{2k-2} \pi^{2k-1}}{3(2k - 3)!} \langle f, f \rangle
\]
to obtain
\[
A(\delta_f^+, \delta_f^-) = \frac{(-1)^k i M \varphi(M) 4^{2k-2}}{3} \frac{\langle f, f \rangle}{\Omega_f^+ \Omega_f^-}.
\]
Now we just argue as we did at the end of the last section to produce the Hecke operator as in the conjecture. We summarize with the following theorem.
Theorem VI.6. Let $f = f_1, f_2, \ldots, f_r$ be a basis of eigenforms of $S_{2k-2}(\Gamma_0(M), \mathcal{O})$ with $k > 2$, $M \geq 4$, $M$ square-free, and $f$ a newform. Suppose that the residual representation $\rho_m$ associated to $f$ is irreducible. There exists a Hecke operator $t \in T_\mathcal{O}$ such that $tf = u \langle f, f \rangle \Omega f$ and $tf_i = 0$ for $i \geq 2$ where $u$ is a unit in $\mathcal{O}$. 
CHAPTER VII

The Congruence

In this chapter we are finally able to combine the results of the previous chapters to produce a congruence between our Saito-Kurokawa lift $F_f$ and a Siegel eigenform $G$ which is not a Saito-Kurokawa lift.

We start by gathering the results of the previous chapters to prove that $F_f$ is congruent to a non Saito-Kurokawa lift Siegel modular form. Once we have done this, we will next show that we actually have a congruence to a cuspidal eigenform.

We fix the following notation throughout this chapter. Let $k$, $M$, and $N$ be positive integers so that $k > 3$, $N > 1$, $M \mid N$ with $M$ odd and square-free. For a prime $p > 2k - 2$ with $\gcd(p, N) = 1$, we fix a system of compatible embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and $\overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$. We let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer $\varpi$. Let $\mathfrak{p}$ be the prime of $\mathcal{O}$ lying over $p$.

7.1 Congruent to a Modular Form

We begin by recalling Theorem IV.9 from the end of Chapter IV which gave an inner product relation between the Eisenstein series studied in that chapter with a Siegel eigenform. We restate the theorem in the situation that we will be interested in.
Theorem VII.1. (Chapter IV, Theorem IV.9) Let $\chi$ be a character as defined in Equation 4.1. Let $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M), \mathcal{O})$ be a newform with real Fourier coefficients and $F_f$ the Saito-Kurokawa lift of $f$. Then we have

$$\langle \mathcal{E}(Z, W), F_f(W) \rangle = \pi^{-3} A_{k,N} L_N(5 - k, \lambda_{F_f}, \chi) F_f(Z)$$

with $\mathcal{E}(Z, W)$ having Fourier coefficients in $\mathbb{Z}_p[\chi]$ and

$$A_{k,N} = \frac{(-1)^k 2^{2k-3} v_N}{3 [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^N(N)]}$$

with $v_N = \pm 1$ depending on $N$.

Recall that $\mathcal{E}(Z, W)$ is a Siegel modular form of weight $k$ and level $N$ in each variable. Shimura has worked out the theory for this situation in [67], but we will just restate a few main points here. The main result we use is the following, applied of course to modular forms.

Lemma VII.2. ([67], Lemma 1.1) Let $A$ be a finite set of indices. For each $\alpha \in A$, let $X_\alpha$ be a set, $U_\alpha$ a finite-dimensional vector space over a field $K$, and $S_\alpha$ a finite-dimensional vector space over $K$ consisting of some $U_\alpha$ vector-valued functions on $X_\alpha$. Put $X = \prod_{\alpha \in A} X_\alpha$ and $U = \bigotimes_{\alpha \in A} U_\alpha$. Further, for each $\beta \in A$, put $Y_\beta = \prod_{\beta \neq \alpha} X_\alpha$ and $V_\beta = \bigotimes_{\beta \neq \alpha} U_\alpha$. Let $f$ be a $U$-valued function on $X$ with the property that, for every $\beta \in A$ and every $y \in Y_\beta$, $f(x, y)$ as a function of $x \in X_\beta$ is a finite sum

$$\sum_i u_i \otimes g_i(x)$$

with $u_i \in V_\beta$ and $g_i \in S_\beta$. Then $f((x_\alpha)_{\alpha \in A})$ is a finite sum of functions of the form $\bigotimes_{\alpha \in A} h_\alpha(x_\alpha)$ with $h_\alpha \in S_\alpha$. Moreover, $(h_\alpha)_{\alpha \in A} \mapsto \bigotimes_{\alpha \in A} h_\alpha(x_\alpha)$ gives an isomorphism of $\bigotimes_{\alpha \in A} S_\alpha$ onto the space of all such functions $f$.

It is also important to note that if $G = g_1 \otimes \cdots \otimes g_t$ and $H = h_1 \otimes \cdots \otimes h_t$, then the inner product is defined so that

$$\langle G, H \rangle = \langle g_1, h_1 \rangle \cdots \langle g_t, h_t \rangle$$
(see ([67], Page 258)). Similarly, the Hecke operators are defined as tensor products of the Hecke operators on \( \Gamma_0^4(N) \), so they act on each term of the tensor products individually ([67], Page 268).

Before we use this lemma we replace \( \mathcal{E}(Z,W) \) with a form of level \( M \). The reason for this will be clear shortly as we will need to apply a Hecke operator that is of level \( M \). We do this by taking the trace. Set

\[
\mathcal{E}_M(Z,W) = \sum_{\gamma \times \delta \in \Gamma_0^4(M)/\Gamma_0^4(N)} \mathcal{E}(Z,W)(\gamma \times \delta).
\]

It is clear that \( \mathcal{E}_M(Z,W) \) is now a Siegel modular form on \( \Gamma_0^4(M) \times \Gamma_0^4(M) \). The Fourier coefficients are seen to still be in \( \mathbb{Z}_p[\chi] \) by applying the \( q \)-expansion principle for Siegel modular forms ([10], Proposition 1.5). Of course if \( M = N \) then there is no change.

Let \( F_0 = F_f, F_1, \ldots, F_r \) be a basis of eigenforms for the Hecke operators \( T(\ell) \) \((\ell \nmid pM)\) of \( \mathcal{M}_k(\Gamma_0^4(M)) \) so that the \( F_i \) is orthogonal to \( F_f \) for \( 1 \leq i \leq r \). We enlarge \( \mathcal{O} \) here if necessary so that

1. \( \mathcal{O} \) contains the values of \( \chi \)
2. the eigenforms \( F_i \) are all defined over \( \mathcal{O} \)
3. the newforms \( f_i \) defined in Conjecture VI.1 are defined over \( \mathcal{O} \).

Following Shimura, we write

\[
(7.2) \quad \mathcal{E}_M(Z,W) = \sum_{i,j} c_{i,j}(F_i(Z) \otimes F_j(W))
\]

with \( c_{i,j} \in \mathbb{C} \) ([72], Equation 7.7).

**Lemma VII.3.** Equation 7.2 can be written in the form

\[
\mathcal{E}_M(Z,W) = c_{0,0}(F_f(Z) \otimes F_f(W)) + \sum_{0 \leq i \leq r} \sum_{0 < j \leq r} c_{i,j}(F_i(Z) \otimes F_j(W)).
\]
Proof. Recall Shimura’s inner product formula as given in Equation 4.10:

\[ \langle \mathcal{E}(Z, W), F_f(W) \rangle_{\Gamma_0(N)} = \pi^{-3} A_{k, N} L_N(5 - k, \lambda_{F_f}, \chi) F_f(Z) \]

and observe that

\[ \langle \mathcal{E}(Z, W), F_f(W) \rangle_{\Gamma_0(N)} = \langle \mathcal{E}_M(Z, W), F_f(W) \rangle_{\Gamma_0(M)} \]

by the way we defined the inner product. Note that we insert the “\(\Gamma_0(N)\)” and “\(\Gamma_0(M)\)” here merely to make explicit which group the inner product is defined on.

On the other hand, if we take the inner product of the right hand side of Equation 7.2 with \(F_f(W)\) we get

\[ \langle \mathcal{E}_M(Z, W), F_f(W) \rangle = \sum_{0 \leq i \leq r} c_{i,0} \langle F_f, F_f \rangle F_i(Z). \]

Equating the two we get

\[ \pi^{-3} A_{k, N} L_N(5 - k, \lambda_{F_f}, \chi) F_f(Z) = \sum_{0 \leq i \leq r} c_{i,0} \langle F_f, F_f \rangle F_i(Z). \]

Since the \(F_i\) form a basis, it must be the case that \(c_{i,0} = 0\) unless \(i = 0\), which gives the result. \(\square\)

Using Lemma VII.3 and Equation 4.10 we write

\[ c_{0,0} \langle F_f, F_f \rangle F_f(Z) = \pi^{-3} A_{k, N} L_N(5 - k, \lambda_{F_f}, \chi) F_f(Z). \]

Equating the coefficient of \(F_f(Z)\) on each side and solving for \(c_{0,0}\) gives us

\[ c_{0,0} = \frac{A_{k, N} L_N(5 - k, \lambda_{F_f}, \chi)}{\pi^3 \langle F_f, F_f \rangle}. \]

We are now in a position to apply the results of Chapter V to study \(c_{0,0}\). Our goal is to show that we can write \(c_{0,0}\) as a product of a unit in \(\mathcal{O}\) and \(\frac{1}{\omega^m}\) for some \(m \geq 1\). Once we have shown we can do this, it will be straightforward to move from this to the congruence we desire.

For convenience we restate the two main theorems of Chapter V here:
Theorem VII.4. (Chapter V, Theorem V.6) Let $M = p_1^{m_1} \ldots p_n^{m_n}$, $f \in S_{2k-2}^{new}(\Gamma_0(M))$ a newform, and $F_f \in S_k^{new}(\Gamma_0(M))$ the Siegel eigenform associated to $f$ via the Saito-Kurokawa correspondence. Let $D$ be a discriminant with $(-1)^{k-1}D > 0$, $\gcd(M,D) = 1$, and $c_{g_f}(|D|) \neq 0$. Then one has

\[ \langle F_f, F_f \rangle = \mathcal{B}_{k, M} \frac{|c_{g_f}(|D|)|^2 L(k, f)}{\pi |D|^{k-3/2} L(k-1, f, \chi_D)} \langle f, f \rangle \]

with

\[ \mathcal{B}_{k, M} = \frac{M^k (k-1) \prod_{i=1}^n (p_i^{2m_i - 2}(p_i^4 - 1))}{2^{v(M)+3} h(p_1, \ldots, p_n)[Sp_4(\mathbb{Z}) : \Gamma_0^4(M)][\Gamma_0(M) : \Gamma_0(4M)]} \]

where $h(p_1, \ldots, p_n)$ is defined as Equation 5.3.

