Frobenius Manifold of K3 LG-model
Rachel Suggs

Abstract

The Landau-Ginzburg mirror symmetry conjecture says that for a quasihomogeneous singularity $W$ and a group $G$ of symmetries of $W$, there is a dual singularity $W^T$ and group $G^T$ such that the Landau-Ginzburg A-model for $W$ and $G$ is isomorphic to the Landau-Ginzburg B-model for $W^T$ and $G^T$. The Landau-Ginzburg A-model is given by the FJRW theory, set forth in [FJR07]. The FJRW construction has been completely computed for a few examples (see [FJR07] and [KS11]). In particular, this construction has a Frobenius manifold structure, determined by the values of genus-zero correlators, which is in general difficult to compute. In this report, we will show how to find the values of all genus-zero correlators for some polynomials defining K3 LG-surfaces.

1 Background

This report deals with the FJRW construction, which uses an invertible polynomial and group of diagonal symmetries fixing that polynomial to construct a Frobenius manifold. For the sake of brevity, we will not give a thorough explanation of the FJRW construction here, but will instead recall those definitions and theorems we will use most frequently. To learn more about the FJRW construction, see [FJR07]. For a simpler introduction to the construction at the Frobenius algebra level, see [KPA+10].

In this paper we will be working with K3 LG-surfaces, which are given by invertible quasihomogeneous polynomials satisfying the Calabi-Yau condition $\sum q_j = 1$. Here, $q_j$ is the $j$th weight of the quasihomogeneous polynomial defining the surface.

We begin with some definitions.

Definition 1.1. For a quasihomogeneous polynomial $W$ in $n$ variables with weights $q_1, q_2, \ldots, q_n$, the central charge $\hat{c}_W$ is defined to be

$$\hat{c}_W = \sum_{j=1}^{n} (1 - 2q_j).$$

Note that for a K3 LG-surface, we have

$$\hat{c}_W = \sum_{j=1}^{n} (1 - 2q_j) = n - 2 \sum_{j=1}^{n} q_j = 4 - 2 = 2.$$

As we said, the FJRW construction requires an invertible polynomial $W$ and a group of diagonal symmetries $G$ that fixes $W$. This group can be written multiplicatively, with elements represented as $\gamma = (e^{2\pi i \Theta_1}, e^{2\pi i \Theta_2}, \ldots, e^{2\pi i \Theta_n})$; however, we will write it additively as $\gamma = (\Theta_1, \Theta_2, \ldots, \Theta_n)$, with the convention that $0 \leq \Theta_j < 1$. The state space $\mathcal{H}_{W,G}$ has the structure of a graded Frobenius algebra. A basis element of $\mathcal{H}_{W,G}$ coming from the sector corresponding to $\gamma \in G$ will be written as $m e_\gamma$, where $m$ is a monomial in the Milnor ring of $W$ restricted to the fixed locus of $\gamma$. We say the sector corresponding to $\gamma = (\Theta_1, \Theta_2, \ldots, \Theta_n)$ is narrow if for each $i$, $\Theta_i \neq 0$; we call it broad otherwise. The grading on $\mathcal{H}_{W,G}$ is given by the following definition.

Definition 1.2. For an element $\alpha$ in $\mathcal{H}_{W,G}$ coming from the sector $\gamma = (\Theta_1, \Theta_2, \ldots, \Theta_n)$, the $W$-degree of $\alpha$ is

$$\deg_W(\alpha) = N + 2 \sum_{j=1}^{n} (\Theta_j - q_j),$$

where $N$ is the dimension of the fixed locus of $\gamma$; in other words, the number of coordinates where $\gamma$ acts trivially.

The FJRW Frobenius manifold structure is determined by genus-zero $k$-point correlators. For $k$ elements $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{H}_{W,G}$, these are defined to be the integrals of certain classes over $\overline{\mathcal{M}}_{0,k}$:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle_{0,k} = \int_{\overline{\mathcal{M}}_{0,k}} \Lambda_W^{\alpha_1, \alpha_2, \ldots, \alpha_k}.$$
In practice, a $k$-point correlator can often be computed from various axioms. Here are three which we will use.

**Axiom 1.3** (Dimension Axiom [KPA$^+$10]). A genus-zero $k$-point correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle_{0,k}$ vanishes unless

$$\frac{1}{2} \sum_{i=1}^{k} \deg_{W}(\alpha_i) = c_W + k - 3.$$ 

A second definition that assists us in determining nonzero correlators is the line bundle degree.

**Definition 1.4.** The $j^{th}$ line bundle degree of a correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle_{0,k}$ is defined to be

$$l_j = q_j(k - 2) - \sum_{i=1}^{k} \Theta_{j}^{\gamma_i}, \quad (1)$$

where $\alpha_i$ comes from the sector corresponding to the element $\gamma_i = (\Theta_1^{\gamma_i}, \Theta_2^{\gamma_i}, \ldots, \Theta_{n}^{\gamma_i})$.

**Axiom 1.5** (Line Bundle Degrees Axiom [KPA$^+$10]). A correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle_{0,k}$ vanishes unless all its line bundle degrees are integers.

Finally, we will use the following axiom to compute those correlators which do not vanish by Axioms 1.3 or 1.5.

**Axiom 1.6** (Concavity Axiom [KPA$^+$10][FJR07]).

1. If a genus-zero three-point correlator $\langle e_{\gamma_7}, e_{\gamma_2}, e_{\gamma_3} \rangle_{0,3}$ does not vanish by the Dimension Axiom (Axiom 1.3) or the Line Bundle Degrees Axiom (Axiom 1.5), and if each $e_{\gamma_i}$ comes from a narrow sector, then if $l_j < 0$ for each $j$, we have $\langle e_{\gamma_1}, e_{\gamma_2}, e_{\gamma_3} \rangle_{0,3} = 1.$

2. Let $\langle e_{\gamma_1}, e_{\gamma_2}, e_{\gamma_3}, e_{\gamma_4} \rangle_{0,4}$ be a genus-zero four-point correlator that does not vanish by the Dimension Axiom (Axiom 1.3) or the Line Bundle Degrees Axiom (Axiom 1.5), with each $e_{\gamma_i}$ coming from a narrow sector. Consider the three graphs in Figure 1.

![Figure 1: The graphs $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ define $g_1$, $g_2$, and $g_3$ respectively.](image)

Each of the graphs $\Gamma_i$ determines a unique element $g_i \in G_{W}^{\max}$ such that the triples corresponding to either half of the graph have integer line bundle degrees. For example, if $i = 1$, $g_1$ satisfies the condition that $\langle \gamma_1, \gamma_2, g_1 \rangle$ and $\langle -g_1, \gamma_3, \gamma_4 \rangle$ have integer line bundle degrees. If all the line bundle degrees corresponding to these six triples are strictly negative, and if the line bundle degrees of the original four-point correlator are also negative, then the value of the correlator is given by the following formula:

$$\langle e_{\gamma_1}, e_{\gamma_2}, e_{\gamma_3}, e_{\gamma_4} \rangle = \frac{1}{2} \sum_{j=1}^{n} \left[ q_j^2 - q_j \right] + \sum_{i=1}^{k} \left( \Theta_j^{\gamma_i} (1 - \Theta_j^{\gamma_i}) \right) - \sum_{l=1}^{3} \left( \Theta_l^{\gamma_l} (1 - \Theta_l^{\gamma_l}) \right) \quad (2)$$

We will now introduce a lemma which will greatly reduce the number of correlators we need to compute using the above Concavity Axiom. Its statement first requires two definitions.

**Definition 1.7.** An element $\alpha \in H_{W,G}$ is primitive if whenever $\alpha = \beta \ast \gamma$ for some $\beta, \gamma \in H_{W,G}$, either $\beta \in \mathbb{C}$ or $\gamma \in \mathbb{C}$.

**Definition 1.8.** A correlator $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle_{0,k}$ is called basic if $\alpha_i$ is primitive for $i \leq k - 2$. 


