Commutators in the Metaplectic Group

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Let $F$ be a local field with nontrivial Hilbert symbol and characteristic not equal to 2 — that is, $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$ (where $q$ is a power of an odd prime) — and fix a nontrivial continuous character $\psi : F \rightarrow \mathbb{T}$. Let $(V, \langle \cdot, \cdot \rangle)$ be a $2n$-dimensional symplectic vector space over $F$.

We will prove that the metaplectic group is equal to its commutator subgroup and apply this result to show that the projective Weil representation of $\text{Sp}(V)$ is not linearizable.

1 Preliminaries

1.1 The Weil index

The Weil index $\gamma$ is a character of the Witt group $W(F)$ which is used in the construction of the metaplectic group. If $\alpha \in F^\times$ and $\langle \alpha \rangle$ is the one-dimensional quadratic space, we denote $\gamma(\langle \alpha \rangle)$ by $\gamma(\alpha)$. We will use the fact that for all $\alpha, \beta \in F^\times$,

$$\gamma(\alpha)\gamma(\beta) \gamma(1) \gamma(\alpha\beta) = (\alpha, \beta),$$

where $(\alpha, \beta)$ is the quadratic Hilbert symbol.

1.2 The Maslov index

Let $\ell_1, \ldots, \ell_r$ be Lagrangian subspaces of $V$. The Maslov index of these Lagrangians is a quadratic space $\tau(\ell_1, \ldots, \ell_r)$. By [2], §2.1, the Maslov index has the following properties (plus some others which we won’t use):

1. Symplectic invariance: For any $g \in \text{Sp}(V)$,

$$\tau(\ell_1, \ldots, \ell_r) = \tau(g\ell_1, \ldots, g\ell_r).$$

2. Dihedral symmetry:

$$\tau(\ell_1, \ldots, \ell_r) = \tau(\ell_2, \ldots, \ell_r, \ell_1),$$

$$\tau(\ell_1, \ell_2, \ldots, \ell_r) = -\tau(\ell_n, \ell_{n-1}, \ldots, \ell_1).$$

3. Chain condition: For any $3 \leq k < n$,

$$\tau(\ell_1, \ldots, \ell_r) = \tau(\ell_1, \ldots, \ell_k) + \tau(\ell_1, \ell_k, \ldots, \ell_n).$$

By [2] (§2.2.4-6, p. 22), the Maslov index $\tau(\ell_1, \ldots, \ell_r)$ is the nondegenerate quotient of the quadratic space $(T, q)$, where

$$T = \{(v_1, \ldots, v_r) \in \ell_1 \oplus \cdots \oplus \ell_r \mid v_1 + \cdots + v_r = 0\}$$
and
\[ q((v_1, \ldots, v_r)) = \sum_{r \geq i > j > 1} \langle v_i, w_j \rangle. \]  

(1.2.1)

By [2] (§2.4.1, p. 23),
\[ \dim T = \frac{(n - 2) \dim V}{2} - \sum_{i \in \mathbb{Z}/r\mathbb{Z}} \dim(\ell_i \cap \ell_{i+1}) + 2 \dim \bigcap_{i \in \mathbb{Z}/r\mathbb{Z}} \ell_i. \]  

(1.2.2)

The following lemma will be useful for computing the Maslov index.

**Lemma 1.1.** Suppose \( \dim V = 2 \), \((p, q)\) is a symplectic basis for \( V \), and \( \ell = \text{span}(p) \). If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( B = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \text{Sp}(2, F) \), then
\[ \tau(\ell, A\ell, B\ell) = \langle cz(az - cx) \rangle. \]

Proof. Let \((T, q) = \tau(\ell, A\ell, B\ell)\) denote the quadratic space. Observe that
\[ (cx - az)p + zAp - cBp = (cx - az) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} a \\ c \end{bmatrix} - c \begin{bmatrix} x \\ z \end{bmatrix} = 0. \]

By equation (1.2.1), the quadratic form is given by \( q(v_1, v_2, v_3) = \langle v_3, v_2 \rangle \). So by basic properties of the symplectic basis,
\[ q((cx - az)p, zAp, -cBp) = \langle -cBp, zAp \rangle = -cz \langle \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} \rangle = -cz(xc - za) = cz(az - cx). \]

By equation (1.2.2), \( \dim T \leq 1 \), hence \( T = \langle cz(az - cx) \rangle \).

\[ \square \]

### 1.3 Construction of the metaplectic group

The metaplectic group \( \text{Mp}(V) \) is explicitly constructed in [2] (§4.3.2, p. 50). (Note: Li uses the notation \( \hat{\text{Sp}}(W) \) instead of \( \text{Mp}(V) \).) We reproduce the key details of the construction here.