Theorem VII.5. (Chapter V, Theorem V.7) Let $f \in S_{2k}^{new}(\Gamma_0(M))$ be a newform and $F_f$ the Saito-Kurokawa lift of $f$. Denote the eigenvalues of $F_f$ by $\lambda_{F_f}$. Then for any $N$ such that $M \mid N$ we have

\[ L_N(2s, \lambda_{F_f}, \chi) = L_N(2s - 2, \chi) L_N(2s + 3, f, \chi) L_N(2s + k - 4, f, \chi). \]

Combining these two theorems with Equation 7.2 we have

\[ c_{0,0} = \mathcal{C}_{k, M, N} \frac{|D|^{k-3/2} L(k-1, f, \chi_D) L_N(3-k, \chi) L_N(1, f, \chi) L_N(2, f, \chi)}{\pi^2 |c_{g_f}(|D|)|^2 L(k, f) \langle f, f \rangle} \]

with

\[ \mathcal{C}_{k, M, N} = \left( \frac{A_{k, N}}{\mathcal{B}_{k, M}} \right) = \frac{(-1)^{k-2k+\nu(M)} v_1 h(p_1, \ldots, p_n)[Sp_4(\mathbb{Z}) : \Gamma_0^4(M)][\Gamma_0(M) : \Gamma_0(4M)]}{M^k (k-1) \prod_{i=1}^n (p_i^{2m_i - 2}(p_i^4 - 1))[Sp_4(\mathbb{Z}) : \Gamma_0^4(N)]}. \]

The main obstacle at this point to studying the $\pi$-valuation of $c_{0,0}$ is the occurrence of transcendental factors $\pi^2$ and $\langle f, f \rangle$ in the denominator. Fortunately, we can apply the results of Chapter VI to remove these factors. We will do this by applying a Hecke operator $t_S$ to Equation 7.2.
Assume that Conjecture VI.1 is satisfied. Recall that we showed this was the case if \( M \geq 4 \) or if \( f \) was ordinary at \( p \). We have a Hecke operator \( t \in T_O \) that acts on \( f \) via the eigenvalue \( u \frac{\langle f, f \rangle}{\Omega_f^+ \Omega_f^-} \) for \( u \) a unit in \( O \) and kills \( f_i \) for all other \( f_i \) in a basis of newforms for \( S^{\text{new}}_{2k-2}(\Gamma_0(M), O) \). Using that the Saito-Kurokawa correspondence is Hecke-equivariant (III.8), we have associated to \( t \) a Hecke operator \( t_S \in T_{S, O} \), so that

\[
(7.8) \quad t_S \cdot F_{f_i} = \begin{cases} 
\frac{u \langle f, f \rangle}{\Omega_f^+ \Omega_f^-} F_f & \text{for } f_i = f \\
0 & \text{for } f_i \neq f
\end{cases}
\]

Applying \( 1 \otimes t_S \) to Equation 7.2 we obtain

\[
(7.9) \quad (1 \otimes t_S) \mathcal{E}_M(Z, W) = c'_{0,0}(F_f(Z) \otimes F_f(W)) + \sum_{0 \leq i \leq r} \sum_{0 < j \leq r} c_{i,j}(F_i(Z) \otimes t_s F_j(W))
\]

and

\[
(7.10) \quad c'_{0,0} = u \frac{\langle f, f \rangle}{\Omega_f^+ \Omega_f^-} c_{0,0} = C'_{k,M,N} \frac{|D|^{k-3/2} L(k-1, f, \chi_D) L_N(3-k, \chi) L_N(1, f, \chi) L_N(2, f, \chi)}{\pi^2 |c_g([|D|])|^2 L(k, f) \Omega_f^+ \Omega_f^-}
\]

where

\[
(7.11) \quad C'_{k,M,N} = \frac{(-1)^k 2^{k+\nu(M)} u \psi h(p_1, \ldots, p_n) [\text{Sp}_4(Z) : \Gamma_0(M)] [\Gamma_0(M) : \Gamma_0(4M)]}{M^k (k-1) \prod_{i=1}^n ((p_i^{2m_i}-2)(p_i^4-1)) [\text{Sp}_4(Z) : \Gamma_0^4(N)]}
\]

Note that we have killed any \( F_j \) that is a Saito-Kurokawa lift other then \( F_f \). Also note that from Lemma II.10 that the Hecke operator preserves the \( p \)-integrality of the Fourier coefficients of \( \mathcal{E}_M \).

Now we need to normalize the \( L \)-values in Equation 7.10 so that we get the algebraic values. Theorem VI.2 of Chapter VI told us that if we divide \( L(m, f, \chi) \) by \( \tau(\chi)(2\pi i)^m \Omega_f^\pm \) we get a value in \( O \) where we choose \( \Omega_f^+ \) if \( \chi(-1) = (-1)^m \) and choose \( \Omega_f^- \) if \( \chi(-1) = (-1)^{m-1} \). It is easy to see that if \( \Omega_f^+ \) is associated to \( L(1, f, \chi) \), then \( \Omega_f^- \) is associated to \( L(2, f, \chi) \) and vice versa. Therefore we have

\[
\frac{L(1, f, \chi)L(2, f, \chi)}{\Omega_f^+ \Omega_f^-} = \tau(\chi)^2 (2\pi i)^3 L_{\text{alg}}(1, f, \chi) L_{\text{alg}}(2, f, \chi).
\]
In particular, we have

\[
\frac{L_N(1, f, \chi)L_N(2, f, \chi)}{\Omega_f^+ \Omega_f^-} = \prod_{\ell \mid N} \left( \frac{L(\ell)(1, f, \chi)L(\ell)(2, f, \chi)}{L(1, f, \chi)L(2, f, \chi)} \right)^{-1} \frac{L(1, f, \chi)L(2, f, \chi)}{\Omega_f^+ \Omega_f^-} = \prod_{\ell \mid N} \frac{\tau(\chi)^2(2\pi i)^3 L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi)}{L(\ell)(1, f, \chi)L(\ell)(2, f, \chi)}.
\]

Next we turn our attention to the ratio \( \frac{L(k - 1, f, \chi_D)}{L(k, f)} \). Since \( L(k, f) \) has no character, it is easy to see that if \( k \) is even then we associate \( \Omega_f^+ \) to \( L(k, f) \) and \( \Omega_f^- \) if \( k \) is odd. We need to associate the same period to \( L(k - 1, f, \chi_D) \). The way to accomplish this is to choose \( D \) so that \( \chi_D(-1) = -1 \). Therefore we have

\[
\frac{L(k - 1, f, \chi_D)}{L(k, f)} = \frac{\tau(\chi_D)L_{\text{alg}}(k - 1, f, \chi_D)}{(2\pi i)L_{\text{alg}}(k, f)}.
\]

Also recall that in Chapter IV we showed that \( L_N(3-k, \chi) \in \mathbb{Z}_p[\chi] \) for \( \gcd(p, N) = 1 \).

Gathering these results together we have:

\[
(7.12) \quad \frac{L(k - 1, f, \chi_D)L_N(3-k, \chi)L_N(1, f, \chi)L_N(2, f, \chi)}{L(k, f)} = \pi^2 D_{k,N,\chi} \mathcal{L}(k, f, D, \chi)
\]

where

\[
\mathcal{L}(k, f, D, \chi) = \frac{L_N(3-k, \chi)L_{\text{alg}}(k - 1, f, \chi_D)L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi)}{L_{\text{alg}}(k, f)}
\]

and

\[
D_{k,N,\chi} = -\frac{2^2 |D|^k \tau(\chi_D) \tau(\chi)^2}{|D|^{3/2} |c_{g_f}(|D|)|^2 \prod_{\ell \mid N} (L(\ell)(1, f, \chi)L(\ell)(2, f, \chi))}.
\]

We substitute this into Equation 7.10:

\[
\zeta_{0,0}' = \zeta_{k,M,N}' D_{k,N,\chi} \mathcal{L}(k, f, D, \chi).
\]

Everything in this equation is now algebraic, so it comes down to studying the \( \varpi \)-divisibility of each of the terms. We would like to show that \( \varpi^m \) divides the denominator for some \( m \geq 1 \) but not the numerator. Note that as long as everything
in the denominator is a \( \star \)-integer, we do not have to worry about anything written in the denominator contributing a "\( \star \)" to the numerator.

We first deal with \( C'_{k,M,N} \). We know from Conjecture VI.1 that \( u \) is a unit of \( \mathcal{O} \) so long as \( \rho_m \) is irreducible. Under this assumption we need not worry about \( u \).

Therefore, it is clear that the numerator will be relatively prime to \( \star \) so long as \( \star \nmid h(p_1, \ldots, p_n) \) where we recall that \( M = p_1^{m_1} \ldots p_n^{m_n} \). If \( M = p_1 \ldots p_n \) is square-free, then this condition is equivalent to \( \star \nmid p_i^2 - 1 \) for \( i = 1, \ldots, n \). If \( M = 1 \) this condition is trivially satisfied.

Next we focus our attention on \( D_{k,N,\chi} \) and show this is relatively prime to \( \star \).

Choosing \( p \) relatively prime to \( D \) takes care of the \( D \)'s that appear. We also can see that \( \star \nmid \tau(\chi) \) and \( \star \nmid \tau(\chi_D) \). For instance, suppose \( \star \mid \tau(\chi) \). Then this would imply that \( \star \mid (\tau(\chi)\overline{\tau(\chi)})^2 = (\sqrt{N})^2 = N \), a contradiction. Similarly for \( \tau(\chi_D) \). Now we need to deal with the \[ \frac{1}{\prod_{\ell \mid N} (L^{(\ell)}(1, f, \chi) L^{(\ell)}(2, f, \chi))} \]

Observe that we can write

\[ \frac{1}{L^{(\ell)}(1, f, \chi)} = \frac{1}{(1 - \lambda_f(\ell) \ell^{-1} + \ell^{2k-5})} = \frac{\ell}{(\ell - \lambda_f(\ell) + \ell^{2k-4})}. \]

Since \( \ell \mid N \) and \( \gcd(p, N) = 1 \), we have that \( p \nmid \ell \). It is also clear now that the denominator is in \( \mathcal{O} \). Since we can do this for each \( \ell \mid N \) and the same argument follows for \( L^{(\ell)}(2, f, \chi) \), we see that \[ \frac{1}{\prod_{\ell \mid N} (L^{(\ell)}(1, f, \chi) L^{(\ell)}(2, f, \chi))} \]

cannot possibly contribute any \( \star \)'s to the numerator. Corollary III.2 gives us that \( |c_g(\|D\|)|^2 \in \mathcal{O} \). Therefore \( D_{k,N,\chi} \) cannot contribute any \( \star \)'s to the numerator.

The term \( \mathcal{L}(k, f, D, \chi) \) is where the divisibility assumption enters into our calculations. We assume here that for some integer \( m \geq 1 \) we have \( \star^m \mid L_{\text{alg}}(k, f) \) and that if \( \star^n \| L_N(3-k, \chi)L_{\text{alg}}(k-1, f, \chi_D)L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi) \) then \( n < m \) so that we end up with a \( \star \) in the denominator of \( c'_{0,0} \).
Under these assumptions we can write

\[(7.13) \quad (1 \otimes t_S)\mathcal{E}_M(Z, W) = \frac{A}{\varpi^{m-n}}(F_f(Z) \otimes F_f(W)) + \sum_{0 \leq i \leq r \atop 0 < j \leq r} c_{i,j}(F_i(Z) \otimes t_S F_j(W)) \]

for some \( \varpi \)-unit \( A \). Recall that Corollary III.9 gave that \( F_f \) has Fourier coefficients in \( \mathcal{O} \) and that we can find a \( T_0 \) so that \( \varpi \nmid A F_f(T_0) \). This allows us to immediately conclude that we must have some \( c_{i,j} \neq 0 \) for at least one of \( i, j \neq 0 \). Otherwise we would have \( (1 \otimes t_S)\mathcal{E}_M(Z, W) = \frac{A}{\varpi^{m-n}}(F_f(Z) \otimes F_f(W)) \) and using the integrality of the Fourier coefficients of \( (1 \otimes t_S)\mathcal{E}_M(Z, W) \) we would get \( F_f(Z) \otimes F_f(W) \equiv 0 \pmod{\varpi^{m-n}} \), a contradiction.