Theorem 1.9 (Reconstruction Lemma [FJR07]). A genus-zero $k$-point correlator $\langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle$ satisfies

$$\langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle = S + \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha, \epsilon, \beta \star \phi \rangle + \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha \star \epsilon, \beta, \phi \rangle - \langle \gamma_1, \ldots, \gamma_{k-3}, \alpha \star \beta, \epsilon, \phi \rangle, \tag{3}$$

where $S$ is a linear combination of genus-zero correlators with fewer than $k$ insertions. In addition, the genus-zero $k$-point correlators are uniquely determined by the pairing, the three-point correlators, and by basic $l$-point correlators for $l \leq k$.

The Reconstruction Lemma allows us to compute just the basic correlators.

2 Some Preliminaries

The purpose of this section is to explain the strategy we will use in the computations in the next two sections. We will begin by reviewing some properties of tensor products of FJRW rings. One fact is the following:

Theorem 2.1 ([FJJS11]). Let $W$ be an invertible polynomial and let $G^\text{max}_W$ be its maximal symmetry group. Suppose $W$ can be written as the sum of two invertible polynomials $W = W_1 + W_2$ with distinct variables. Then

$$H_{W,G^\text{max}} \cong H_{W_1,G^\text{max}_{W_1}} \otimes H_{W_2,G^\text{max}_{W_2}}$$

via the isomorphism $(m_1 e_{\gamma_1}, m_2 e_{\gamma_2}) \mapsto m_1 m_2 e_{\gamma_1 + \gamma_2}$. When we write the sum $\gamma_1 + \gamma_2$, we are thinking of the $\gamma_i$ as elements of $G^\text{max}$ via inclusion.

If $W = W_1 + W_2 + \ldots + W_r$, we can identify $H_{W,G^\text{max}}$ with $H_{W_1,G^\text{max}_{W_1}} \otimes \ldots \otimes H_{W_r,G^\text{max}_{W_r}}$ via this isomorphism. One consequence of this theorem is that to calculate a basis for $W$, we only need calculate bases for $W_1, \ldots, W_r$. Another consequence is that primitive elements in $H_{W,G^\text{max}_W}$ will be of the form $(1, \ldots, 1, p_{W_i}, 1, \ldots, 1)$, where 1 represents the identity in the appropriate ring, and $p_{W_i}$ is a primitive element in $H_{W_i,G^\text{max}_{W_i}}$. Also, from the definition of the $W$-degree of an element $\alpha \in H_{W,G^\text{max}_W}$, it is clear that if $\alpha \mapsto (\alpha_1, \ldots, \alpha_r)$ in the isomorphism of Theorem 2.1, then $\deg_W(\alpha) = \sum_{i=1}^r \deg_{W_i}(\alpha_i)$.

We will want to make use of the following lemma:

Lemma 2.2 ([FJR07]). Assume $\deg_W(\alpha) \leq \hat{c}$ for all $\alpha \in H_{W,G^\text{max}_W}$, and let $P$ be the maximum $W$-degree of any primitive element. Then all genus-zero correlators are determined by the pairing, the three-point correlators, and the basic $k$-point correlators with

$$k \leq 2 + \frac{1 + \hat{c}}{1 - \frac{c}{2}}.$$

To compute $P$, we can use the fact discussed above that when $W$ is a sum of polynomials $W_1 + W_2 + \ldots + W_r$, the $W$-degree is additive. Since the degree of the identity element 1 is 0, this means that the degrees of primitive elements in $W$ are just the degrees of primitive elements in the $W_i$ summands. Thus, to find $P$ for a polynomial $W = W_1 + \ldots + W_r$, we can calculate $P$ for each of the $W_i$ and then take the maximum of these.

Finally, the following theorem is a useful combination of Axioms 1.5 and 1.3:

Lemma 2.3. A non-vanishing genus-zero $k$-point correlator corresponding to a polynomial $W$ in $n$ variables must satisfy

$$\frac{1}{2} \sum_{i=1}^k N_i - \sum_{j=1}^n l_j = k + n - 3 \tag{4}$$

where $l_j$ is the $j^{th}$ line bundle degree, and $N_i$ is the dimension of the fixed locus of the group element corresponding to the $i^{th}$ insertion in the correlator.
Proof. We can combine Axiom 1.3 and Definition 1.2 to get the following, which must be satisfied by any nonvanishing $k$-point correlator:

$$\frac{1}{2} \sum_{i=1}^{k} \left( N_i + 2 \sum_{j=1}^{n} (\Theta_j^{i} - q_j) \right) = \hat{c} + k - 3 \tag{5}$$

Also, we can rewrite Equation (1), which defines the line bundle degrees, in the following way:

$$\sum_{i=1}^{k} \Theta_j^{i} = (k - 2)q_j - l_j. \tag{6}$$

Hence, when we combine Equations (5) and (6), using the fact that $\sum_j q_j = 1$ for a K3 surface, we find that a nonvanishing correlator must satisfy

$$\frac{1}{2} \sum_{i=1}^{k} N_i + \sum_{j=1}^{n} \left( \sum_{i=1}^{k} \Theta_j^{i} - \sum_{i=1}^{k} q_j \right) = \hat{c} + k - 3$$

$$\frac{1}{2} \sum_{i=1}^{k} N_i + \sum_{j=1}^{n} ((k - 2)q_j - l_j - kq_j) = \hat{c} + k - 3$$

$$\frac{1}{2} \sum_{i=1}^{k} N_i + \sum_{j=1}^{n} (-2q_j - l_j) = \sum_{j=1}^{n} (1 - 2q_j) + k - 3$$

$$\frac{1}{2} \sum_{i=1}^{k} N_i - \sum_{j=1}^{n} l_j = \hat{c} + k - 3$$

Because it is helpful to work with tensor products of FJRW rings, we will find it useful to think of correlators as grids, with rows corresponding to insertions in the correlator and columns corresponding to atomic polynomials in a decomposition of $W$. If $W = W_1 + W_2 + \ldots + W_r$, and if $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir}) \in H_{W_1,G^\text{max}_W} \otimes H_{W_2,G^\text{max}_W} \otimes \ldots \otimes H_{W_r,G^\text{max}_W}$ maps to $\alpha_i \in H_{W,G^\text{max}_W}$, we will represent the genus-zero correlator $\langle \alpha_{11}, \alpha_{12}, \ldots, \alpha_{kr} \rangle$ with the grid

$$\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1r} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{k1} & \alpha_{k2} & \ldots & \alpha_{kr}
\end{bmatrix}$$

Note that in this notation, it is easy to identify a basic correlator. Such a correlator will have exactly one primitive element in each of the top $k - 2$ rows, and the remaining entries in these top rows will all be $1$. In what follows, a basic column refers to a column in such a grid as above, which corresponds to a basic correlator. The basic elements of a basic column are the first $k - 2$ entries of the column, and must be either $1$ or primitive.

Now that we have the language of these grids, we want to reconsider the statement of Lemma 2.3 looks like when we have a sum $W = W_1 + W_2 + \ldots + W_r$ of invertible polynomials. Suppose we have a nonvanishing correlator $\langle \alpha_{11}, \alpha_{12}, \ldots, \alpha_{kr} \rangle$, and suppose $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir})$ maps to $\alpha_i$ in the isomorphism of Theorem 2.1. Note that we can break up the left hand side of Equation (4) as a sum over the polynomials in the following way: if $W_l$ corresponds to coordinates $a_{l1}$ through $b_{lr}$, then

$$\frac{1}{2} \sum_{i=1}^{k} N_i - \sum_{j=1}^{n} l_j = \sum_{l=1}^{r} \left( \frac{1}{2} \sum_{i=1}^{k} N_i^l - \sum_{j=1}^{b_{l}} l_j \right)$$
where $N_i^l$ is the dimension of the fixed locus of the element $\alpha_{l, l}$. Since each atomic polynomial $W_i$ corresponds to a column in the correlator, we call the quantity
\[
\frac{1}{2} \sum_{i=1}^{k} N_i^l - \sum_{j=\alpha_{l, l}}^{l} l_j
\]
the contribution of the $l^b$ column. Then using the fact that for a K3 LG-surface, $n = 4$, Lemma 2.3 says that in this case a nonvanishing correlator must have the contributions of its columns summing to $k + 1$.