Let \( \Lambda(V) \) be the set of Lagrangian subspaces of \( V \). For every \( g \in \text{Sp}(V) \) and \( \ell \in \Lambda(V) \) (with an arbitrary orientation fixed on \( \ell \)), define
\[ m_g(\ell) := \gamma(1)^{\dim \ell - \dim(\ell \cap g\ell)} \gamma(A_{g\ell, \ell}), \]

where \( A_{g\ell, \ell} \) is the pairing of the orientations on \( g\ell \) and \( \ell \) defined in [2] (§1.3.13, p. 18). Also, for any \( g, h \in \text{Sp}(V) \), denote
\[ c_{g,h}(\ell) := \gamma(\tau(\ell, g\ell, gh\ell)). \]

Define \( \text{Mp}(V) \) to be the set of all pairs of the form \((g, t)\), where \( g \in \text{Sp}(V) \) and \( t : \Lambda(V) \rightarrow \mathbb{C}^\times \) is a map such that

- \( t(\ell)^2 = m_g(\ell)^2 \) for all \( \ell \in \Lambda(V) \),
- \( t(\ell') = \gamma(\tau(\ell, g\ell, g\ell'))t(\ell) \) for all \( \ell, \ell' \in \Lambda(V) \).

Multiplication in \( \text{Mp}(V) \) is defined by
\[ (g, s) \cdot (h, t) = (gh, st \cdot c_{g,h}). \]

The unit element is \((1, 1)\), and \((g, t)^{-1} = (g^{-1}, t^{-1}) \) since \( c_{g,g^{-1}}(\ell) = 1 \). Also, we have the following:

**Proposition 1.2** ([2], §4.3.3, p. 51). The projection map \( \text{Mp}(V) \rightarrow \text{Sp}(V) \) defined by \((g, t) \mapsto g \) makes \( \text{Mp}(V) \) a two-fold covering of \( \text{Sp}(V) \).
2 Main theorem

This section is devoted to the proof of the following:

Theorem 2.1. The metaplectic group \( \text{Mp}(V) \) is equal to its commutator subgroup.

Throughout the proof, let \( M \) denote the commutator subgroup \([\text{Mp}(V), \text{Mp}(V)]\).

2.1 Initial reductions

Suppose \((1, -1) \in M\). The group operation on \( \text{Mp}(V) \) is simply multiplication in the first coordinate and \( \text{Sp}(V) = [\text{Sp}(V), \text{Sp}(V)] \), so for each \( x \in \text{Sp}(V) \), some \((x, \sigma)\) is in \( \text{Mp}(V) \). Thus \((x, -\sigma) = (x, -\sigma \cdot c_{x,1}) = (x, \sigma)(1, -1) \in M\).

Since \( \text{Mp}(V) \) is a double cover of \( \text{Sp}(V) \) and \( \sigma \neq -\sigma \), it follows that \( \text{Mp}(V) = M \).

Thus, it suffices to show that \((1, -1) \in M\). Moreover, we need only show that \((1, \sigma) \in M\) for some \( \sigma : \Lambda(V) \longrightarrow \mathbb{C}^\times \) such that \( \sigma(\ell) = -1 \) for some \( \ell \in \Lambda(V) \). Hence, in a slight abuse of notation, we will fix a Lagrangian \( \ell \) and only keep track of the value for \( \ell \) in the second coordinate.

For each \( x, y \in \text{Sp}(V) \), denote \( \tau[x, y] := \tau(\ell, x\ell, xy\ell, xyx^{-1}\ell, [x, y]\ell) \).

The following will be useful later:

Lemma 2.2. For all \( x, y \in \text{Sp}(V) \),

\[ ([x, y], \gamma(\tau[x, y])) \in M. \]

Proof. Let \((x, s), (y, t) \in \text{Mp}(V)\) be arbitrary. Then

\[
[(x, s), (y, t)] = (x, s)(y, t)(x^{-1}, s^{-1})(y^{-1}, t^{-1})
\]

\[
= (xy, xstc_{xy})(x^{-1}, s^{-1})(y^{-1}, t^{-1})
\]

\[
= (xyx^{-1}, xstc_{xy}s^{-1}c_{xy,x^{-1}})(y^{-1}, t^{-1})
\]

\[
= (xyx^{-1}y^{-1}, xstc_{xy}s^{-1}c_{xy,x^{-1}}t^{-1}c_{xyx^{-1},y^{-1}})
\]

\[
= ([x, y], xstc_{xy}s^{-1}c_{xy,x^{-1}}c_{xyx^{-1},y^{-1}})
\]

\[
= ([x, y], \gamma(\tau(\ell, x\ell, xy\ell))\gamma(\tau(\ell, xy\ell, xyx^{-1}\ell))\gamma(\tau(\ell, xyx^{-1}\ell, [x, y]\ell)))
\]

\[
= ([x, y], \gamma(\tau[x, y]))
\]

by the chain condition. \( \square \)

2.2 Proof for \( n = 1 \)

First, we will show the \( n = 1 \) case, that is, that \((1, -1) \in M := [\text{Mp}(2, F), \text{Mp}(2, F)]\).