The \( \mathcal{O} \)-integrality of the Fourier coefficients of \( (1 \otimes t_S)\mathcal{E}_M(Z, W) \) allows us to write

\[ F_f(Z) \otimes F_f(W) \equiv -\frac{\varpi^{m-n}}{A} \sum_{0 \leq i \leq r \atop 0 < j \leq r} c_{i,j}(F_i(Z) \otimes \tilde{F}_j(W)) \pmod{\varpi^{m-n}}, \]

where this is a congruence of Fourier coefficients and we write \( \tilde{F}_j = t_S F_j \). Recalling that each side is a modular form in \( Z \) and \( W \) independently, we can expand each side in its Fourier expansion in terms of \( Z \). This gives us

\[ \sum_T F_f(W)A F_f(T)e^{2\pi i \text{Tr}(TZ)} \equiv -\frac{\varpi^{m-n}}{A} \sum_T \sum_{0 \leq i \leq r \atop 0 < j \leq r} c_{i,j} \tilde{F}_j(W)A F_i(T)e^{2\pi i \text{Tr}(TZ)} \pmod{\varpi^{m-n}}. \]

Equating the \( T_0^{th} \) Fourier coefficients we get

\[ A F_f(T_0)F_f(W) \equiv -\frac{\varpi^{m-n}}{A} \sum_{0 \leq i \leq r \atop 0 < j \leq r} c_{i,j}A F_i(T_0)\tilde{F}_j(W) \pmod{\varpi^{m-n}}, \]

i.e., we have a congruence \( F_f \equiv G \pmod{\varpi^{m-n}} \) for \( G \in \mathcal{M}_k(\Gamma_0(M)) \) where

\[(7.14) \quad G(W) = -\frac{\varpi^{m-n}}{A \cdot A F_f(T_0)} \sum_{0 \leq i \leq r \atop 0 < j \leq r} c_{i,j}A F_i(T_0)\tilde{F}_j(W). \]
It is clear from Equation 7.8 that $G$ is a sum of forms not in the Maass space. Since $\mathcal{M}_k^*(\Gamma_0(M))$ is a vector subspace of $\mathcal{M}_k(\Gamma_0(M))$, it must be that $G$ is not in the Maass space, i.e., $G$ is a not a Saito-Kurokawa lift.

In the next section we will show how $G$ can be used to produce a cusp form with eigenvalues that are congruent to the eigenvalues of $F_f$ away from the level, but before we do we gather our results into the following theorem.

**Theorem VII.6.** Let $M = p_1 \ldots p_n$ and suppose $p \nmid \prod_{i=1}^{n}(p_i^2 - 1)$. (Note that this condition is the condition that $p \nmid h(p_1, \ldots, p_n)$ since $M$ is square-free.) Let $f \in S_{2k-2}^{\text{new}}(\Gamma_0(M), \mathcal{O})$ be a newform with real Fourier coefficients and $F_f$ the Saito-Kurokawa lift of $f$. Suppose that $\rho_m$ is irreducible and that Conjecture VI.1 is satisfied. Let $D$ be a fundamental discriminant so that $\gcd(M, D) = 1$, $(-1)^{k-1}D > 0$, $\chi_D(-1) = -1$, and $c_{g_f}(|D|) \neq 0$. If

$$\varpi^m \mid L_{\text{alg}}(k, f)$$

with $m \geq 1$ and

$$\varpi^n \mid L_N(3 - k, \chi) L_{\text{alg}}(k - 1, f, \chi_D) L_{\text{alg}}(1, f, \chi) L_{\text{alg}}(2, f, \chi)$$

with $n < m$, then there exists $G \in \mathcal{M}_k(\Gamma_0(M))$ that is not a Saito-Kurokawa lift so that

$$F_f \equiv G(\text{mod } \varpi^{m-n}).$$

### 7.2 Congruent to a Cuspidal Eigenform

In this section we will show how given a congruence

$$(7.15) \quad F_f \equiv G(\text{mod } \varpi^m)$$

for $m \geq 1$ as in Theorem VII.6, we can find a cuspidal eigenform that has the same eigenvalues as $F_f$ modulo $\varpi$ away from the level.
**Notation VII.7.** If $F_1$ and $F_2$ have eigenvalues that are congruent modulo $\varpi$ away from the level, we will write

$$F_1 \equiv_{ev} F_2 \pmod{\varpi}$$

where the $ev$ stands for the congruence being a congruence of eigenvalues away from the level.

We begin by showing that given a congruence as in Theorem VII.6, there must be an eigenform $F$ so that $F \equiv_{ev} F_f \pmod{\varpi}$. Once we have shown this, we will show that we can actually obtain an eigenvalue congruence to a cusp form. Applying the first result again we obtain our final goal of an eigenvalue congruence between $F_f$ and a cuspidal eigenform that is not a Saito-Kurokawa lift.

**Lemma VII.8.** Let $G \in \mathcal{M}_k(\Gamma_0^4(M))$ be as in Equation 7.14 so that we have the congruence $G \equiv F_f \pmod{\varpi}$. Then there exists an eigenform $F$ so that $F$ is not a Saito-Kurokawa lift and $F_f \equiv_{ev} F \pmod{\varpi}$.

**Proof.** Write $G = \sum c_i F_i$ with each $F_i$ an eigenform away from the level and $c_i \in \mathcal{O}$. It is clear from the construction of $G$ that $F_i \neq F_f$ and $F_i$ is not a Saito-Kurokawa lift for all $i$. Recall that we have the decomposition

$$\mathbb{T}_{S,\mathcal{O}} \cong \prod \mathbb{T}_{S,\mathcal{O},\mathfrak{m}}$$

where the $\mathfrak{m}$ are maximal ideals of $\mathbb{T}_{S,\mathcal{O}}$ containing $\varpi$. Let $\mathfrak{m}_{F_f}$ be the maximal ideal corresponding to $F_f$. There is a Hecke operator $t \in \mathbb{T}_{S,\mathcal{O}}$ so that $tF_f = F_f$ and $tF = 0$ for any eigenform $F$ that does not correspond to $\mathfrak{m}_{F_f}$, i.e., if $F \not\equiv_{ev} F_f \pmod{\varpi}$. If $F_i \not\equiv_{ev} F_f \pmod{\varpi}$ for every $i$ then applying $t$ to the congruence $G \equiv F_f \pmod{\varpi}$ would then yield $F_f \equiv 0 \pmod{\varpi}$, clearly a contradiction. Thus there must be an $i$ so that $F_f \equiv_{ev} F_i \pmod{\varpi}$. \qed
We now show that we actually have an eigenvalue congruence to a cusp form. Let $F_f \equiv_{ev} F(\mod \varpi)$ with $F$ the non Saito-Kurokawa eigenform constructed in Lemma VII.8. Suppose that $F$ is not a cusp form so that $\Phi(F|\gamma) \neq 0$ for some $\gamma \in \text{Sp}_4(\mathbb{Z})$. We will assume that $\Phi(F) \neq 0$ as the exact same argument will work for any $\gamma$. Let $g = \Phi(F)$. Since $F$ is an eigenform, so is $g$ (Theorem II.13). We denote the $n^{th}$ eigenvalue of $g$ as $\lambda_g(n)$. Let $\ell$ be a prime so that $\ell \nmid Mp$. Note that since $F$ has eigenvalues in $\mathcal{O}$ and Theorem ?? gives that $\lambda_F(\ell) = (1 - \ell^{2-k})\lambda_g(\ell)$, we must have $(1 - \ell^{2-k})\lambda_g(\ell) \in \mathcal{O}$. Theorem II.12 gives

\[
\Phi(T_S(\ell)F) = (1 - \ell^{2-k})T(\ell)\Phi(F) = (1 - \ell^{2-k})T(\ell)g = (1 - \ell^{2-k})\lambda_g(\ell)g.
\]

On the other hand, the congruence gives us that

\[
\Phi(T_S(\ell)F) \equiv_{ev} \Phi(T_S(\ell)F_f)(\mod \varpi)
\]

\[
= \Phi(\lambda_{F_f}(\ell)F_f)
\]

\[
= \Phi((\ell^{k-1} + \ell^{k-2} + \lambda_f(\ell))F_f)
\]

\[
= (\ell^{k-1} + \ell^{k-2} + \lambda_f(\ell))\Phi(F_f)
\]

\[
\equiv (\ell^{k-1} + \ell^{k-2} + \lambda_f(\ell))g(\mod \varpi).
\]

Thus we have that

\[(7.16) \quad (\ell^{k-1} + \ell^{k-2} + \lambda_f(\ell)) \equiv (1 - \ell^{2-k})\lambda_g(\ell)(\mod \varpi).
\]

Now we investigate what this says in terms of Galois representations. For the reader unfamiliar with Galois representations, Section 8.1 of Chapter VIII gives an introduction to Galois representations.
Denote the Galois representation associated to \( f \) by \( \rho_f \) and similarly for \( g \). Denote the residual representations after reducing modulo \( \varpi \) by \( \overline{\rho}_f \) and \( \overline{\rho}_g \). Equation 7.16 and the Tchebotarov Density Theorem show that we have the following equivalence of 4-dimensional Galois representations

\[
\begin{pmatrix}
\omega^{k-1} \\
\omega^{k-2} \\
\overline{\rho}_f
\end{pmatrix}
= \begin{pmatrix}
\overline{\rho}_g \\
\omega^{2-k}\overline{\rho}_g
\end{pmatrix},
\]

where \( \omega \) is the \( p \)th cyclotomic character reduced modulo \( \varpi \). It is clear from this that \( \overline{\rho}_f \) must be reducible. However, we assumed before that this was not the case. This contradiction shows that \( \Phi(F) = 0 \). We have proved the following theorem.

**Theorem VII.9.** Let \( M = p_1 \ldots p_n \) and suppose \( p \nmid \prod_{i=1}^n (p_i^2 - 1) \). Let \( f \in S_{2k-2}^\text{new}(\Gamma_0(M), \mathcal{O}) \) be a newform with real Fourier coefficients and \( F_f \) the Saito-Kurokawa lift of \( f \). Suppose that \( \rho_m \) is irreducible and that Conjecture VI.1 is satisfied. Let \( D \) be a fundamental discriminant so that \( \gcd(M, D) = 1 \), \(( -1 )^{k-1}D > 0\), \( \chi_D(-1) = -1 \), and \( c_{g_f}(|D|) \neq 0 \). If

\[ \varpi^m \mid L_{\text{alg}}(k, f) \]

with \( m \geq 1 \) and

\[ \varpi^n \parallel L_N(3 - k; \chi)L_{\text{alg}}(k - 1, f; \chi_D)L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi) \]

with \( n < m \), then there exists \( G \in S_k(\Gamma_0(M)) \) that is an eigenform and is not a Saito-Kurokawa lift so that

\[ F_f \equiv_{\text{ev}} G(\text{mod } \varpi). \]
CHAPTER VIII

Selmer Groups and Galois Representations

We begin this chapter with some basic facts and definitions about Galois representations. It is not easy to find all of these definitions and theorems in one place, so it is my hope that this collection will be helpful to the reader who is not an expert in this field.

In the second section we define the Selmer group we wish to study as well as state the Bloch-Kato conjecture for modular forms in the form for which we will provide evidence.

The final section consists of Galois representation arguments relying on the congruence established in Theorem VII.9. Here we state the main theorem of the thesis giving the implication $\varpi | L_{alg}(k, f)$ implies $p$ divides the order of the Selmer group.

8.1 Galois Representations: Definitions and Basic Facts

In this section we state the relevant theorems giving us the existence of the Galois representations we will be interested in. We will also take some space to state some of the relevant theorems from linear algebra and representation theory that allow us to work with these representations.

Definition VIII.1. Let $d$ be a positive integer. A $d$-dimensional $p$-adic Galois
representation is a continuous homomorphism
\[ \rho : \text{Gal}(\overline{Q}/Q) \to \text{GL}_d(K) \]
where \( K/Q_p \) is a finite extension. If \( \rho' : \text{Gal}(\overline{Q}/Q) \to \text{GL}_d(K) \) is another such representation and there exists a matrix \( M \in \text{GL}_d(K) \) such that \( \rho'(\sigma) = M^{-1}\rho(\sigma)M \) for all \( \sigma \in \text{Gal}(\overline{Q}/Q) \), then \( \rho \) and \( \rho' \) are equivalent.