The remainder of this paper will look at two different types of K3 LG-surfaces. For each type, our strategy will be to first compute the state space $\mathcal{H}_{W, G}$ of the polynomial, and then to use Lemma 2.2 to find an upper bound on the number of insertions we need to consider. For each $k$ less than this bound, we will compute all basic columns that have integer line bundle degrees. We will then compute the contribution of each column, and using the fact that the contributions of the columns must sum to $k + 1$, we will write down a list of possibly nonzero basic correlators. Next, we will use the Reconstruction Lemma to further reduce the list of correlators we actually need to compute. Finally, we will use the Concavity Axiom to compute the remaining basic correlators.

### 3 K3 LG-Surfaces of the Form $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4}$

In this section we will examine the so-called Fermat K3 LG-surfaces, those with $W = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4}$. There are 14 of these, listed in Table 1 [Pri11]. They can be calculated from a list of possible quasihomogeneous weight systems such as that found in [Ska96].

\[
\begin{array}{cccc}
  x^4 + y^4 + z^4 + w^4 & x^{10} + y^{15} + z^6 + w^3 & x^{12} + y^2 + z^4 + w^6 \\
  x^{12} + y^{12} + z^2 + w^3 & x^2 + y^6 + z^6 + w^6 & x^2 + y^4 + z^8 + w^8 \\
  x^3 + y^4 + z^4 + w^6 & x^3 + y^3 + z^5 + w^6 & x^20 + y^2 + z^4 + w^5 \\
  x^{42} + y^2 + z^4 + w^7 & x^{10} + y^2 + z^5 + w^5 & x^{18} + y^2 + z^3 + w^9 \\
  x^{24} + y^2 + z^3 + w^8 & x^{12} + y^3 + z^3 + w^4 &
\end{array}
\]

Table 1: All K3 LG-surfaces of the form $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4}$

#### 3.1 The state space and its product structure

Since we can write $W = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4}$ as the sum of four invertible polynomials, each of the form $W_i = x_i^{a_i}$, by the above discussion on tensor products, we only need to understand the state space and product structure of $W = x^a$. We will compute this briefly here; for more details about constructing FJRW rings, see [KPA10].

Let $W = x^a$. Then the group $G_{\text{max}}$ is generated by the element $e^{2\pi i a^{-1}}$. The Milnor ring of $W$ is spanned by the elements $\{1, x, \ldots, x^{a-2}\}$. Thus, the action of $e^{2\pi i a^{-1}}$ on an element $x^r e_0$ coming from this sector will be $\frac{x^r}{a^2}$, which cannot be an integer, since $0 \leq r \leq a - 2$. Thus this sector contributes nothing to $\mathcal{H}_{W, G_{\text{max}}}$, and a basis for the state space is given by the elements
\[
\left\{ e^{\frac{a}{2}}, e^{\frac{a}{3}}, \ldots, e^{\frac{a-1}{a}} \right\}.
\]

It remains to determine the product structure. By definition,
\[
e^{\frac{a}{2}} \star e^{\frac{a}{3}} = \sum_{\sigma, \tau} (e^{\frac{a}{2}}, e^{\frac{a}{3}}, \sigma) \eta^{\sigma \tau}.
\]

If $r_1 + r_2 > a$, then for any $\sigma = e^{\frac{a}{2}}$, we calculate the line bundle degree of the correlator $(e^{\frac{a}{2}}, e^{\frac{a}{3}}, \sigma)$, using 1 to be
\[
l_1 = q_j(k - 2) - \sum_{j=1}^k \Theta_j^k = \frac{1 - r_1 - r_2 - r_3}{a} < -1,
\]
since we also know \( r_3 > 1 \). Thus, if \( l_1 \in \mathbb{Z} \) as required by Axiom 1.5, we must have \( l_1 \leq -2 \). On the other hand, Lemma 2.3 requires
\[
\frac{1}{2} \sum_{i=1}^{k} N_i - \sum_{j=1}^{n} l_j = k + n - 3
\]
which implies
\[-l_1 = 3 + 1 - 3 = 1.\]
This is a contradiction; hence all such correlators must vanish, and \( e_{\frac{a}{2}} \ast e_{\frac{b}{2}} = 0 \).

Now, if \( r_1 + r_2 \leq a \), using line bundle degrees and the definition of the pairing we see that the only nonzero term in the sum defining their product is
\[
\left\langle e_{\frac{a_1}{2}}, e_{\frac{a_2}{2}}, e_{\frac{a_1-a_2}{2}}, e_{\frac{a_1}{2}+1} \right\rangle \eta e_{\frac{a_1-a_2}{2}} \ast e_{\frac{a_1}{2}+1} = e_{\frac{a_1+1}{2}-1}.
\]
We can use part (1) of the Concavity Axiom (Axiom 1.6) to conclude that the correlator in the equation above is 1, so since the pairing is also 1, \( e_{\frac{a_1}{2}} \ast e_{\frac{b_1}{2}} = e_{\frac{a_1+1}{2}-1} \).

With this information, it is easy to see that the identity is given by \( 1 = e_{\frac{1}{2}} \), and that if \( a > 2 \) the unique primitive element is given by \( p_W = e_{\frac{a}{2}} \). If \( a = 2 \), the ring \( \mathcal{H}_{W,GW^*} \) contains only the identity.

The element of highest \( W \)-degree is \( h_W = e_{\frac{a}{2}} \), and this element has the property that for \( \alpha \in \mathcal{H}_{W,GW^*} \),
if \( \alpha \neq 1 \), \( h_W \ast \alpha = 0 \).

### 3.2 Bound on \( k \)

In order to use Lemma 2.2 to bound \( k \), we need to calculate the maximum \( W \)-degree of any primitive element in \( \mathcal{H}_{W_j,GW_j^*} \). There is only one primitive element in \( \mathcal{H}_{W_j,GW_j^*} \), and that is \( e_{2/a_j} \), with \( W \)-degree
\[
deg_W (e_{\frac{a_j}{2}}) = N + 2 \sum_j (\Theta_j - q_j) = 0 + 2 \left( \frac{2}{a_j} - \frac{1}{a_j} \right) = \frac{2}{a_j}.
\]
Hence, the value of \( P \) for the larger polynomial \( W = \sum_j W_j \) is equal to the maximum of \( \frac{2}{a_j} \). Since the smallest \( a_j \) can be is 2, and we can check in Table 1 that this value does occur, we can use \( P = \frac{2}{2} = 1 \) to calculate a bound on \( k \) for any Fermat K3 LG-surface:
\[
k \leq 2 + \frac{1 + \hat{e}}{1 - \frac{2}{2}} = 2 + \frac{1 + 2}{1 - \frac{2}{2}} = 2 + 3 \cdot 2 = 8.
\]

### 3.3 Four-point correlators

Following the strategy outlined at the end of Section 2, we want to find all basic columns with integer line bundle degrees and compute their contributions. For a column corresponding to a Fermat type polynomial, the dimension of the fixed locus of the elements is always zero, and the column corresponds to only one coordinate. Thus, for a polynomial \( W = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4} \), there are four columns, and the contribution of a column is just the negative of the line bundle degree of that coordinate. We need to find columns yielding \(- \sum_{j=1}^{4} l_j = k + 1 \).

**Proposition 3.1.** Consider a four-point correlator associated with to \( \mathcal{H}_{W_j,GW_j^*} \). For some fixed \( j \), assume \( l_j \in \mathbb{Z} \), and assume \( x_j^{a_j} \) appears as a Fermat summand of \( W \). Then,
- \( l_j = -1 \) or \(-2 \)
- If \( a_j = 2 \), then \( l_j = -1 \)
- If \( a_j \neq 2 \), \((pW_j, pW_j, hW_j, hW_j)^T\) is the unique basic column with \( l_j = -2 \).
Proof. Denote the \(j^{th}\) column by \((e_{r_j^1/aj}, e_{r_j^2/aj}, e_{r_j^3/aj}, e_{r_j^4/aj})^T\), where \(1 \leq r_j^i \leq a_j - 1\). By the definition of line bundle degrees, we have

\[
l_j = a_j(k - 2) - \sum_{i=1}^{k} \Theta_j^i = \frac{1}{a_j}(2 - 4)r_j^i).
\]

Since each \(r_j^i\) is at least one, this sum is clearly negative, and hence must be at most \(-1\) if it is an integer. If \(a_j = 2\), \(r_j^i = 1\) for each \(i\), so \(l_j = \frac{1}{2}(2 - 4) = 1\). If \(a_j \neq 2\), then \(l_j\) is minimized by the column \((e_{\frac{1}{aj}}, e_{\frac{2}{aj}}, e_{\frac{a_j-1}{aj}}, e_{\frac{a_j}{aj}})^T\). By Lemma 2.3, when \(a_j = 2\), \(r_j^i = 1\) for each \(i\), so \(l_j = \frac{1}{2}(2 - 4) = 1\). If \(a_j \neq 2\), then \(l_j\) is minimized by the column \((e_{\frac{1}{aj}}, e_{\frac{2}{aj}}, e_{\frac{a_j-1}{aj}}, e_{\frac{a_j}{aj}})^T\). In this case, \(l_j = -2\).