Let \( a \in F^\times \) be arbitrary, and let \( c \in F^\times \) such that \( c^2 \neq 1 \). By direct computation,

\[
\left[ \begin{bmatrix} c^{-1} & 1 \\ 0 & 1 \end{bmatrix}, \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a/(c^2 - 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \right] = \begin{bmatrix} 1 \\ a \end{bmatrix}. \]
Since \( \ell \) is the span of the first basis vector, \( A\ell \) only depends on the first column of \( A \) (for any \( A \in \text{Sp}(V) \)). So

\[
\gamma \left( \tau \left( \begin{bmatrix} c^{-1} & 1 \\ a/(c^2 - 1) & 1 \end{bmatrix} \right) \right) = \gamma \left( \tau \left( \begin{bmatrix} \ell & 1 \\ \ell & 1 \end{bmatrix} \right) \right)
\]

\[
= \gamma \left( \tau \left( \begin{bmatrix} \ell & 1 \\ \ell & 1 \end{bmatrix} \right) \right)
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\]

\[
= \gamma \left( \tau \left( \begin{bmatrix} \ell & 1 \\ \ell & 1 \end{bmatrix} \right) \right)
\]

using Lemma 1.1 for the last step. Since the Weil index is invariant under multiplication by elements of \((F^\times)^2\), we have

\[
\gamma \left( \begin{bmatrix} a ac^2/c^2-1 & a - ac^2/c^2-1 \\ c^2/c^2-1 & c^2/c^2-1 \end{bmatrix} \right) = \gamma \left( \begin{bmatrix} a c^2/c^2-1 & c^2/c^2-1 \end{bmatrix} \right) = \gamma \left( \begin{bmatrix} c^2/c^2-1 & c^2/c^2-1 \end{bmatrix} \right) = \gamma(-a).
\]

Therefore, for any \( a \in F^\times \), by Lemma 2.2,

\[
\left( \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix} , \gamma(-a) \right) \in M. \tag{2.2.1}
\]

Using analogous commutators for unit-diagonal upper-triangular matrices, we find that

\[
\left( \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix} , 1 \right) \in M, \tag{2.2.2}
\]

because any upper-triangular matrix in \( \text{Sp}(V) \) leaves \( \ell \) fixed.

Let \( \alpha \in F^\times \) be arbitrary, and denote

\[
g_\alpha := \begin{bmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{bmatrix}.
\]

By direct computation,

\[
g_\alpha = \begin{bmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

By multiplying the commutators from (2.2.1) and (2.2.2), we obtain

\[
(g_\alpha, \sigma_\alpha) \in M,
\]

where

\[
\sigma_\alpha(\ell) = 1 \cdot \gamma(\alpha^{-1}) \cdot 1 \cdot 1 \cdot \gamma(-1) \cdot 1 \cdot \gamma(T) = \gamma(\alpha)\gamma(-1)\gamma(T),
\]

\[
T = \tau \left( \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix} \ell, \begin{bmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{bmatrix} \ell, \begin{bmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{bmatrix} \ell, \begin{bmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{bmatrix} \ell, \begin{bmatrix} \alpha & \alpha^{-1} \\ \alpha^{-1} & \alpha \end{bmatrix} \ell \right)
\]

\[
= \tau \left( \begin{bmatrix} \ell & \ell \\ \ell & 1 \end{bmatrix} \right) = 0.
\]
So $\sigma_\alpha(\ell) = \gamma(\alpha)\gamma(-1)$.

By (1.1.1), for any $\alpha, \beta \in F^\times$, 
\[
\frac{\gamma(\alpha)\gamma(\beta)\gamma(-1)}{\gamma(\alpha\beta)} = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha\beta)\gamma(1)} = (\alpha, \beta),
\]
and so 
\[
\gamma(\alpha)\gamma(\beta)\gamma(-1) = (\alpha, \beta)\gamma(\alpha\beta) = (\alpha, \beta)\gamma(\alpha\beta).
\]

Hence 
\[
(g_{\alpha\beta}, \sigma_\alpha \sigma_\beta \cdot c_{g_{\alpha\beta}}) = (g_\alpha, \sigma_\alpha) \cdot (g_\beta, \sigma_\beta) \in M.
\]

Since $g_\alpha, g_\beta$ are upper triangular, 
\[
c_{g_{\alpha\beta}}(\ell) = \gamma(\tau(\ell, g_\alpha \ell, g_\alpha g_\beta \ell)) = \gamma(\tau(\ell, \ell, \ell)) = \gamma(0) = 1.
\]