It will sometimes be convenient to write our Galois representations
\[ \rho : \text{Gal}(\overline{Q}/Q) \to \text{GL}(V) \]
for a \( d \)-dimensional \( K \)-vector space \( V \). We will pass between the two notations depending on which is most convenient for the circumstance. Also note that to give an \( p \)-adic Galois representation is equivalent to giving a finite-dimensional \( \mathbb{Q}_p \)-vector space \( V \) with a continuous action of \( \text{Gal}(\overline{Q}/Q) \).

**Proposition VIII.2.** ([17], Proposition 9.3.5) Let \( \rho : \text{Gal}(\overline{Q}/Q) \to \text{GL}_d(K) \) be an \( p \)-adic Galois representation. Then \( \rho \) is equivalent to a Galois representation \( \rho' : \text{Gal}(\overline{Q}/Q) \to \text{GL}_d(\mathcal{O}_K) \).

In general we will just write \( \rho = \rho' \) in this situation when no confusion will arise.

**Definition VIII.3.** Let \( \rho : \text{Gal}(\overline{Q}/Q) \to \text{GL}_d(\mathcal{O}_K) \) be a \( p \)-adic Galois representation and \( \varpi \) a uniformizer of \( \mathcal{O}_K \). The representation \( \overline{\rho} : \text{Gal}(\overline{Q}/Q) \to \text{GL}_d(\mathcal{O}_K/\varpi\mathcal{O}_K) \) is called the residual representation of \( \rho \).

**Definition VIII.4.** Let \( \rho \) be a \( p \)-adic Galois representation and let \( \ell \) be a prime. We say \( \rho \) is unramified at \( \ell \) if \( I_\lambda \subset \ker \rho \) for any maximal ideal \( \lambda \subset \mathbb{Z} \) lying over \( \ell \), where \( I_\lambda \) is the absolute inertia group of \( \lambda \).

**Example VIII.5.** Let \( \varepsilon_p \) be the \( p \)-th cyclotomic character. This is an odd 1-dimensional Galois representation \( \varepsilon_p : \text{Gal}(\overline{Q}/Q) \to \text{GL}_1(\mathbb{Z}_p) \) that is unramified at \( \ell \neq p \) and such
that for \( \ell \neq p \) one has \( \varepsilon_p(\text{Frob}_\ell) = \ell \). Write \( \mathbb{Q}_p(n) \) for the 1-dimensional space over \( \mathbb{Q}_p \) on which \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts by \( \varepsilon^n_p \) and similarly for \( \mathbb{Z}_p(n) \). We denote the residual representation of \( \varepsilon_p \) by \( \omega_p \). We will drop the \( p \) when it is clear from the context.

Before stating the existence of the particular Galois representations we are interested in, we state a couple of theorems that will be helpful in our study of Galois representations.

**Definition VIII.6.** A representation \( \rho : G \to \text{GL}(M) \) is said to be **semi-simple** if \( M \) can be written as a direct sum of \( G \)-modules \( M_i \)

\[
M = M_1 \oplus \cdots \oplus M_n
\]

such that each \( M_i \) is simple.

Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(V) \) be a Galois representation. Then \( V \) has a composition series given by

\[
V = V_0 \supset V_1 \supset \cdots \supset V_{d-1} = 0
\]

of \( \rho \)-invariant subspaces with \( V_i/V_{i+1} \) irreducible. The Galois representation \( \rho^{ss} \) defined by \( V' = \sum_{i=0}^{d-1} V_i/V_{i+1} \) is semi-simple: \( \rho^{ss} \) is the semi-simplification of \( \rho \). Note that the semi-simplification of a representation is unique up to isomorphism (\([60], \text{Page I-10}\)).

The semi-simplification of a representation will turn out to be very important for us. For example, we have the following theorems.

**Theorem VIII.7.** (\([60], \text{Lemma Page I-11}\)) Let \( K \) be a field of characteristic 0 and let \( \rho : \Gamma \to \text{GL}(V) \) and \( \rho' : \Gamma \to \text{GL}(W) \) be two finite dimensional \( K \)-linear representations of a group \( \Gamma \). If \( \rho \) and \( \rho' \) are semi-simple and have the same trace, then they are isomorphic.
For fields of characteristic larger then 0, we have the Brauer-Nesbitt theorem, stated in the form we will need it.

**Theorem VIII.8.** *(Brauer-Nesbitt Theorem, [11] Theorem 30.16)* Let \( \rho : \Gamma \to \text{GL}_d(\mathbb{F}) \) and \( \rho' : \Gamma \to \text{GL}_d(\mathbb{F}) \) be two \( \mathbb{F} \)-linear \( d \)-dimensional representations of a group \( \Gamma \) with \( \mathbb{F} \) a field. Then \( \rho \) and \( \rho' \) have the same semi-simplification if and only if \( \det(xI - \rho(\sigma)) \) and \( \det(xI - \rho'(\sigma)) \) have the same roots counted with multiplicity.

We will primarily be interested in 2-dimensional and 4-dimensional Galois representations. We have the following theorem of Deligne providing the existence of 2-dimensional Galois representations attached to elliptic modular forms.

**Theorem VIII.9.** *(cf: [17], Theorem 9.6.5)* Let \( f \in S_k(\Gamma_1(M), \psi) \) be a normalized eigenform with Fourier coefficients generating the number field \( K_f \). For each maximal ideal \( p \) of \( \mathcal{O}_{K_f} \) lying over \( p \) there is an odd irreducible 2-dimensional Galois representation \( \rho_{f,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(K_{f,p}). \)

This representation is unramified at all primes \( \ell \nmid pM \) and for such a prime \( \ell \) the characteristic equation of \( \rho_{f,p}(\text{Frob}_\ell) \) is

\[
x^2 - a_f(\ell)x + \chi(\ell)\ell^{k-1}.
\]

When we wish to refer to the representation \( \rho_{f,p} \) as a vector space with a continuous \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-action, we will write \( V_{f,p} \).

Similarly, for 4-dimensional Galois representations attached to Siegel modular forms we have the following theorem. The proof is essentially due to Weissauer ([86]) and Laumon ([44]), but one can see Skinner-Urban ([76]) for a complete statement.
Theorem VIII.10. ([76], Theorem 3.1.3) Let $F \in \mathcal{S}_k(\Gamma_0(M))$ be an eigenform, $K_F$ the number field generated by the Hecke eigenvalues of $F$, and $p$ a prime of $K_F$ over $p$. There exists a finite extension $E$ of the completion $K_{F,p}$ of $K_F$ at $p$ and a continuous semi-simple Galois representation

$$\rho_{F,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_4(E)$$

unramified at all primes $\ell \nmid pN$ and so that for all $\ell \nmid pN$, we have

$$\det(X \cdot I - \rho_{F,p}(\text{Frob}_\ell)) = I_{\text{spin}}(\ell)$$

(we are using arithmetic Frobenius here as opposed to geometric which is more prevalent in the literature.)

When we wish to refer to the representation $\rho_{F,p}$ as a vector space with a continuous $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, we will write $V_{F,p}$.

8.2 Selmer Groups

In this section we will define the relevant Selmer group following Bloch and Kato [7] and Diamond, Flach, and Guo [15]. The definition of the Selmer group in the case of elliptic curves is given in [28] and [75]. We will conclude the section by stating a version of the Bloch-Kato conjecture for modular forms.

For a field $K$ and a topological $\text{Gal}(\overline{K}/K)$-module $M$, we write $H^1(K,M)$ for $H^1_{\text{cont}}(\text{Gal}(\overline{K}/K), M)$ to ease notation, where “cont” indicates continuous cocycles. We write $D_\ell$ to denote a decomposition group at $\ell$. We identify $D_\ell$ with $\text{Gal}(\overline{\mathbb{Q}}_{\ell}/\mathbb{Q}_{\ell})$.

Let $E$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ the ring of integers of $E$, and $\varpi$ a uniformizer. Let $V$ be a $p$-adic Galois representation defined over $E$. Let $T \subseteq V$ be a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}$-lattice. Set $W = V/T$. For $n \geq 1$, put

$$W_n = W[\varpi^n] = \{ x \in W : \varpi^n x = 0 \} \cong T/\varpi^n T.$$
In the following section we will construct non-zero cohomology classes in $H^1(Q, W)$ and we would like to know that they remain non-zero when we map them under the natural map into $H^1(Q, W)$.

**Lemma VIII.11.** If $T/\varpi T$ is irreducible as an $(O/\varpi O) [\text{Gal}(\overline{Q}/Q)]$-module then $H^1(Q, W_1)$ injects into $H^1(Q, W)$.

**Proof.** Consider the exact sequence

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W \longrightarrow 0.$$ 

This short exact sequence gives rise to the long exact sequence of cohomology groups

$$0 \longrightarrow H^0(Q, W_1) \longrightarrow H^0(Q, W) \longrightarrow H^0(Q, W) \longrightarrow H^1(Q, W_1) \longrightarrow H^1(Q, W) \longrightarrow \cdots.$$ 

We show that $\psi$ is injective. Recalling that $H^0(G, M) = M^G$, it is clear that $H^0(Q, W_1) = 0$ since we have assumed that $T/\varpi T$ is irreducible. Since $W$ is torsion, $H^0(Q, W)$ is necessarily torsion as well. If $H^0(Q, W)$ contains a non-zero element, multiplying by a suitable $\varpi^m$ makes it a non-zero element in $W_1$. This would give us a non-zero element in $H^0(Q, W_1)$, a contradiction. Thus, we obtain that $H^0(Q, W) = 0$ and so $\psi$ is an injection. \hfill $\blacksquare$

We also have that $H^1(Q_\ell, W_1)$ injects into $H^1(Q_\ell, W)$ when $T/\varpi T$ is irreducible as an $(O/\varpi O) [D_\ell]$-module, by an analogous argument.

We write $\mathcal{B}_{\text{cris}}$ to denote Fontaine’s ring of $p$-adic periods ([24]). For a $p$-adic representation $V$, set

$$D = (V \otimes_{Q_p} \mathcal{B}_{\text{cris}})^{D_p}$$

and

$$\text{Cris}(V) = H^0(Q_p, V \otimes_{Q_p} \mathcal{B}_{\text{cris}}).$$
Definition VIII.12. A p-adic representation $V$ is called crystalline if

$$\dim_{\mathbb{Q}_p} V = \dim_{\mathbb{Q}_p} \text{Cris}(V).$$

Definition VIII.13. A crystalline representation $V$ is called short if the following hold

1. $\text{Fil}^0 D = D$ and $\text{Fil}^p D = 0$;
2. if $V'$ is a nonzero quotient of $V$, then $V' \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(p - 1)$ is ramified

where $\text{Fil}^i D$ is a decreasing filtration of $D$.

Theorem VIII.14. ([22], [81]) Let $F$ be as in Theorem VIII.10 with $p$ a prime not dividing the level of $F$. The restriction of $\rho_{F,p}$ to the decomposition group $D_p$ is crystalline at $p$. In addition if $p > 2k - 2$ then $\rho_{F,p}$ is short.

Following Bloch-Kato ([7], Page 353), we define spaces $H^1_f(\mathbb{Q}_\ell, V)$ by

$$H^1_f(\mathbb{Q}_\ell, V) = \begin{cases} H^1_{ur}(\mathbb{Q}_\ell, V) & \ell \neq p, \infty \\ \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\text{cris}})) & \ell = p \end{cases}$$

where

$$H^1_{ur}(\mathbb{Q}_\ell, M) = \ker(H^1(\mathbb{Q}_\ell, M) \to H^1(I_\ell, M))$$

for any $D_\ell$-module $M$. The Bloch-Kato groups $H^1_f(\mathbb{Q}_\ell, W)$ are defined by

$$H^1_f(\mathbb{Q}_\ell, W) = \text{im}(H^1_f(\mathbb{Q}_\ell, V) \to H^1(\mathbb{Q}_\ell, W)).$$

One should note here that the $f$ appearing in these definitions has nothing to do with the elliptic modular form $f$ we have been working with and is merely standard notation in the literature: (standing for “finite part”).