In what follows, we will identify elements \(\alpha \in \mathcal{H}_{W_j,G_{W_j}^{max}}\) with their images

\[
(1, \ldots, \alpha, \ldots, 1) \in \mathcal{H}_{W_j,G_{W_j}^{max}} = \mathcal{H}_{W_1,G_{W_1}^{max}} \otimes \cdots \otimes \mathcal{H}_{W_j,G_{W_j}^{max}} \otimes \cdots \otimes \mathcal{H}_{W_k,G_{W_k}^{max}}.
\]

Also, note that the element \(h_W\) of highest \(W\)-degree in \(\mathcal{H}_{W_j,G_{W_j}^{max}}\) is equal to \((h_W, h_{W_2}, h_{W_3}, h_{W_4}) \in \mathcal{H}_{W_1,G_{W_1}^{max}} \otimes \cdots \otimes \mathcal{H}_{W_4,G_{W_4}^{max}}\).

Theorem 3.2. For \(W = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4}\), the genus-zero four-point correlators are uniquely determined by the pairing, the three-point correlators, and the correlators

\[
\langle p_{W_j}, p_{W_j}, h_{W_j}, h_W \rangle
\]

for \(a_j \neq 2\).

Proof. By Lemma 2.3, when \(k = 4\), we want the contributions of our columns to add up to 5. Then Proposition 3.1 implies that exactly one line bundle degree must be \(-2\), while the others must be \(-1\). For a moment, let us take \(l_1 = -2\) (note that this implies \(a_1 \neq 2\)). The other cases are analogous. Proposition 3.1 then allows us to fill in the first column of the correlator grid:

\[
\begin{pmatrix}
p_{W_1} \\
p_{W_1} \\
h_{W_1} \\
h_{W_1}
\end{pmatrix}.
\]

Since we want this to be a basic correlator, the primitive elements in the first column determine the remaining elements in the first two rows:

\[
\begin{pmatrix}
p_{W_1} & 1 & 1 & 1 \\
p_{W_1} & 1 & 1 & 1 \\
h_{W_1} & r_{11} & r_{12} & r_{13} \\
h_{W_1} & r_{21} & r_{22} & r_{23}
\end{pmatrix}.
\]

Here, we identify \(r_{ij}\) with \(e_{r_{ij}}\). By line bundle degrees, \(-1 = \frac{1}{aj}(2 - 1 - 1 - r_{ij})\) which implies \(r_{ij} + r_{2j} = a_j\) But as seen in section 3.1, this is exactly the requirement that \(r_{ij} \ast r_{2j} = h_{W_j}\). This allows us to use the Reconstruction Lemma (Theorem 1.9). Set \(\alpha = (p_{W_1}, 1, 1, 1)\), \(\beta = (h_{W_1}, r_{21}, r_{22}, r_{23})\), \(\epsilon = (h_{W_1}, 1, 1, 1)\), and \(\phi = (h_{W_1}, r_{11}, r_{12}, r_{13})\). Then \(\epsilon \ast \phi = (h_{W_1}, r_{11}, r_{12}, r_{13})\), as required. Also, \(\alpha \ast \beta = 0\), \(\alpha \ast \epsilon = 0\), and \(\beta \ast \phi = (h_{W_1}, h_{W_2}, h_{W_3}, h_{W_4})\). Thus, all correlators of the form \(7\) are determined by the pairing, the three-point correlators, and the correlator

\[
\begin{pmatrix}
p_{W_1} & 1 & 1 & 1 \\
p_{W_1} & 1 & 1 & 1 \\
h_{W_1} & 1 & 1 & 1 \\
h_{W_1} & h_{W_2} & h_{W_3} & h_{W_4}
\end{pmatrix}.
\]

All that remains is to compute the values of the correlators \(\langle p_{W_j}, p_{W_j}, h_{W_j}, h_W \rangle\).
Theorem 3.3. The four-point correlator
\[ \langle p_{W_1}, p_{W_j}, h_{W_j}, h_W \rangle \]
takes the value \( \frac{1}{a_j} \). In particular it is nonvanishing.

Proof. Again, for simplicity we will take \( j = 1 \); the other cases are analogous. We want to calculate the correlator
\[ \left\langle e \left( \frac{2}{a_1} \cdot \frac{1}{a_2} \cdot \frac{1}{a_3} \cdot \frac{1}{a_4} \right) \right\rangle \left\langle e \left( \frac{2}{a_1} \cdot \frac{1}{a_2} \cdot \frac{1}{a_3} \cdot \frac{1}{a_4} \right) \right\rangle \left\langle e \left( \frac{2}{a_1} \cdot \frac{1}{a_2} \cdot \frac{1}{a_3} \cdot \frac{1}{a_4} \right) \right\rangle \left\langle e \left( \frac{2}{a_1} \cdot \frac{1}{a_2} \cdot \frac{1}{a_3} \cdot \frac{1}{a_4} \right) \right\rangle . \]

All the group elements are narrow, and the line bundle degrees are \(-2, -1, -1, -1\). Since the correlator satisfies Equation (4), it also satisfies the Dimension Axiom, so we can begin to apply the second part of the Concavity Axiom (Axiom 1.6). The graphs \( \Gamma \) described in Axiom 1.6 are depicted in Figure 2. Here, we are identifying elements of \( H_{W,G_m} \) with the sectors they came from.

Using the line bundle degrees axiom, we can calculate that \( g_1 = \left( \frac{a_1 - 3}{a_1}, \frac{a_2 - 1}{a_2}, \frac{a_3 - 1}{a_3}, \frac{a_4 - 1}{a_4} \right) \) and \( g_2 = g_3 = \left( 0, \frac{a_2 - 1}{a_2}, \frac{a_3 - 1}{a_3}, \frac{a_4 - 1}{a_4} \right) \). All of the line bundle degrees associated with the triples described in Axiom 1.6 are strictly negative, so we can apply that axiom to conclude that the value of the correlator is equal to
\[ \frac{1}{2} \sum_{j=1}^{4} \left( q_j^2 - q_j \right) + \sum_{i=1}^{k} \left( \Theta_j^i (1 - \Theta_j^i) \right) - \sum_{i=1}^{3} \left( \Theta_j^0 (1 - \Theta_j^0) \right) . \]

We will compute the summand corresponding to each value of \( j \) over each coordinate individually, and then divide by 2. When \( j = 1 \), we have
\[
(q_1^2 - q_1) + \sum_{i=1}^{k} (\Theta_j^i (1 - \Theta_j^i)) - \sum_{i=1}^{3} (\Theta_j^0 (1 - \Theta_j^0)) \\
= \frac{1}{a_1^2} - \frac{1}{a_1} + \left[ \frac{2}{a_1} \left( \frac{a_1 - 2}{a_1} \right) + \frac{2}{a_1} \left( \frac{a_1 - 2}{a_1} \right) + \frac{1}{a_1} \left( \frac{a_1 - 1}{a_1} \right) + \frac{1}{a_1} \left( \frac{a_1 - 1}{a_1} \right) - \left[ \frac{3}{a_1} \left( \frac{a_1 - 3}{a_1} \right) + 0 + 0 \right] \\
= \frac{1}{a_1^2} (1 - a_1 + 4a_1 - 8 + 2a_1 - 2 - 3a_1 + 9) \\
= \frac{2a_1}{a_1^2} = \frac{2}{a_1}.
\]