Moreover, 
\[
\sigma_\alpha(\ell)\sigma_\beta(\ell) = \gamma(\alpha)\gamma(-1)\gamma(\beta)\gamma(-1) = (\alpha, \beta)\gamma(\alpha\beta)\gamma(-1) = (\alpha, \beta)\sigma_\alpha(\ell).
\]

Thus, denoting $g := g_{\alpha\beta}$, 
\[
(g, \sigma_\alpha\beta), (g, (\alpha, \beta)\sigma_\alpha\beta) \in M.
\]

So 
\[
(1, (\alpha, \beta)) = \left( gg^{-1}, (\alpha, \beta)\sigma_\alpha\beta\sigma_\alpha^{-1} g g^{-1} \right) = (g, \sigma_\alpha\beta) \cdot (g, (\alpha, \beta)\sigma_\alpha\beta)^{-1} \in M.
\]

Since $F$ has nontrivial Hilbert symbol, there exist $\alpha, \beta \in F^\times$ such that $(\alpha, \beta) = -1$, completing the proof for $\dim V = 2$.

### 2.3 Proof of the general case

The general case easily follows. Let $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ be a symplectic basis for $V$. Then $\text{Sp}(2, F)$ embeds as a subgroup of $\text{Sp}(V)$ by 
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [A_{i,j}]_{i,j=1}^{2n}, \quad (2.3.1)
\]

where $A_{1,1} = a$, $A_{1,n+1} = b$, $A_{n+1,1} = c$, $A_{n+1,n+1} = d$, and $A_{i,j} = \delta_{ij}$ for all other $i, j$.

Let $\ell = \text{span}(p_1, \ldots, p_n)$. Since any matrices $A, B$ as in Equation (2.3.1) act trivially on the subspace $\text{span}(p_2, \ldots, p_n)$, the Maslov index $\tau(\ell, A\ell, B\ell)$ is 1-dimensional, and the same proof used for Lemma 1.1 applies.

Observe that the proof that $(1, -1)$ is in the commutator subgroup of $\text{Mp}(2, F)$ only relies on Lemma 1.1 and multiplication in $\text{Sp}(2, F)$. So, using the same argument, but replacing each $2 \times 2$ matrix with the associated $2n \times 2n$ matrix given by Equation (2.3.1), we see that $(1, -1)$ is in the commutator subgroup of $\text{Mp}(V)$.

### 3 Implications for the Weil representation

A key property of the Weil representation follows from the above result.

**Corollary 3.1.** The central extension 
\[
1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Mp}(V) \longrightarrow \text{Sp}(V) \longrightarrow 1
\]
does not split.
Proof. Suppose the extension splits. Then $Mp(V)$ is isomorphic to a semidirect product of $\mathbb{Z}/2\mathbb{Z}$ and $Sp(V)$. Since the extension is central, $Sp(V)$ acts trivially on $\mathbb{Z}/2\mathbb{Z}$, and so $Mp(V) \cong \mathbb{Z}/2\mathbb{Z} \times Sp(V)$. But $\mathbb{Z}/2\mathbb{Z}$ is abelian, so this contradicts the fact that $Mp(V)$ is equal to its commutator subgroup.

Corollary 3.2. The projective Weil representation $\omega : Sp(V) \rightarrow PGL(W)$ does not lift to a linear representation of $Sp(V)$.

Proof. We have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
1 & \rightarrow & \mathbb{C}^\times & \rightarrow & \widehat{Sp}_\psi(V) & \xrightarrow{\pi} & Sp(V) & \rightarrow & 1 \\
\| & & \downarrow{\sim} & & \downarrow{\omega} & & \downarrow{\omega} & & \downarrow{1} \\
1 & \rightarrow & \mathbb{C}^\times & \rightarrow & GL(W) & \rightarrow & PGL(W) & \rightarrow & 1
\end{array}
$$

Here, $\widehat{Sp}_\psi(V)$ is as defined in [2] (§4.1.1, p. 45). Suppose there exists a homomorphism $\eta : Sp(V) \rightarrow GL(W)$ lifting $\omega$. Since $\sim$ is the projection onto the second coordinate of $\widehat{Sp}_\psi(V) \subset Sp(V) \times GL(W)$, the map

$$
Sp(V) \rightarrow \widehat{Sp}_\psi(V) \\
g \mapsto (g, \eta(g))
$$

is a homomorphism splitting $\pi$. Then $\widehat{Sp}_\psi(V) \cong Sp(V) \times \mathbb{C}^\times$. But by [2] (§4.3.4, p. 52), $Mp(V)$ embeds as a subgroup of $\widehat{Sp}_\psi(V)$, so this contradicts Corollary 3.1. Hence $\omega$ does not lift to a linear representation of $Sp(V)$.

References