Lemma VIII.15. If $V$ is unramified at $\ell$, then

$$H^1_f(\mathbb{Q}_\ell, W) = H^1_{ur}(\mathbb{Q}_\ell, W).$$
Proof. Note that we have a natural inclusion of $H^1_f(Q_{\ell}, W)$ in $H^1_{ur}(Q_{\ell}, W)$. Therefore we need only show that $H^1_{ur}(Q_{\ell}, V)$ surjects onto $H^1_{ur}(Q_{\ell}, W)$. The short exact sequence

$$0 \longrightarrow T \longrightarrow V \longrightarrow W \longrightarrow 0$$

gives rise to the long exact sequence in cohomology

$$0 \longrightarrow H^0(F_{\ell}, T) \longrightarrow H^0(F_{\ell}, V) \longrightarrow H^0(F_{\ell}, W) \longrightarrow H^1(F_{\ell}, T) \longrightarrow H^1(F_{\ell}, V) \psi \longrightarrow H^1(F_{\ell}, W) \longrightarrow H^2(F_{\ell}, T) \longrightarrow \cdots$$

where we identify $\text{Gal}(F_{\ell}/F_{\ell})$ with $D_{\ell}/I_{\ell}$. Since $\text{Gal}(F_{\ell}/F_{\ell}) \cong \hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}$ has cohomological dimension 1 ([61], Chapter 5, Proposition 18), we have that $H^2(F_{\ell}, T) = 0$, i.e., $\psi$ is a surjection. Observing that for any $D_{\ell}$-module $M$ we have a natural isomorphism

$$H^1_{ur}(Q_{\ell}, M) \cong H^1(F_{\ell}, M^{I_{\ell}})$$

and using the fact that $V$ is assumed to be unramified at $\ell$ and so $T$ is unramified at $\ell$ as well, we obtain the result.

Remark VIII.16. Let $R$ be a ring and let $M$ and $N$ be $R$-modules. Recall that an $(R$-linear) extension of $M$ by $N$ is a short exact sequence of $R$-modules

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0.$$ 

There is a bijection between $\text{Ext}^1_R(M, N)$ and the set of equivalence classes of extensions of $M$ by $N$ ([20], Chapter 17, Theorem 12). Let $\alpha \in H^1(Q_{\ell}, V)$. It is well
known that \( H^1(\mathbb{Q}_\ell, V) \cong \text{Ext}^1_{E[D_\ell]}(E, V) \) ([33], Theorem 6.12). Therefore we have that \( \alpha \) corresponds to an extension \( X \) of \( E \) by \( V \):

\[
0 \longrightarrow V \longrightarrow X \longrightarrow E \longrightarrow 0.
\]

For \( \ell \neq p \), one has that \( X \) is an unramified representation if and only if \( \alpha \in H^1_{ur}(\mathbb{Q}_\ell, V) \). If \( \ell = p \), then \( X \) is a crystalline representation if and only if \( \alpha \in H^1_f(\mathbb{Q}_p, V) \).

We are now in a position to define the Selmer group of interest to us.

**Definition VIII.17.** Let \( W \) and \( H^1_f(\mathbb{Q}_\ell, W) \) be defined as above. The *Selmer group* of \( W \) is given by

\[
H^1_f(\mathbb{Q}, W) = \ker \left( H^1(\mathbb{Q}, W) \to \bigoplus_{\ell} H^1(\mathbb{Q}_\ell, W) \right),
\]

i.e., it consists of the cocycles \( c \in H^1(\mathbb{Q}, W) \) that when restricted to \( D_\ell \) lie in \( H^1_f(\mathbb{Q}_\ell, W) \) for each \( \ell \).

Lemma VIII.15 allows us to identify \( H^1_f(\mathbb{Q}_\ell, W) \) with \( H^1_{ur}(\mathbb{Q}_\ell, W) \) for \( \ell \neq p \). Define \( H^1_f(\mathbb{Q}_\ell, W_n) = H^1_{ur}(\mathbb{Q}_\ell, W_n) \) for \( \ell \neq p \). At the prime \( p \), we define \( H^1_f(\mathbb{Q}_p, W_n) \subseteq H^1(\mathbb{Q}_p, W_n) \) to be the subset of classes of extensions of \( D_p \)-modules

\[
0 \longrightarrow W_n \longrightarrow X \longrightarrow \mathcal{O}/\mathcal{O}n \longrightarrow 0
\]

so that \( X \) is in the essential image of \( V \) where \( V \) is the functor defined in Section 1.1 of [15]. We will not define the functor here; we will be content with stating the relevant properties that we will need. This essential image is stable under direct sums, subobjects, and quotients ([15], Section 2.1). This gives that \( H^1_f(\mathbb{Q}_p, W_n) \) is an \( \mathcal{O} \)-submodule of \( H^1(\mathbb{Q}_p, W_n) \). We also have that \( H^1_f(\mathbb{Q}_p, W_n) \) is the preimage of \( H^1_f(\mathbb{Q}_p, W_{n+1}) \) under the natural map \( H^1(\mathbb{Q}_p, W_n) \to H^1(\mathbb{Q}_p, W_{n+1}) \). For our purposes, it will be enough to note the following fact.
Lemma VIII.18. ([15], Page 670) If \( V \) is a short crystalline representation at \( p \), \( T \) a \( D_p \)-stable lattice, and \( X \) a subquotient of \( T/\varpi^nT \) that gives an extension of \( D_p \)-modules

\[
0 \to W_n \to X \to O/\varpi^nO \to 0,
\]

then the class of this extension is in \( H^1_f(Q_p, W_n) \).

We have a natural map \( \phi_n : H^1(Q_p, W_n) \to H^1(Q_p, W) \). On the level of extensions this map is given by pushout via the map \( \varpi^{-n}T/T \to V/T \), pullback via the map \( O \to O/\varpi^nO \), and the isomorphism \( H^1(Q_p, W) \cong \text{Ext}^1_{\mathcal{O}[D_p]}(\mathcal{O}, V/T) \). In the next chapter we will be interested in the situation where we have a non-zero cocycle \( h \in H^1(Q, W_1) \) that restricts to be in \( H^1_f(Q_\ell, W_1) \). We want to be able to conclude that this gives a non-zero cocycle in \( H^1(Q, W) \) that restricts to be in \( H^1_f(Q_\ell, W) \). We saw above that \( H^1(Q, W_1) \) injects into \( H^1(Q, W) \), so it only remains to show that the restriction is in \( H^1_f(Q_\ell, W) \). This is accomplished via the following proposition.

**Proposition VIII.19.** ([15], Proposition 2.2) The natural isomorphism

\[
\lim_{\mathbb{R}} H^1(Q_\ell, W_n) \cong H^1(Q_\ell, W)
\]

induces isomorphisms

\[
\lim_{\mathbb{R}} H^1_{\text{ur}}(Q_\ell, W_n) \cong H^1_{\text{ur}}(Q_\ell, W)
\]

and

\[
\lim_{\mathbb{R}} H^1_f(Q_p, W_n) \cong H^1_f(Q_p, W).
\]

This proposition shows that the map \( \phi_n \) gives a map from \( H^1_f(Q_p, W_n) \) to \( H^1_f(Q_p, W) \).

We summarize with the following proposition.

**Proposition VIII.20.** Let \( h \) be a non-zero cocycle in \( H^1(Q, W_1) \) and assume that \( T/\varpi T \) is irreducible. If \( h|_{D_\ell} \in H^1_f(Q_\ell, W_1) \) is non-zero, then \( h|_{D_\ell} \) gives a non-zero
\( \varpi \)-torsion element of \( H^1_f(\mathbb{Q}_\ell, W) \). If \( h|_{D_\ell} \in H^1_f(\mathbb{Q}_\ell, W_1) \) for every prime \( \ell \), then \( h \) is a non-zero \( \varpi \)-torsion element of \( H^1_f(\mathbb{Q}, W) \).

We conclude this section with a short discussion of the Bloch-Kato conjecture for modular forms. The reader interested in more details or a more general framework should consult [7] where the conjecture is referred to as the “Tamagawa number conjecture”.

For each prime \( p \) let \( V_p := V_{f,p} \) be the \( p \)-adic Galois representation arising from a newform \( f \) of weight \( 2k - 2 \), \( T_p := T_{f,p} \) a Gal(\( \mathbb{Q} / \mathbb{Q} \))-stable lattice, and \( W_p := W_{f,p} = V/T \). The \( W_p \) here should not be confused with our use of \( W_n \) earlier. Denote the \( j \)-th Tate twist of \( W_p \) by \( W_p(j) \). Let \( \pi \) be the natural map \( H^1(\mathbb{Q}, V_p(j)) \to H^1(\mathbb{Q}, W_p(j)) \) used to define the groups \( H^1_f(\mathbb{Q}_\ell, W_p(j)) \). We define the Tate-Shafarevich group to be

\[
\text{III}(j) = \bigoplus_\ell H^1_f(\mathbb{Q}, W_\ell(j))/\pi_* H^1_f(\mathbb{Q}, V_\ell(j)).
\]

Define the set of “global points” by

\[
\Gamma_\mathbb{Q}(j) = \bigoplus_\ell H^0(\mathbb{Q}, W_\ell(j)).
\]

One should think of these as the analogue of the rational torsion points on an elliptic curve.

**Conjecture VIII.21.** (Bloch-Kato) With the notation as above, one has

\[
L(k, f) = \frac{\prod_{\ell} c_{\ell}(k)) \text{vol}_\mathbb{Q}(k) \# \text{III}(1-k)}{\# \Gamma_\mathbb{Q}(k) \# \Gamma_\mathbb{Q}(k-2)}
\]

where \( c_p(j) \) are “Tamagawa factors” and \( \text{vol}_\infty(k) \) is a certain real period. See [14] for a careful treatment of \( \text{vol}_\infty(k) \).

**Remark VIII.22.** 1. It is known that away from the central critical value the Selmer group is finite ([34], Theorem 14.2). Therefore we can identify the \( \varpi \)-part of
the Selmer group with the $\varpi$-part of the Tate-Shafarevich group.

2. If $T_\ell / \varpi T_\ell$ is irreducible, then $H^0(\mathbb{Q}, W_\ell(j)) = 0$.

3. The Tamagawa factors are integers. See ([7], Section 5) for definitions and discussion.

4. The real period $\text{vol}_\infty(k)$ should be $\pi^k \Omega_f^\pm$ up to $p$-adic unit with the $\pm$ depending on the parity of $k$.

In the next section we will prove that if $\varpi \mid L_{\text{alg}}(k, f)$, then $p \mid \# H^1_f(\mathbb{Q}, W_{f,p}(1 - k))$. Using Remark VIII.22 this divisibility gives evidence for the Bloch-Kato conjecture.

8.3 Galois Arguments

In this section we will combine the results of the previous chapters to imply a divisibility result on the Selmer group $H^1_f(\mathbb{Q}, W_{f,p}(1 - k))$. We show that the congruence leads to a non-zero class in $H^1_f(\mathbb{Q}, W_{f,p}(1 - k))$, thereby giving the result that $p$ divides the order. For the remainder of this chapter we restrict ourselves to level 1. This restriction is necessary to avoid the situation in which the congruence produced in Chapter VII is to a CAP form. It may be possible to remove such a condition with a more in-depth investigation of CAP forms, but we do not attempt such an investigation here.

Note that in this section entries of matrices denoted by *’s can be anything and are assumed to be of the appropriate size. Similarly, a 1 as a matrix entry is assumed to be of the appropriate size. A blank space in a matrix is assumed to be 0.