On the other hand, when \( j \neq 1 \), we have
\[
(q_j^2 - q_j) + \sum_{i=1}^{k} (\Theta_j^i (1 - \Theta_j^i)) - \sum_{i=1}^{3} (\Theta_j^0 (1 - \Theta_j^0)) \\
= \frac{1}{a_j^2} - \frac{1}{a_j} + \left[ 4 \cdot \frac{1}{a_j} \left( \frac{a_j - 1}{a_j} \right) \right] - \frac{1}{2} \left[ 3 \cdot \frac{1}{a_j} \left( \frac{a_j - 1}{a_j} \right) \right] \\
= \frac{1}{a_j^2} (1 - a_j + a_j - 1) = 0.
\]

Thus, when we sum over all coordinates and divide by two, we find \( \langle p_{W_1}, p_{W_j}, h_{W_j}, h_W \rangle = \frac{1}{a_1} \). \( \square \)
3.4 Higher-point correlators

The goal of this section is to prove the following:

**Theorem 3.4.** All basic genus-zero \(k\) point correlators for \(5 \leq k \leq 8\) vanish.

First, we generalize Proposition 3.1 with the following:

**Proposition 3.5.** Let \(5 \leq k \leq 8\). For some fixed \(j\), assume \(l_j \in \mathbb{Z}\), and assume \(x_{aj}^j\) appears as a Fermat summand of \(W\). Then,

- \(-1 \geq l_j \geq \frac{4-k-2a_j}{a_j}\)
- If \(a_j = 2\), \(l_j = -1\).
- A basic column with \(l_j = -2\) (and \(a_j \neq 2\)) can have at most \(k-4\) elements equal to \(1\).

**Proof.** Denote the \(j\)th column by \((e_{r_1^j/a_j}, e_{r_2^j/a_j}, \ldots, e_{r_k^j/a_j})^T\), where \(1 \leq r_i^j \leq a_j-1\). By the definition of line bundle degrees, we have

\[
l_j = q_j^j(k-2) - \sum_{i=1}^{k} \Theta_i^j = \frac{1}{a_j}((k-2) - \sum_{i=1}^{k} r_i^j).
\]

Since \(r_i^j \geq 1\), this is clearly always negative, and hence less than or equal to \(-1\) if it is an integer. On the other hand, this quantity is minimized by the column \((e_{\frac{2}{a_j}}, e_{\frac{2}{a_j}}, \ldots, e_{\frac{a_j-1}{a_j}}, e_{\frac{a_j-1}{a_j}})^T\), in which case

\[
l_j = \frac{1}{a_j}((k-2) - 2(k-2) - 2(a_j-1)) = \frac{4-k-2a_j}{a_j},
\]

so we have the desired lower bound on \(l_j\).

If \(a_j = 2\), then \(r_i^j = 1\), and \(l_j = \frac{1}{2}((k-2) - k) = -1\).

Finally, note that \(l_j = -2\) is realized by the column \((e_{\frac{2}{a_j}}, e_{\frac{2}{a_j}}, 1, \ldots, 1, e_{\frac{a_j-1}{a_j}}, e_{\frac{a_j-1}{a_j}})^T\) with \(k-4\) identity elements. Now, if we wish to use another identity element, we must increase the subscript on one of the other elements in order to preserve \(l_j = -2\). But we cannot increase the subscript on the \(e_{\frac{2}{a_j}}\) elements, since that would make the column no longer basic, and we cannot increase the subscript on the \(e_{\frac{a_j-1}{a_j}}\) elements, because those would no longer be in the ring \(H_{W,G_{max}}\). Thus \(k-4\) is the maximum number of identity elements.

Now we prove Theorem 3.4:

**Proof.** Our proof will be a case-by-case consideration for each value of \(k\).

- **Case \(k = 5\):**

  By Lemma 2.3 we need \(-\sum_{j=1}^{4} l_j = 6\). Using the first part of Proposition 3.5, we see that integer values of \(l_j\) are \(-1\) or \(-2\), so the sum must split as \(2 + 2 + 1 + 1\). But, using the third part of the proposition, we see that we have at most one \(1\) in the first two columns of our correlator. This is not enough \(1\)'s to make a basic correlator, as is clear in the diagram below:

  \[
  \begin{bmatrix}
  p_{W_1} & p_{W_1} & * & * \\
  p_{W_1} & 1 & * & * \\
  1 & p_{W_1} & * & * \\
  h_{W_1} & h_{W_1} & * & * \\
  h_{W_1} & h_{W_1} & * & * 
  \end{bmatrix}
  \]

  Already, the top row of the correlator cannot correspond to a primitive element of \(H_{W,G_{max}}\). 


Case $k = 6$:
By Lemma 2.3 we need $-\sum_{j=1}^{4} l_j = 7$. Using the first part of Proposition 3.5, we see that integer values of $l_j$ are $-1$ or $-2$, so the sum must split as $2 + 2 + 2 + 1$. As with the case where $k = 5$, we can use the third part of the proposition to show that in this case, the first three columns do not have enough $1$'s to fill out a basic correlator.

Case $k = 7$:
By Lemma 2.3 we need $-\sum_{j=1}^{4} l_j = 8$. Using the first part of Proposition 3.5, we see that integer values of $l_j$ are $-1$, $-2$, or $-3$ (the latter can happen only in the case $a_j = 3$). Thus, the sum can split as $2 + 2 + 2 + 2$, as $3 + 2 + 2 + 1$, or as $3 + 3 + 1 + 1$. The first case can be ruled out by an argument similar to those used in cases $k = 5$ and $k = 6$. For the final two cases, note that $l_j = -3$ is in fact the absolute minimum value for $l_j$, so that $l_j = -3$ corresponds to the column $(e_{2/3}, e_{2/3}, \ldots, e_{2/3})^T$. This column has no $1$'s in it; by an argument similar to those already employed, the final two options will not have enough $1$'s for a basic correlator.

Case $k = 8$:
By Lemma 2.3 we need $-\sum_{j=1}^{4} l_j = 9$. Using the first part of Proposition 3.5, we see that integer values of $l_j$ are $-1$, $-2$, or $-3$ (if $a_j = 3, 4$). This means the sum can split as $3+3+2+1$ or $3+2+2+2$. It is easy to check that $l_j = -3$ is realized only for the columns $(e_{2/4}, \ldots, e_{2/4}, e_{3/4}, e_{3/4})^T$ (for $a_j = 4$), or $(e_{1/3}, e_{2/3}, \ldots, e_{2/3})^T$ (for $a_j = 3$). Once again, these columns do not have enough $1$'s to fill out a basic correlator.

Already the top row cannot be a primitive element of $H_{W,G_{\text{max}}}$. \qed

4 K3 LG-Surfaces of the Form $x^2 y + y^n + z_1^{a_1} + z_2^{a_2}$

There are 15 distinct K3 LG-surfaces of the form $x^2 y + y^n + z_1^{a_1} + z_2^{a_2}$. There are 15 of these, listed in Table 2 [Pri11]. They can be calculated from a list of possible quasihomogeneous weight systems such as that found in [Ska96].

\[
\begin{align*}
  x^2 y + y^3 + z^{12} + w^6 & , & x^2 y + y^5 + z^{24} + w^3 & , & x^2 y + y^6 + z^4 + w^6 \\
  x^2 y + y^3 + z^8 + w^8 & , & x^2 y + y^4 + z^4 + w^8 & , & x^2 y + y^3 + z^3 + w^9 \\
  x^2 y + y^3 + z^{20} + w^5 & , & x^2 y + y^5 + z^{15} + w^3 & , & x^2 y + y^6 + z^2 + w^5 \\
  x^2 y + y^3 + z^{12} + w^4 & , & x^2 y + y^5 + z^5 + w^5 & , & x^2 y + y^3 + z^3 + w^8 \\
  x^2 y + y^3 + z^6 + w^6 & , & x^2 y + y^6 + z^{12} + w^3 & , & x^2 y + y^6 + z^2 + w^3 \\
  x^2 y + y^3 + z^6 + w^6 & , & x^2 y + y^6 + z^{12} + w^3 & , & x^2 y + y^6 + z^2 + w^3
\end{align*}
\]

Table 2: All K3 LG-surfaces of the form $x^2 y + y^n + z_1^{a_1} + z_2^{a_2}$

4.1 The state space and its product structure ($n > 2$)

Using Theorem 2.1 on tensor products, we only need to understand the state spaces of the summands $D_{n+1} = x^2 y + y^n$, $W_1 = z_1^{a_1}$, and $W_2 = z_2^{a_2}$. From Section 3.2 above, we already understand the state space and product structure of $H_{W_1,G_{\text{max}}}^{\text{max}}$ and $H_{W_2,G_{\text{max}}}^{\text{max}}$. It remains to calculate $H_{D_{n+1},G_{\text{max}}}^{\text{max}}$; this is described on pages 48-49 of [FJR07], which gives an explicit isomorphism from the Milnor ring $Q_{x^2 y + y^n}$ to $H_{D_{n+1},G_{\text{max}}}^{\text{max}}$. 