Recall that for a Saito-Kurokawa lift one has a decomposition of the Spinor $L$-function: for $F_f$ we have

$$L_{\text{spin}}(s, F_f) = \zeta(s - k + 1)\zeta(s - k + 2)L(s, f).$$
This decomposition gives us that the Galois representation $\rho_{F_j,p}$ has a very simple form. In particular, using that $\rho_{F_j,p}$ is semi-simple and applying the Brauer-Nesbitt theorem (Theorem VIII.8) we have that

$$
\rho_{F_j,p} = \begin{pmatrix}
\varepsilon^{k-2} & \rho_{f,p} \\
0 & \varepsilon^{k-1}
\end{pmatrix}
$$

where $\varepsilon$ is the $p^{th}$ cyclotomic character.

Under the conditions of Theorem VII.9, we have a cuspidal Siegel eigenform $G$ such that $G \equiv_{ev} F_f (\mod p)$. This congruence clearly gives a congruence between the Spinor $L$-functions as well. Let $p$ be a prime of a sufficiently large finite extension $E/\mathbb{Q}_p$ so that $O_E$ contains the $O$ needed for the congruence and so that $\rho_{F_j,p}$ and $\rho_{G,p}$ are defined over $O_E$. We set $O = O_E$ and let $\varpi$ be a uniformizer of $O$. Therefore, again applying the Brauer-Nesbitt theorem we get that

$$
\overline{\rho}_{G,p}^{ss} = \overline{\rho}_{F_j,p} = \begin{pmatrix}
\omega^{k-2} & \rho_{f,p} \\
0 & \omega^{k-1}
\end{pmatrix}
$$

where we use $\omega$ to denote the reduction of the cyclotomic character $\varepsilon$ modulo $\varpi$.

The goal now is to use this information on the semi-simplification of $\overline{\rho}_{G,p}$ to deduce the form of $\overline{\rho}_{G,p}$. Our first step is to show that there is a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice $T$ so that the reduction of $\rho_{G,p}$ is of the form

$$
\overline{\rho}_{G,p} = \begin{pmatrix}
\omega^{k-2} & *_1 & *_2 \\
*_3 & \overline{\rho}_{f,p} & *_4 \\
& & \omega^{k-1}
\end{pmatrix}
$$
where either \( *_1 \) or \( *_3 \) is zero. We proceed by brute force, working our way backwards from the semi-simplification using the definition as given above.

We begin by noting some conjugation formulas that will be important. Expanding on the notation used in [57], write

\[
P_1 = \begin{pmatrix}
\varpi \\
1 \\
1 \\
1 \\
\end{pmatrix},
\]

\[
P_2 = \begin{pmatrix}
1 \\
\varpi \\
\varpi \\
1 \\
\end{pmatrix},
\]

and

\[
P_3 = \begin{pmatrix}
1 \\
1 \\
1 \\
\varpi \\
\end{pmatrix}.
\]

We have the following conjugation formulas

\[
P_1 \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\
\varpi a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\
\varpi a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\
\varpi a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\
\end{pmatrix} P_1^{-1} = \begin{pmatrix}
a_{1,1} & \varpi a_{1,2} & \varpi a_{1,3} & \varpi a_{1,4} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\
\end{pmatrix},
\]
and

(8.5) \[
\begin{pmatrix}
\omega a_{1,1} & \omega a_{1,2} & a_{1,3} & \omega a_{1,4} \\
\omega a_{2,1} & a_{2,2} & a_{2,3} & \omega a_{2,4} \\
\omega a_{3,1} & a_{3,2} & a_{3,3} & \omega a_{3,4} \\
\omega a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4}
\end{pmatrix}
\]

Recalling the notation used in Equation 8.1, in our case we have \( V := V_{G,p} = V_0 \supset V_1 \supset V_2 \supset V_3 = 0 \) with each of \( V_0/V_1, V_1/V_2, \) and \( V_2 \) irreducible components of \( \overline{p}_{G,p} \). Since we know \( \overline{p}^ss_{G,p} \) explicitly, we can say that \( V_0/V_1, V_1/V_2, \) and \( V_2 \) consist of two 1-dimensional spaces and one 2-dimensional space, corresponding to \( \omega^{k-1}, \omega^{k-2} \) and \( \overline{p}_{f,p} \). The difficulty is that we do not know which \( V_i/V_{i+1} \) corresponds to which of \( \omega^{k-1}, \omega^{k-2}, \) and \( \overline{p}_{f,p} \). Therefore we must consider all situations and see what this implies for the form of \( \overline{p}_{G,p} \). We split this into several cases.

**Case 1:** \( \dim V_2 = 1 = \dim V_0/V_1, \dim V_1/V_2 = 2 \).

This case corresponds to the situation where we have either

\[
\overline{p}_{G,p} = \begin{pmatrix}
\omega^{k-2} & * & * \\
* & \overline{p}_{f,p} & * \\
* & * & \omega^{k-1}
\end{pmatrix}
\]

or

\[
\overline{p}_{G,p} = \begin{pmatrix}
\omega^{k-1} & * & * \\
* & \overline{p}_{f,p} & * \\
* & * & \omega^{k-2}
\end{pmatrix}
\]
The first of these is already in the form we want, so we can leave this one alone. As for the second case, we have

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
\omega^{k-1} & * & * \\
\bar{p}_{f,p} & * \\
\omega^{k-2} & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
^{-1}
= 
\begin{pmatrix}
\omega^{k-2} \\
* \\
\bar{p}_{f,p}
\end{pmatrix}.
\]

Now by changing to the basis as in Equation 8.5, we obtain

\[
\bar{p}_{G,p} = 
\begin{pmatrix}
\omega^{k-2} & * \\
* & \bar{p}_{f,p} & * \\
\omega^{k-1}
\end{pmatrix},
\]

which is also in the form we seek.

**Case 2:** \(\dim V_2 = 2, \dim V_0/V_1 = \dim V_1/V_2 = 1\).

This case corresponds to the situation where we have either

\[
\bar{p}_{G,p} = 
\begin{pmatrix}
\bar{p}_{f,p} & * & * \\
\omega^{k-2} & * \\
\omega^{k-1}
\end{pmatrix}
\]

or

\[
\bar{p}_{G,p} = 
\begin{pmatrix}
\bar{p}_{f,p} & * & * \\
\omega^{k-1} & * \\
\omega^{k-2}
\end{pmatrix}.
\]

The first case is handled by observing

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
\bar{p}_{f,p} & * & * \\
\omega^{k-2} & * \\
\omega^{k-1}
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
^{-1}
= 
\begin{pmatrix}
\omega^{k-2} & * \\
* & \bar{p}_{f,p} & * \\
\omega^{k-1}
\end{pmatrix}.
\]
The second case is handled similarly:
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\rho_{f,p} & \ast & \ast \\
\ast & \omega^{k-1} & \ast \\
\ast & \ast & \omega^{k-2} \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
\end{pmatrix}
^{-1}
\begin{pmatrix}
1 \\
1 \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
\omega^{k-2} & \ast \\
\ast & \rho_{f,p} & \ast \\
\ast & \ast & \omega^{k-1} \\
\end{pmatrix}.
\]

Next we change bases as in Equation 8.4 and then as in Equation 8.5 to obtain
\[
\bar{\rho}_{G,p} = 
\begin{pmatrix}
\omega^{k-2} & \ast & \ast \\
\ast & \rho_{f,p} & \ast \\
\ast & \ast & \omega^{k-1} \\
\end{pmatrix}.
\]

**Case 3:** \( \dim V_2 = \dim V_1 \cap V_2 = 1, \dim V_0 \cap V_1 = 2. \)

Case 3 corresponds to the situation where we have either
\[
\bar{\rho}_{G,p} = 
\begin{pmatrix}
\omega^{k-1} & \ast & \ast \\
\ast & \omega^{k-2} & \ast \\
\ast & \omega^{k-1} & \ast \\
\end{pmatrix}
\]
or
\[
\bar{\rho}_{G,p} = 
\begin{pmatrix}
\omega^{k-2} & \ast & \ast \\
\ast & \omega^{k-1} & \ast \\
\end{pmatrix}
\]

To put the first case into the form we desire, we begin with the conjugation formula
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\omega^{k-1} & \ast & \ast \\
\ast & \omega^{k-2} & \ast \\
\ast & \ast & \bar{\rho}_{f,p} \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
\end{pmatrix}
^{-1}
\begin{pmatrix}
1 \\
1 \\
1 \\
\end{pmatrix}
= 
\begin{pmatrix}
\omega^{k-2} & \ast \\
\ast & \bar{\rho}_{f,p} & \ast \\
\ast & \ast & \omega^{k-1} \\
\end{pmatrix}.
\]

Now apply Equation 8.5 to obtain
\[
\bar{\rho}_{G,p} = 
\begin{pmatrix}
\omega^{k-2} & \ast & \ast \\
\ast & \bar{\rho}_{f,p} & \ast \\
\omega^{k-1} \\
\end{pmatrix}.
\]
The second form is handled by observing
\[
\begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & \omega^{k-1} & \ast & \\
& & \ast & \omega^{-1} & \overline{p}_{f,p} \\
& & & 1 & \\
& & & & 1
\end{pmatrix}
\begin{pmatrix}
\ast & \ast & & & \\
\ast & & & & \\
& & \ast & \ast & \\
& & & \ast & \\
& & & & \ast
\end{pmatrix}
^{-1}
= \begin{pmatrix}
\ast & \ast & & & \\
\ast & & & & \\
& & \ast & \ast & \\
& & & \ast & \\
& & & & \ast
\end{pmatrix}.
\]

Now apply Equation 8.3 and then Equation 8.5 to obtain
\[
\overline{p}_{G,p} = \begin{pmatrix}
\omega^{k-2} & \ast & & \\
& \ast & \overline{p}_{f,p} & \ast \\
& \ast & & \\
& & & \omega^{k-1}
\end{pmatrix}.
\]

We have proven that there is a lattice so that we have
\[
\overline{p}_{G,p} = \begin{pmatrix}
\omega^{k-2} & \ast & \ast_1 & \ast_2 \\
& \ast & \overline{p}_{f,p} & \ast_4 \\
& \ast & & \\
& & & \omega^{k-1}
\end{pmatrix}
\]
where either \ast_1 or \ast_3 is zero.

Now that we have the matrix in the appropriate form, we would like to further limit the possibilities. We begin with the following Proposition.

**Proposition VIII.23.** Let \( \rho_{G,p} \) be such that it does not have a sub-quotient of dimension 1 and \( \overline{p}^{\text{ss}}_{G,p} = \omega^{k-2} \oplus \overline{p}_{f,p} \oplus \omega^{k-1} \). Then there exists a \( \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \)-stable \( \mathcal{O} \)-lattice in \( V_G \) having an \( \mathcal{O} \)-basis such that the corresponding representation \( \rho = \rho_{G,p} : \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_4(\mathcal{O}) \) has reduction of the form

\[
(8.6) \quad \overline{p}_{G,p} = \begin{pmatrix}
\omega^{k-2} & \ast_1 & \ast_2 \\
& \ast_3 & \overline{p}_{f,p} & \ast_4 \\
& & & \\
& & & \omega^{k-1}
\end{pmatrix}
\]
and such that there is no matrix of the form

\[
U = \begin{pmatrix}
1 & n_1 \\
1 & n_2 \\
1 & n_3 \\
1 & 1
\end{pmatrix} \in \text{GL}_4(\mathcal{O})
\]

such that \( \rho' = U\rho U^{-1} \) has reduction of type (8.6) with \( *_2 = *_4 = 0 \)

**Proof.** Fix a \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-stable lattice and an \( \mathcal{O} \)-basis giving rise to a representation \( \rho_0 \) of type (8.6). Suppose there exists a \( U_0 \) as in (8.7). Set \( \rho_0' = U_0\rho_0 U_0^{-1} \). Recall we defined

\[
P_3 = \begin{pmatrix}
1 & \\
1 & \\
1 & \\
\varpi & \\
\end{pmatrix}
\]

before. Then it is clear that \( \rho_1 = P_3\rho_0' P_3^{-1} \) is a representation into \( \text{GL}_4(\mathcal{O}) \) of the form (8.6). Let \( M_1 = P_3U_0 \). Inductively define \( M_2, M_3, \ldots \) and \( \rho_2, \rho_3, \ldots \) starting with \( M_1 \) and \( \rho_1 \). Put \( N_n = M_n \ldots M_1 \) so that \( \rho_n = N_n\rho_0 N_n^{-1} \). Note that we have \( N_n = P_3^n U_n \) with \( U_n \) of the form (8.7). Furthermore, we have that \( U_n \) converges to \( U_\infty \) of the form (8.7). We have that \( \rho_n'' = U_n\rho_0 U_n^{-1} \) is of the form that the first three entries of the rightmost column are all divisible by \( \varpi^n \) since \( \rho_n = P_3^n \rho_n'' P_3^{-n} \) has entries in \( \mathcal{O} \). Thus, \( \rho_\infty = U_\infty\rho_0 U_\infty^{-1} \) is such that the first three entries of the rightmost column are zero. This gives a 1-dimensional subquotient of \( \rho_{G,p} \), a contradiction. Thus no such \( U_0 \) can exist.

\( \square \)

In light of this proposition our next step is to show that \( \rho_{G,p} \) does not have a sub-quotient of dimension 1 for \( G \) as in Theorem VII.9. There are three possibilities
for how $\rho_{G,p}$ could split up with a sub-quotient of dimension 1. It could have a sub-
quotient of dimension 3 and of dimension 1, a 2-dimensional sub-quotient and two
1-dimensional ones, or four 1-dimensional sub-quotients. The case of a 3 dimensional
sub-quotient cannot occur, see ([80], Page 512) or ([76], Proof of Theorem 3.2.1). The
case of splitting into four 1-dimensional sub-quotients is not possible either. Indeed,
if $\rho_{G,p} = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$ for characters $\chi_i$, then $\overline{\rho}_{G,p}$ splits into four 1-dimensional
sub-quotients as well but this gives a contradiction as we know $\overline{\rho}_{f,p}$ is not completely
reducible ([57], Proposition 2.1).

The last case to worry about is if $\rho_{G,p}$ splits into a 2-dimensional sub-quotient and
two 1-dimensional sub-quotients. In this case $G$ must be a CAP form ([76], Proof of
Theorem 3.2.1). This is where our restriction to level one is necessary. If $G$ is a CAP
form, then it necessarily has the same eigenvalues as a Saito-Kurokawa lift for some
elliptic cusp form $g$. However, Theorem II.14 gives us that $G$ would have to be in the
Maass space. This gives a contradiction as the Saito-Kurokawa isomorphism would
then imply that $G$ is a Saito-Kurokawa lift, a contradiction. Therefore we have that
$G$ is not a CAP form.

Summarizing to this point, we now have that there exists a Gal($\overline{Q}/Q$)-stable
lattice $T_{G,p}$ so that the reduction $\overline{\rho}_{G,p}$ is of the form

\[
\overline{\rho}_{G,p} = \begin{pmatrix}
\omega^{k-2} & *_1 & *_2 \\
*_3 & \overline{\rho}_{f,p} & *_4 \\
& \omega^{k-1}
\end{pmatrix}
\]

where $*_1$ or $*_3$ is zero and so that $\overline{\rho}_{G,p}$ is not equivalent to a representation with $*_2$
and $*_4$ both zero. Write $W_{G,p}$ for $V_{G,p}/T_{G,p}$.

We now show that $*_4$ gives us a non-zero class in $H_{f}^{1}(Q, W_{f,p}(1-k))$. Note that
the fact that $\overline{\rho}_{G,p}$ is a homomorphism gives that $*_4$ necessarily gives a cohomology
class in $H^1(Q, W_{f,p}(1-k)[\omega])$.

First we suppose we are in the situation where $*_3 = 0$. Our first step is to show that the quotient extension

$$
\rho_{f,p} = 
\begin{pmatrix}
\rho_{f,p} & *_4 \\
0 & \omega^{k-1}
\end{pmatrix}
$$

is not split. Suppose it is split. Then by Proposition VIII.23 we know that the extension

$$
\omega^{k-2} *_2 \\
0 & \omega^{k-1}
$$

cannot be split as well. We show this gives a contradiction by showing it gives a non-trivial quotient of the $\omega^{-1}$-isotypical piece of the $p$-part of the class group of $Q(\mu_p)$. However, Herbrand's Theorem (see for example, [85], Theorem 6.17) says that we must then have $p \mid B_2$, which clearly cannot happen.

Consider the non-split representation

$$
\begin{pmatrix}
\omega^{-1} & h \\
0 & 1
\end{pmatrix}
$$

which arises from twisting the non-split representation

$$
\begin{pmatrix}
\omega^{k-2} & *_2 \\
0 & \omega^{k-1}
\end{pmatrix}
$$

by $\omega^{1-k}$. Note that $\rho$ is unramified away from $p$ because $\rho_{G,p}$ is unramified away from $p$.

We claim that this representation gives us a non-trivial finite unramified abelian $p$-extension $K/Q(\mu_p)$ with the action of $\text{Gal}(K/Q)$ on $\text{Gal}(K/Q(\mu_p))$ given by $\omega^{-1}$. 
Note that $\mathbb{Q}(\mu_p) = \overline{\mathbb{Q}}^{\ker \omega^{-1}}$, so when we restrict $\overline{\rho}$ to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))$ we get

$$\overline{\rho} |_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},$$

i.e., we get a non-trivial homomorphism $h : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p)) \to \mathbb{F}$. Set $K = \mathbb{Q}(h) = \overline{\mathbb{Q}}^{\ker h}$, the splitting field of $h$.

The fact that $\text{Gal}(K/\mathbb{Q}(\mu_p))$ is abelian of $p$-power order follows from the fact that

$$\text{Gal}(K/\mathbb{Q}(\mu_p)) \cong \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))/\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(h))$$

$$= \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))/\ker h$$

$$\cong \text{Image}(h)$$

and $\text{Image}(h)$ is a subgroup of $\mathbb{F}$, which is of $p$-power order. The fact that $K/\mathbb{Q}(\mu_p)$ is unramified away from $p$ also follows easily from the fact that $\overline{\rho}$ is unramified away from $p$. This shows that $h(I_{\ell}) = 0$ for all $\ell \neq p$. In particular, $h(I_{\ell}(K/\mathbb{Q}(\mu_p))) = 0$ for all $\ell \neq p$. Since we have the isomorphism above to a subgroup of $\mathbb{F}$, it must be that $I_{\ell}(K/\mathbb{Q}(\mu_p)) = 1$ for all $\ell \neq p$.

The fact that $\text{Gal}(K/\mathbb{Q})$ acts on $\text{Gal}(K/\mathbb{Q}(\mu_p))$ via $\omega^{-1}$ follows from the fact that for $\sigma \in \text{Gal}(K/\mathbb{Q}(\mu_p))$ and $g \in \text{Gal}(K/\mathbb{Q})$, we have

$$\overline{\rho}(g\sigma g^{-1}) = \overline{\rho}(g)\overline{\rho}(\sigma)\overline{\rho}(g^{-1}),$$

i.e., we have

$$h(g\sigma g^{-1}) = \omega^{-1}(g)h(\sigma).$$

Our next step is to show that the extension $K/\mathbb{Q}(\mu_p)$ that we have constructed is actually unramified at $p$. We have that $h|_{D_p} \in H^1(\mathbb{Q}_p, \mathbb{F}(-1))$. Therefore, we have that $h$ gives an extension $X$ of $\mathcal{O}/\mathfrak{w}\mathcal{O}$ by $\mathbb{F}(-1)$:
Applying Lemma VIII.14 and Lemma VIII.18 we have that $h|_{D_p} \in H^1_f(Q_p, \mathbb{F}(-1))$.

A calculation in [7] shows that $H^1_f(Q_p, E(-1)) = 0$ where $E$ is the field of definition for $\rho_{G,p}$. Actually, it is shown that $H^1_f(Q_p, Q_p(r)) = 0$ for every $r < 0$, but it is clear that this implies $H^1_f(Q_p, E(-1)) = 0$ since $E$ is a finite extension ([7], Example 3.9).

Since we define $H^1_f(Q_p, E/\mathcal{O}(-1))$ to be the image of the $H^1_f(Q_p, E(-1))$, we have $H^1_f(Q_p, E/\mathcal{O}(-1)) = 0$. Since $h|_{D_p} \in H^1_f(Q_p, \mathbb{F}(-1))$, Proposition VIII.20 gives that $h|_{D_p} \in H^1_f(Q_p, E/\mathcal{O}(-1))$ and hence is 0. Thus we have that $h$ vanishes on the entire decomposition group $D_p$, so in particular, it must be unramified at $p$ as claimed.

Therefore, we have an unramified extension $K$ of $\mathbb{Q}(\mu_p)$ that is of $p$-power order such that $\text{Gal}(K/\mathbb{Q})$ acts via $\omega^{-1}$. Let $C$ be the $p$-part of the class group of $\mathbb{Q}(\mu_p)$. Class field theory tells us that we have

$$C/C^p \cong \text{Gal}(F/\mathbb{Q}(\mu_p))$$

where $F$ is the maximal unramified elementary abelian $p$-extension of $\mathbb{Q}(\mu_p)$. Therefore we have that $\text{Gal}(K/\mathbb{Q}(\mu_p))$ is a non-trivial subgroup of the $\omega^{-1}$-isotypical piece of the $p$-part of the class group of $\mathbb{Q}(\mu_p)$, a contradiction as observed above.

Therefore, we must have that the quotient extension

$$\begin{pmatrix} \rho_{f,p} & \ast_4 \\ 0 & \omega^{k-1} \end{pmatrix}$$

is not split if $\ast_3 = 0$.

Now suppose that $\ast_1 = 0$. Then the extension

$$\begin{pmatrix} \omega^{k-2} & \ast_2 \\ 0 & \omega^{k-1} \end{pmatrix}$$
is a quotient extension and as above must necessarily be split. Therefore again we get that the subextension

\[
\begin{pmatrix}
\overline{\rho}_{f,p} & *_{4} \\
0 & \omega^{k-1}
\end{pmatrix}
\]

cannot be split.

It remains to show that \( *_{4} \) actually lies in \( H^1_f(\mathbb{Q}, W_{f,p}(1-k)) \) since we have shown it is not zero. Write \( h = *_{4} \) to ease notation. As noted above, we have that \( h \) gives a non-zero class in \( H^1(\mathbb{Q}, W_{f,p}(1-k)[\varpi]) \), so in particular, in \( H^1(\mathbb{Q}, W_{f,p}(1-k)) \) since we have shown in the previous section that \( H^1(\mathbb{Q}, W_{f,p}(1-k)[\varpi]) \) injects in \( H^1(\mathbb{Q}, W_{f,p}(1-k)) \). It remains to show that \( h|_{D_\ell} \in H^1_{ur}(\mathbb{Q}_\ell, W_{f,p}(1-k)) \) for each \( \ell \neq p \) and \( h|_{D_p} \in H^1_f(\mathbb{Q}_p, W_{f,p}(1-k)) \). The fact that \( h|_{D_\ell} \in H^1_{ur}(\mathbb{Q}_\ell, W_{f,p}(1-k)[\varpi]) \) for \( \ell \neq p \) is clear from the fact that \( \rho_{G,p} \) is unramified away from \( p \). Therefore, we can appeal to Proposition VIII.19, to obtain that \( h \in H^1_{ur}(\mathbb{Q}_\ell, W_{f,p}(1-k)) \) for all \( \ell \neq p \).