At times we will need to consider the case \( n = 2 \) separately. For \( n > 2 \), as described in that paper, the group \( G_{\text{max}}^{n+1} \) is generated by the element \( g = (-\frac{1}{2n}; \frac{1}{n}) \). If \( e_a \) indicates a basis element coming from the sector \( ag \), then a basis for the state space of \( H_{D_n+1}G_{\text{max}}^{n+1} \) is given by the elements

\[
\{ xe_0, e_1, e_2, \ldots, e_{n-1}, e_{n+1}, \ldots, e_{2n-1} \}.
\]

The identity \( 1 \) in the ring is equal to \( e_{n+1} \), corresponding to the sector \((\frac{n-1}{2n}; \frac{1}{n}) \). The unique primitive element, which will be denoted by \( p_{n+1} \), is equal to \( e_{n+2} \), and comes from the sector \((\frac{1-2}{2n}; \frac{2}{n}) \). The element of highest \( W \)-degree, denoted \( h_{n+1} \), is equal to \( e_{n-1} \), and comes from the sector \((\frac{2}{2n+1}; \frac{n-1}{n}) \). The computations in [FJR07] also tell us that when two elements come from inverse sectors, including the identity sector, their product is a scalar multiple of \( h_{D_n+1} \).

For \( n = 2 \), a basis for the state space

\[
\{ e_{(1/4,1/2)}, xe(0,0), e_{(3/4,1/2)} \}.
\]

The element \( 1 = e_{(1/4,1/2)} \) is the identity; \( p_{D_3} = xe(0,0) \) is primitive; and \( p_{D_3} \ast p_{D_3} \) is a scalar multiple of \( h_{D_3} = e_{(3/4,1/2)} \), the element of top degree. Thus, the primitive element is the same as the element from a sector with nontrivial fixed locus; this is the reason we will occasionally need to consider the case \( n = 2 \) separately.

### 4.2 Bound on \( k \) (all \( n \))

In order to use Lemma 2.2 to bound \( k \), we need to calculate the maximum \( W \)-degree of any primitive element in the rings corresponding to the summand polynomials. In Section 3.1 we saw that for the Fermat summands \( x_j^{a_j} \), the maximum \( W \)-degree of a primitive element was \( \frac{3}{1-n} \). One can check from Table 2 that for K3 LG-surfaces of the form \( x^2y + y^n + z_1^{a_1} + z_2^{a_2} \), the largest \( \frac{3}{1-n} \) will ever be is \( \frac{2}{3} \). In the \( D_{n+1} \) summand, the primitive element \( e_{(\frac{3}{1-n}, \frac{3}{1-n})} \) has \( W \)-degree \( 2(\frac{1}{2n} + \frac{1}{n}) = \frac{2}{2n} = \frac{1}{n} \). The largest this will ever be is \( \frac{1}{n} \), hence, a safe value for \( P \) is \( \frac{2}{3} \), for any \( n \) appearing in Table 2. We can now calculate a bound on \( k \) for any surface of this type:

\[
k \leq 2 + \frac{1 + \frac{2}{3}}{1 - \frac{1}{2}} = 2 + \frac{2}{1 - \frac{1}{3}} = 2 + \frac{3}{2} = 6 \frac{1}{2}.
\]

### 4.3 Four-point correlators

Following the strategy in Section 3.3, we want to think of correlator grids with three columns, each corresponding to an atomic polynomial. We want to find all possible basic columns that give us integer line bundle degrees. The last two columns correspond to Fermat atomic polynomials; the possibilities for these were discussed in Section 3.3 above. We still need to calculate columns corresponding to the polynomial \( D_{n+1} \) and their contributions. It is a matter of combinatorics to produce Tables 3 and 4, which list these columns for a four-point correlator.

<table>
<thead>
<tr>
<th>( l_1 )</th>
<th>Basic elements</th>
<th>Remaining elements</th>
<th>( l_2 )</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1, 1</td>
<td>( xe_0, x_0 )</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1, ( e_{n+2} )</td>
<td>( e_{2n-1}, xe_0 )</td>
<td>-1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( e_{n+2}, e_{n+2} )</td>
<td>( e_{2n-2}, xe_0 )</td>
<td>-1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( e_{n+2}, e_{n+2} )</td>
<td>( e_{2n-1}, 2n-1 )</td>
<td>-2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>1, 1</td>
<td>( e_m, 2n-m ) for ( 0 &lt; m &lt; 2n, m \neq n )</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>1, ( e_{n+2} )</td>
<td>( e_m, 2n-m-1 ) for ( 0 &lt; m &lt; 2n-1, m \neq n, n-1 )</td>
<td>-2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( e_{n+2}, e_{n+2} )</td>
<td>( e_m, 2n-m-2 ) for ( 0 &lt; m &lt; 2n-2, m \neq n, n-1, n-2 )</td>
<td>-1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( e_{n+2}, e_{n+2} )</td>
<td>( e_{n-1}, e_{n-1} ) (above case with ( m = n-1 ))</td>
<td>-2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Possible \( D_{n+1} \) columns for a nonzero basic four-point correlator, \( n > 2 \)
Theorem 4.1. For $W = x^2y + y^2 + z_1^a + z_2^a$, the genus-zero four-point correlators are completely determined by the pairing, the three-point correlators, and the correlators

$$\langle p_{W_1}, p_{W_1}, h_{W_1}, h_W \rangle,$$

$$\langle p_{W_2}, p_{W_2}, h_{W_2}, h_W \rangle,$$

and

$$\langle p_{D_{n+1}}, p_{D_{n+1}}, h_{D_{n+1}}, h_W \rangle.$$  

It is important to note that because of the broad sectors, in general we will not be able to compute all the three-point correlators. Thus, even though we will be able to compute the three correlators listed above for most values of $n$, we still will not be able to determine the value of every four-point correlator, because we don’t know the values of the three-point correlators.

Proof. The contributions of the three columns must add to 5, by Lemma 2.3. From Section 3.3 we know a Fermat column can only contribute 1 or 2; hence the contributions are partitioned either as $1 + 2 + 2$, $2 + 2 + 1$, or $3 + 1 + 1$, where the first summand always refers to the contribution of the Fermat column which contributes 2 looks like $(p_{W_2}, p_{W_1}, h_{W_1}, h_W)^T$, so that two of these columns cannot appear in a basic correlator. Hence the sum breaks up as $2 + 2 + 1$ or as $3 + 1 + 1$. We will consider each case separately.