The case at \( p \) is easily handled by appealing to our work in the previous section. Since \( h|_{D_p} \in H^1(\mathbb{Q}_p, W_{f,p}(1-k)[\varpi]) \), we get an extension \( X \) of \( \mathcal{O}/\varpi\mathcal{O} \) by \( W_{f,p}(1-k)[\varpi] \):

\[
0 \longrightarrow W_{f,p}(1-k)[\varpi] \longrightarrow X \longrightarrow \mathcal{O}/\varpi\mathcal{O} \longrightarrow 0.
\]

Appealing to Lemma VIII.14 and Lemma VIII.18 we have that \( h|_{D_p} \) lies in \( H^1_f(\mathbb{Q}_p, W_{f,p}(1-k)[\varpi]) \) as desired. Proposition VIII.20 gives that \( h|_{D_p} \) lies in \( H^1_f(\mathbb{Q}_p, W_{f,p}(1-k)) \).

Therefore, we have that \( h \) is a non-zero torsion element of \( H^1(\mathbb{Q}, W_{f,p}(1-k)) \) that lies in \( H^1_f(\mathbb{Q}_\ell, W_{f,p}(1-k)) \) for every \( \ell \). Applying Proposition VIII.20 to \( h \) we have that \( h \) is a non-zero \( \varpi \)-torsion element of \( H^1_f(\mathbb{Q}, W_{f,p}(1-k)) \). Therefore, it must be
that $p \mid \# \text{H}_f^1(\mathbb{Q}, W_{f,p}(1-k))$. We summarize with the following theorem.

**Theorem VIII.24.** Let $k > 3$ be an integer and $p > 2k - 2$ a prime. Let $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}), \mathcal{O})$ be a newform with real Fourier coefficients so that $\rho_{mf}$ is irreducible and Conjecture VI.1 holds (for instance, if $f$ is ordinary at $p$). Let

$$\varpi^m \mid L_{\text{alg}}(k, f)$$

with $m \geq 1$. If there exists an integer $N > 1$ with $\gcd(p, N) = 1$, a fundamental discriminant $D$ so that $(-1)^{k-1}D > 0$ and $\chi_D(-1) = -1$, and a Dirichlet character $\chi$ of conductor $N$ so that

$$\varpi^n \parallel L_N(3-k, \chi)L_{\text{alg}}(k-1, f, \chi_D)L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi)$$

with $n < m$, then

$$p \mid \# \text{H}_f^1(\mathbb{Q}, W_{f,p}(1-k)).$$

### 8.4 Numerical Example

In this concluding section we provide a numerical example of Theorem VIII.24.

We used the computer software MAGMA, Stein’s Modular Forms Database ([78]), and Dokchitser’s PARI program ComputeL ([19]).

Let $p = 516223$. We consider level 1 and weight 54 newforms in $S_{54}(\text{SL}_2(\mathbb{Z}))$. There is one Galois conjugacy class of such newforms, consisting of four newforms which we label $f_1, f_2, f_3, f_4$. Using the software Stein wrote for MAGMA we find that

$$p \mid \prod_{i=1}^4 L_{\text{alg}}(28, f_i).$$

The $q$-expansions of each $f_i$ are defined over a number field $K_i$. Appealing to
MAGMA again we find each $K_i$ is generated by a root of

$$g(x) = x^4 + 68476320x^3 - 19584715019010048x^2 - 10833127246634489297121280x$$

$$+ 39446133467662904714689328971776.$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $g(x)$. Note that two of the $\alpha_i$ are real and the other two are a complex conjugate pair. Relabelling the $f_i$ if necessary, we may assume $K_i = \mathbb{Q}(\alpha_i)$. Let $\mathcal{O}_{K_i}$ be the ring of integers of $K_i$. Note that $L_{\text{alg}}(28, f_i) \in \mathcal{O}_{K_i}$ for each $i$. Therefore, using Equation 8.10 we see that there exists $j \in \{1, 2, 3, 4\}$ and a prime $\wp_j \subset \mathcal{O}_{K_j}$ over $p$ so that $\wp_j | L_{\text{alg}}(28, f_j)$. Since the $f_i$ are all Galois conjugate, there is a conjugate prime $\wp_i \subset \mathcal{O}_{K_i}$ over $p$ for each $i \in \{1, 2, 3, 4\}$ so that $\wp_i | L_{\text{alg}}(28, f_i)$.

Let $\chi = \chi_{-3}$ where we define $\chi_{-3}$ as in Chapter II Section 2.3. It is easy to check that this $\chi$ and $D = -3$ satisfy the conditions of Theorem VIII.24. Using MAGMA we find that

$$p \nmid \prod_{i=1}^{4} L_{\text{alg}}(j, f_i, \chi),$$

for $j = 1, 2$ and

$$p \nmid \prod_{i=1}^{4} L_{\text{alg}}(27, f_i, \chi_D).$$

We use ComputeL to show that

$$p \nmid L_{3}(-25, \chi).$$

Let $F_i = K_{i, \wp_i}$ with ring of integers $\mathcal{O}_i$ and uniformizer $\varpi_i$. Set $F_i = \mathcal{O}_i/\varpi_i = \mathbb{F}_p[\overline{\alpha}_i]$ where $\overline{\alpha}_i = \alpha_i (\text{mod } \wp_i)$. Let $\rho_i : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O}_i)$ be the Galois representation associated to $f_i$. This representation is unramified away from $p$ and crystalline at $p$. Let $\overline{\rho}_i : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_i)$ be the residual representation. Suppose that $\overline{\rho}_i$
is reducible. Standard arguments show that $\overline{\rho}_i$ is non-split and we can write

$$
\overline{\rho}_i = \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}
$$

with $* \neq 0$ (see [57]). Let $\omega : \text{Gal}(\overline{Q}/Q) \to \mathbb{F}_p^\times$ be the mod $p$ cyclotomic character. Since $\varphi \psi = \omega^{53}$ and $\varphi$ and $\psi$ are necessarily unramified away from $p$ and of order prime to $p$, we can write $\varphi = \omega^a$ and $\psi = \omega^b$ with $0 \leq a < b < p - 1$ and $a + b = 53$ or $a + b = p - 1 + 53$. Arguing as in the previous section where we proved that

$$
\begin{pmatrix} \overline{\rho}_{f,p} & * \\ 0 & \omega^{k-1} \end{pmatrix}
$$

cannot be split, we have that $*$ gives a non-zero cocycle class in $H^1(Q, \mathbb{F}_p(a-b))$ since $a - b < 0$. As before, this shows that we must have that $p$ divides the class number of $Q(\mu_p)$, i.e., $p \mid B_{b-a+1}$ where we recall that $B_n$ is the $n$th Bernoulli number ([85], Theorem 6.17). Appealing to the tables of Buhler ([9]), we see that the only Bernoulli number that 516223 divides is $B_{451304}$. Therefore, we must have $b - a + 1 = 451304$, which in turn implies that $a + b = p - 1 + 53$ since necessarily $a > 0$. Solving this system of equations for $a$ and $b$ we get $a = 32486$ and $b = 483789$. Observe that we have

$$
\text{Tr}(\overline{\rho}_i(Frob_2)) = 2^a + 2^b \pmod{p}
$$

$$
= 258573 \pmod{p}.
$$

Using Stein’s tables we see that $\text{Tr}(\overline{\rho}_i(Frob_2)) = \alpha_i$, so we must have that $\overline{\alpha}_i \equiv 258573 \pmod{p}$. This also shows that $\overline{\alpha}_i$ must belong to $\mathbb{F}_p$ and so must be a root of one of the linear factors of $g(x)$ modulo $p$. Using Maple to compute the linear roots of $g(x)$ modulo $p$ we find that they are 287487 and 85284, neither of which is
congruent to 258573 modulo $p$. This provides a contradiction so we may conclude that $\mathfrak{p}_i$ is irreducible.

Due to the size of the prime under consideration, it was not possible with the computer we used to compute the $p^{th}$ Fourier coefficients of the $f_i$ to check ordinarity. So, instead we show that in this case the ordinarity assumption is not necessary. We do this by showing there are no congruences between the $f_i$. Let $E$ be a large number field containing all of the $K_i$. Let $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Let $q$ be any prime of $E$ over $p$. As in Chapter VI, $f_i$ and $f_j$ each give a map from $\mathbb{T}_{\mathcal{O}_{E_q}}$ to $\mathcal{O}_{E_q}$ given by $T(\ell) \mapsto a_{f_i}(\ell)$ and $T(\ell) \mapsto a_{f_j}(\ell)$ respectively. Let $m_i$ and $m_j$ be the respective maximal ideals defined as the inverse image of $q$ under these maps. (These are the maximal ideals associated to $f_i$ and $f_j$ of $\mathbb{T}_{\mathcal{O}_{E_q}}$ as in Chapter VI.) There is a congruence between $f_i$ and $f_j$ modulo $q$ if and only if the maximal ideals $m_i$ and $m_j$ are the same. This is equivalent to the statement that

$$a_{f_i}(\ell) \equiv a_{f_j}(\ell) \pmod{q}$$

for all $\ell \neq p$. In particular, looking at the case when $\ell = 2$, if a congruence exists between $f_i$ and $f_j$ we have

$$q \mid (a_{f_i}(2) - a_{f_j}(2)),$$

i.e.,

$$q \mid (\alpha_i - \alpha_j).$$

Therefore we have that

$$\text{Nm}(q) \mid \text{Nm}(\alpha_i - \alpha_j).$$

The left hand side is a power of $p$ where as the right hand side divides a power of the discriminant of $g(x)$, so that necessarily $p$ divides the discriminant of $g(x)$. Comput-
ing the discriminant with Maple we find the prime factorization of the discriminant,

\[ \text{disc}(g(x)) = -2^{48}3^35^6 \cdot 11 \cdot 59 \cdot 15909926723 \cdot 4581597403 \]

\[ \cdot 61912455248726091228769884731066259290896074682396020673553. \]

Therefore we have that \( p \) does not divide this discriminant. Therefore we must have that there is no congruence modulo \( q \) between any of the \( f_j \)'s. We can now appeal to the same argument used in the proof of Lemma VII.8 to conclude that there exists a Hecke operator \( t \) so that \( t \cdot f_i = u \cdot \frac{(f_i, f_i)}{\Omega_i} f_i \) and \( t \cdot f_j = 0 \) for \( j \neq i \). In this way we have avoided needing to check the ordinarity of each \( f_j \) to get the existence of the Hecke operator conjectured in Conjecture VI.1.

If we choose \( f_i \) to be one of the two newforms with real Fourier coefficients, then we satisfy all of the hypotheses of Theorem VIII.24 and so obtain the result that

\[ 516223 \mid \# H_f^1(\mathbb{Q}, W_{f_i, \varphi_i}(-27)). \]
BIBLIOGRAPHY


120


ABSTRACT

Saito-Kurokawa Lifts, $L$-values for GL$_2$, and Congruences Between Siegel Modular Forms

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Let $f$ be a newform of weight $2k - 2$ and level 1. There is a conjecture of Bloch and Kato that states that the special values of the $L$-function attached to $f$ should be the size of the Selmer group attached to $f$. In this thesis we provide evidence for this conjecture by demonstrating an implication that if $\varpi | L_{\text{alg}}(k, f)$, then $\varpi$ must divide the order of the Selmer group as well where $\varpi$ is the uniformizer of a finite extension of $\mathbb{Q}_p$.

The method we employ is the following. Let $F_f$ be the Saito-Kurokawa lift of $f$. Given a prime $p$ so that $\varpi | L_{\text{alg}}(k, f)$, we construct a cuspidal Siegel eigenform $G$ that is not a CAP form so that Hecke eigenvalues of $G$ are congruent to the Hecke eigenvalues of $F_f$ modulo $\varpi$.

Associated to a Siegel eigenform we have a 4-dimensional Galois representation. Once we have such an eigenform $G$, we use the explicit nature of the Saito-Kurokawa
correspondence to make deductions about the Galois representation associated to $G$. These deductions allow us to construct a non-zero element of $p$-power order in the Selmer group, thereby providing the divisibility we seek.