- Case $2+2+1$:
  Since the Fermat column which contributes 2 looks like $(p_{W_2}, p_{W_1}, h_{W_1}, h_W)^T$, the $D_{n+1}$ column needs to have its first two elements equal to 1. This eliminates all but rows 1 and 5 in Table 3, which are the same as rows 1 and 2 in Table 4. Then the correlator grid is

$$\begin{bmatrix}
1 & p_{z_1} & 1 \\
1 & p_{z_1} & 1 \\
s_1 & h_{z_1} & t_1 \\
s_2 & h_{z_1} & t_2
\end{bmatrix},$$

where $s_1 * s_2$ is a scalar multiple of $h_{D_{n+1}}$ and $t_1 * t_2$ is a scalar multiple of $h_{s_2}$. (Here we are using the facts, discussed in Sections 3.1 and 4.1, that in $H^+,\text{max}_{D_{n+1}}$ and $H^+,\text{max}_{z_1^a, z_2^a}$, when two elements come from inverse sectors, their product is a scalar multiple of the element of highest weight.) Notice that we have taken the second column to be the one that contributes 2; if the third column contributes 2, the argument is exactly the same. Now we can use the Reconstruction Lemma (Theorem 1.9) to reduce the number of these correlators we need to compute. Set $\alpha = (1, p_{z_1}^2, 1), \beta = (s_2, h_{z_1}^a, t_2), \epsilon = (1, h_{z_1}^a, 1), \text{ and } \phi = (s_1, 1, t_1)$. Then $\phi \star \epsilon = (s_1, h_{z_1}^a, t_1)$, while $\alpha \star \epsilon = (1, 0, 1) = 0, \alpha \star \beta = (s_2, 0, t_2) = 0, \text{ and } \beta \star \phi = (k_1 h_{W'}, h_{z_1}^a, k_2 h_{z_2}^a)$ for some scalars $k_1, k_2 \in \mathbb{C}$. Thus, all these nonzero basic four-point correlators are determined by the pairing, the three-point correlators, and the correlator

$$\begin{bmatrix}
1 & p_{z_1} & 1 \\
1 & p_{z_1} & 1 \\
h_{z_1} & 1 \\
h_{W'} & h_{z_1}^a & h_{z_2}^a
\end{bmatrix}.$$  

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>Elements</th>
<th>$l_2$</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1, 1, p_{D_1}, p_{D_3}$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>$1, 1, 1, h_{D_3}$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$p_{D_1}, p_{D_3}, h_{D_3}, h_{D_3}$</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4: Possible $D_3$ columns for a four-point correlator ($n = 2$).
• Case 3+1+1:
Now, assume that the contribution of the \( D_{n+1} \) column is 3; in other words, we are using the final column in Table 3. Since \( e_{n-1} = h_{D_{n+1}} \), any basic correlator is as follows:

\[
\begin{bmatrix}
  p_{D_{n+1}} & 1 & 1 \\
  p_{D_{n+1}} & 1 & 1 \\
  h_{D_{n+1}} & s_1 & t_1 \\
  h_{D_{n+1}} & s_2 & t_2 \\
\end{bmatrix},
\]

where \( s_1 \ast s_2 \) is a scalar multiple of \( h_{s_1} \) and \( t_1 \ast t_2 \) is a scalar multiple of \( h_{s_2} \). By using the Reconstruction Lemma (Theorem 1.9) in an argument identical to the one above, we find that these correlators are determined by the pairing, the three-point correlators, and the single correlator

\[
\begin{bmatrix}
  p_{D_{n+1}} & 1 & 1 \\
  p_{D_{n+1}} & 1 & 1 \\
  h_{D_{n+1}} & h_{s_1} & h_{s_2} \\
\end{bmatrix},
\]

It remains to calculate the values of these three four-point correlators.

**Theorem 4.2.** For \( n \geq 2 \), the four-point correlator

\[
\langle p_{W_1}, p_{W_1}, h_{W_1}, h_W \rangle
\]

takes the value \( \frac{1}{a_j} \). In particular it is nonvanishing.

**Proof.** For simplicity we will take \( j = 1 \); the other cases are analogous. We want to calculate the correlator

\[
\langle e \left( \frac{a_{-1}}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_3} \right), e \left( \frac{a_{-1}}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_3} \right), e \left( \frac{a_{-1}}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_3} \right) \rangle.
\]

All the sectors are narrow, and since the correlator satisfies Equation (4), it also satisfies the Dimension Axiom. The line bundle degrees are \(-1, -1, -2, -1\), so in particular they are all strictly negative, so we can begin to apply the Concavity Axiom (Axiom 1.6). The graphs \( \Gamma_i \) described there are depicted in Figure 3. Here, we are identifying elements of \( H_{W,G}^{max} \) with the sectors they came from.

![Figure 3: Graphs defining \( g_1 \) and \( g_2 = g_3 \)](image)

Using the line bundle degrees axiom, we can calculate that \( g_1 = \left( \frac{n+1}{2n}, \frac{n-1}{n}, \frac{a_{-1}}{a_1}, \frac{a_2-1}{a_2} \right) \) and \( g_2 = \left( \frac{n+1}{2n}, \frac{n-1}{n}, 0, \frac{a_2-1}{a_2} \right) \). All of the line bundle degrees associated with the triples described in Axiom 1.6 are strictly negative, so we can apply that axiom here. As in the proof of Theorem 3.3, we will calculate the sum in Equation (2) by computing the summand corresponding to each value of \( j \) separately, and then divide the total sum by two. For \( j = 3 \) and \( j = 4 \), we found in that proof that the sum is \( \frac{2}{a_1} \) and 0, respectively. When \( j = 1 \), we get

\[
\begin{align*}
(q_1^2 - q_2) + \sum_{i=1}^{k} \left( \Theta_i^j (1 - \Theta_i^j) \right) - \sum_{i=1}^{3} \left( \Theta_i^j (1 - \Theta_i^j) \right) \\
= \frac{(n-1)^2}{4n^2} - \frac{n-1}{2n} + 4 \cdot \frac{n-1}{2n} - 3 \cdot \frac{n-1}{2n} \\
= \frac{1}{4n^2} \left( n^2 - 2n + 1 - 2n^2 + 2n + n^2 - 1 \right) = 0.
\end{align*}
\]
When \( j = 2 \), we get

\[
\left( \frac{1}{n^2} - \frac{1}{n} \right) + 4 \cdot \frac{1}{n} - 3 \cdot \frac{1}{n} = \frac{1}{n^2} (1 - n + n - 1) = 0.
\]

This means the total value of the correlator is \( \frac{1}{2} \left( 0 + \frac{2}{a_1} + 0 \right) = \frac{1}{a_1} \). \( \square \)

It turns out that when \( n = 2 \) or \( 3 \), as is the case for the first five polynomials in Table 2, we cannot apply the Concavity Axiom to calculate the value of the other correlator; in fact, we do not know how to compute this correlator.

**Theorem 4.3.** For \( n > 3 \), the four-point correlator

\[
\langle p_{D_{n+1}}, p_{D_{n+1}}, h_{D_{n+1}}, h_{W} \rangle
\]

takes the value \( \frac{1}{n} \). In particular it is nonvanishing.

**Proof.** We want to calculate the correlator

\[
\left\langle \Theta_{p_{D_{n+1}}} \right. \Theta_{p_{D_{n+1}}}, \Theta_{h_{D_{n+1}}}, \Theta_{h_{W}} \left. \right\rangle.
\]

If \( n > 2 \), all the sectors are narrow, and since the correlator satisfies Equation (4), it also satisfies the Dimension Axiom. The line bundle degrees are \( -1, -2, -1, -1 \), so we can begin to apply the Concavity Axiom (Axiom 1.6). The graphs \( \Gamma_i \) described in that axiom are exactly the same as in Figure 3, except that \( p_{W_1} \) is replaced with \( p_{D_{n+1}} \) and \( h_{W_1} \) with \( h_{D_{n+1}} \).

Using the line bundle degrees axiom, we can calculate that \( g_1 = \left( \frac{n-3}{2n}, \frac{n-3}{2n}, \frac{a_1-1}{a_1}, \frac{a_2-1}{a_2} \right) \) and \( g_2 = \left( \frac{1}{2}, \frac{a_1-1}{a_1}, \frac{a_2-1}{a_2} \right) \). All of the line bundle degrees associated with the triples described in Axiom 1.6 are strictly negative, given \( n > 3 \), so we can apply that axiom here. Note that if \( n = 3 \), \( l_1 = 0 \) on the graph defining \( g_1 \), so that we cannot apply the Concavity Axiom. As in the proof of Theorem 3.3, we will calculate the sum in Equation (2) by computing the summand corresponding to each value of \( j \) separately, and then divide the total sum by two. For \( j = 3 \) and \( j = 4 \), we found in that proof that the sum is 0. When \( j = 1 \), we get

\[
\left( q_j^2 - q_j \right) + \sum_{i=1}^{k} \left( \Theta^j (1 - \Theta^j) \right) - \sum_{i=1}^{3} \left( \Theta^p (1 - \Theta^q) \right)
\]

\[
= \left( \frac{1}{n^2} \right) - \frac{1}{2n} + \frac{n-2}{2n} - \frac{n-2}{2n} + \frac{n-1}{2n} - \frac{n-2}{2n} - \frac{n-1}{2n} - \frac{n-3}{2n} + \frac{n-3}{2n} - \frac{2}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}
\]

\[
= \frac{1}{4n^2} (n^2 - 2n + 1 - 2n^2 - 2n + 2n^2 - 8 + 2n^2 - 2 - n^2 + 9 - n^2) = 0.
\]

When \( j = 2 \), we get

\[
= \left( \frac{1}{n^2} - \frac{1}{n} \right) + \left( \frac{2n}{n^2} - \frac{2n}{n} + \frac{2n}{n^2} + \frac{1}{n} \right) - \left( \frac{3n}{n} + 0 \right)
\]

\[
= \frac{1}{n^2} (7 + 3n - n + 7) = \frac{2n}{n^2} = \frac{2}{n}.
\]

This means the total value of the correlator is \( \frac{1}{2} \left( 0 + \frac{2}{n} + 0 + 0 \right) = \frac{1}{n} \). \( \square \)


4.4 Higher-point correlators

We will now show that all five- and six-point basic correlators vanish. Our method will be similar to that used in the four-point correlator case: we will examine all possible $D_{n+1}$ columns. For a five-point correlator, these are listed in Tables 5 and 6; for a six-point correlator, they are found in Tables 7 and 8.

$$
\begin{array}{|c|c|c|c|c|}
\hline
l_1 & \text{i's} & p_{D_{n+1}}'s & \text{Remaining elements} & l_2 & \text{Contribution} \\
\hline
0 & 3 & 0 & x_{e_0}, x_{e_0} & 0 & 2 \\
2 & 1 & e_{2n-1}, x_{e_0} & -1 & 2 \\
1 & 2 & e_{2n-2}, x_{e_0} & -1 & 2 \\
1 & 2 & e_{2n-1}, e_{2n-1} & -2 & 2 \\
0 & 3 & e_{2n-3}, x_{e_0}, \text{if } n \neq 3 & -1 & 2 \\
0 & 3 & e_{2n-1}, e_{2n-2} & -2 & 2 \\
-1 & 3 & e_{m}, e_{2n-m} \text{ for } 0 < m < 2n, m \neq n & -1 & 2 \\
2 & 1 & e_{m}, e_{2n-m-1} \text{ for } 0 < m < 2n-1, m \neq n, n-1 & -1 & 2 \\
1 & 2 & e_{m}, e_{2n-m-2} \text{ for } 0 < m < 2n-2, m \neq n, n-1, n-2 & -1 & 2 \\
1 & 2 & e_{n-1}, e_{n-1} \text{ (above case with } m = n-1) & -2 & 3 \\
0 & 3 & e_{m}, e_{2n-m-3} \text{ for } 0 < m < 2n-3, m \neq n-3, \ldots, n & -1 & 2 \\
0 & 3 & e_{n-1}, e_{n-2} \text{ (above case with } m = n-1 \text{ or } m = n-2) & -2 & 3 \\
\hline
\end{array}
$$

Table 5: Possible $D_{n+1}$ columns for a nonzero basic five-point correlator, $n > 2$. The first two columns show the distribution of the basic elements.

$$
\begin{array}{|c|c|c|c|}
\hline
l_1 & \text{Elements} & l_2 & \text{Contribution} \\
\hline
0 & p_{D_3}, p_{D_3}, p_{D_3}, p_{D_3}, h_{D_3} & -1 & 3 \\
1, 1, 1, p_{D_3}, p_{D_3} & 0 & 2 \\
-1 & 1, p_{D_3}, p_{D_3}, h_{D_3}, h_{D_3} & 0 & 3 \\
1, 1, 1, 1, h_{D_3} & -1 & 2 \\
\hline
\end{array}
$$

Table 6: Possible $D_3$ columns for a five-point correlator ($n = 2$).

**Theorem 4.4.** If $W = x^2y + y^n + z_1^n + z_2^n$ for $n > 2$, then all basic genus-zero five- and six-point correlators associated with $H_{W,G_{n}^{0,n}}$ vanish.

**Proof.** We will prove the cases $k = 5$ and $k = 6$ separately.

- **Case $k = 5$:**
  By Lemma 2.3, we know that the contributions of the columns must sum to 6. Thus, the sum must split up as $2 + 2 + 2$ or as $3 + 2 + 1$, where the first summand is always the contribution of the $D_{n+1}$ column. The first option does not work because there are not enough i’s:

  $$
  \begin{bmatrix}
  * & p_{W_1} & p_{W_2} \\
  * & p_{W_1} & 1 \\
  * & 1 & p_{W_2} \\
  * & h_{W_1} & h_{W_2} \\
  * & h_{W_1} & h_{W_2}
  \end{bmatrix}
  $$

  Already, the top row cannot be a primitive element of $H_{W,G_{n}^{0,n}}$. Thus, the sum must split as $3 + 2 + 1$, so $D_{n+1}$ column must contribute 3. But from Proposition 3.5 we know that the Fermat column contributing 2 has at most one 1, so that the $D_{n+1}$ column must have at least two 1’s. From Tables 5 and 6 it is clear that no column satisfies both these requirements.

- **Case $k = 6$:**
  By Lemma 2.3, we know that the contributions of the columns must sum to 7. Thus, the sum must
Table 7: Possible $D_{n+1}$ columns for a nonzero basic six-point correlator, $n > 2$. The first two columns show the distribution of the basic elements.

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>1's</th>
<th>$p_{D_{n+1}}$'s</th>
<th>Remaining elements</th>
<th>$l_2$</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>0</td>
<td>$xe_0, xe_0$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td>$e_{2n-1}, xe_0$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td>$e_{2n-2}, xe_0$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td>$e_{2n-1}, e_{2n-1}$</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td>$e_{2n-3}, xe_0$, if $n \neq 3$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td>$e_{2n-1}, e_{2n-2}$</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
<td>$e_{2n-4}, xe_0$, if $n \neq 3, 4$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
<td>$e_{2n-1}, e_{2n-3}$, if $n \neq 3$</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
<td>$e_{2n-2}, e_{2n-2}$</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>0</td>
<td>$e_{m}, e_{2n-m}$ for $0 &lt; m &lt; 2n$, $m \neq n$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td>$e_{m}, e_{2n-m-1}$ for $0 &lt; m &lt; 2n-1$, $m \neq n, n-1$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td>$e_{m}, e_{2n-m-2}$ for $0 &lt; m &lt; 2n-2$, $m \neq n, n-1, n-2$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td>$e_{n-1}, e_{n-1}$ (above case with $m = n-1$)</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td>$e_{m}, e_{2n-m-3}$ for $0 &lt; m &lt; 2n-3$, $m \neq n-3, \ldots, n$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td>$e_{n-1}, e_{n-2}$ (above case with $m = n-1$ or $m = n-2$)</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
<td>$e_{m}, e_{2n-m-4}$ for $0 &lt; m &lt; 2n-4$, $m \neq n-4, \ldots, n$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
<td>$e_{n-1}, e_{n-4}$ for $n \neq 3$ (above case with $m = n-1$ or $m = n-3$)</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td></td>
<td>$e_{n-2}, e_{n-2}$ (above case with $m = n-2$)</td>
<td>-2</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 8: Possible $D_3$ columns for a six-point correlator ($n = 2$).

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>Elements</th>
<th>$l_2$</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1, 1, 1, 1, p_{D_3}, p_{D_3}$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$1, p_{D_3}, p_{D_3}, p_{D_3}, p_{D_3}, h_{D_3}$</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>$1, 1, p_{D_3}, p_{D_3}, h_{D_3}, h_{D_3}$</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$1, 1, 1, 1, 1, h_{D_3}$</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

split up as $3 + 2 + 2$. We can fill in the last two columns of the correlator as follows:

$$
\begin{bmatrix}
* & p_{W_1} & 1 \\
* & p_{W_1} & 1 \\
* & 1 & p_{W_2} \\
* & 1 & p_{W_2} \\
* & h_{W_1} & h_{W_2} \\
* & h_{W_1} & h_{W_2}
\end{bmatrix}
$$

This implies that the $D_{n+1}$ column must contribute 3 and have all 1’s for its basic elements. From Tables 7 and 8 we see that this never happens. Hence all basic six-point correlators vanish.

\[\square\]
References


