The Caldero degenerations of the Grassmannian of 3-planes in 6-space and the positive tropical $G(3,6)$

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September 4, 2009

Abstract

We find that out of the 48 maximal cones of $\text{Trop}^+(G(3,6))$, only 29 of these cones are realized by the toric degenerations of Caldero.

1 Introduction

This paper is concerned in relating the toric degenerations discovered by Caldero [C] with the maximal cones of $\text{Trop}^+(G(3,6))$. Provided below is an introduction to tropical geometry, the positive portion of the tropical Grassmanian, the concept of reduced words, and Caldero toric degenerations. More detailed information and proof of certain claims included below can be found in the references section of this paper.

1.1 Tropical Geometry

The field of Tropical Geometry first introduces the tropical semiring. This consists of the real numbers with a point at infinity. Instead of the usual operations on the real numbers, however, the tropical semiring uses tropical addition, the act of taking the minimum of input values, and tropical multiplication, the act of normal addition. In general, most theoretical work done with Tropical Geometry is over some algebraically closed field $K$ with some valuation into the reals, i.e. $\text{deg}: K^* \to \mathbb{R}$. In application, however, we assume $K = \mathbb{C}$.

Tropical Geometry has objects known as initial ideals defined as follows. First, fix $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$. Let $f \in K[x]$. For any term of $f$, $c \cdot x_1^{a_1} \cdots x_n^{a_n}$ with $c \in K$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we call $\text{deg}(c) + a_1w_1 + \cdots + a_nw_n$ the $w$-weight of $f$. Let $f(x_1, \ldots, x_n) = f(t^{w_1}x_1, \ldots, t^{w_n}x_n)$. The initial form of $f$, $\text{in}_w(f)$, is the sum of the terms of $f$ having minimal $w$-weight. Given any ideal $I \subset K[x]$, its initial ideal is defined as so:

$$\text{in}_w(I) = \langle \text{in}_w(f) | f \in I \rangle$$
One of the most important ideas expressed in tropical geometry is the concept of tropicalization. Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}((t^{1/n}))$. As explained by Speyer and Williams ([SpW]), for each element of $\mathcal{C}$, there exists a unique lowest term at $t^n$ with $a \in \mathbb{C}$ and $u \in \mathbb{Q}$. Define the valuation map $\text{val} : (\mathbb{C}^*)^n \rightarrow \mathbb{Q}^n$, $(x_1, x_2, \ldots, x_n) \mapsto (\text{val}(x_1), \text{val}(x_2), \ldots, \text{val}(x_n))$ with $\text{val}(at^n) = u$. The tropicalization of a variety $V(I)$ is denoted $\text{Trop}(V(I))$ and is equal to the closure of the image under $\text{val}$ of $V(I) \cap (\mathbb{C}^*)^n$. The positive part of $\text{Trop}(V(I))$, denoted $\text{Trop}^+(V(I))$, is equal to the closure of the image under $\text{val}$ of $V(I) \cap (\mathbb{R}^+)^n$. From Proposition 2.2 (see [SpW]), we also know that:

$$\text{Trop}^+(V(I)) = \{ w \in \mathbb{Q}^n | \text{in}_w(I) \text{ does not contain any nonzero polynomials from } \mathbb{R}^+[x_1, x_2, \ldots, x_n] \}.$$  

There is much more to Tropical Geometry than put forth in this section (see [SpStu]), but, as shown in the later portions of the next section, all the tools required for talking about the Tropical Grassmannian have been put forward.

1.2 The Grassmannian

The Grassmannian is a projective variety whose points are in natural bijection with the linear $d$ dimensional subspaces ($d$-planes for short) in an $n$ dimensional space. This is often denoted $G(d, n)$ or $\text{Gr}(d, n)$. It is convenient to think of the Grassmannian as the space of all rank $d$ matrices of order $d \times n$ modulo $\text{GL}_d$, the group of general linear $d \times d$ matrices. Consider the polynomial ring $\mathbb{Z}[p] = \mathbb{Z}[p_1, p_2, \ldots, p_{\binom{n}{d}} | 1 \leq i_1 < i_2 < \ldots < i_d \leq n]$ of $\binom{n}{d}$ variables. It is known that there exist relations between the determinants of the $d \times d$ minors of rank $d$ matrices. The Plücker Ideal $I_{d,n}$ is the prime homogeneous ideal in $\mathbb{Z}[p]$ that consists of these relations.

In their paper on the tropical Grassmannian, designated $G_{d,n}$, Speyer and Sturmfels show that $G_{3,6} = \text{Trop}(G_{3,6}) = \{ w \in \mathbb{R}^{\binom{6}{3}} | \text{in}_w(I_{3,6}) \text{ contains no monomials } \}$. They go on to explain how $\text{Trop}(G(3,6))$ and $\text{Trop}^+(G(3,6))$ are ten dimensional polyhedral complexes inside of $\mathbb{R}^9$. $\text{Trop}^+(G(3,6))$ is isomorphic to $P \times \mathbb{R}^6$ where $P$ is a four dimensional polyhedral complex that is combinatorically equivalent to $\mathbb{R}^4$. Table 2 describes the parameterization of $P$.

1.3 The Symmetric Group and Reduced Words

The symmetric group $S_n$ is the representation of automorphisms on a set of $n$ objects with specified positions. It is well known that $S_n$ is generated by adjacent transpositions, $s_i = (i, i + 1)$, i.e. the transpositions that exchange two adjacent objects in the $i$ and $i + 1$ positions and leave all other objects alone. A word is a sequence $\alpha = (s_{i_1}, s_{i_2}, \ldots, s_{i_l})$ of these adjacent transpositions. $l$, the number of transpositions, is the length of the word. The set of words $W_n$ is defined as follows: $W_n = \{ \alpha = (s_{i_1}, s_{i_2}, \ldots, s_{i_k}) | 1 \leq i_k \leq n - 1 \forall k \}$. There exists a map $\pi : W_n \rightarrow S_n$ such that $\pi = s_1 \cdot s_2 \cdot \ldots \cdot s_l$. Call two words $\alpha, \beta$ equivalent if $\pi = \beta$. A reduced word is a word with a length less than or equal the length of all other equivalent words.
There are three different operations that can be used on a word to produce a different equivalent word. These operations all change a local portion of the word while leaving the ordering and numbers of the rest of the word intact. Equation (1) demonstrates the usage of these operations on an example word. The *braid operation* recognizes the equivalence of the three transposition segments \((i)(i+1)(i)\) and \((i+1)(i)(i+1)\) and switches between them accordingly. The legitimacy of this equivalency is easy confirmed by considering three objects \(a, b,\) and \(c\) initially in positions \(i, i+1,\) and \(i+2\) and working out that both \((i)(i+1)(i)\) and \((i+1)(i)(i+1)\) produce the same permutation. The *commuting operation* is one that commutes two transpositions \(i\) and \(j\) (i.e. interchanging \(ij\) and \(ji\)) but only if \(|i - j| \geq 2\). If \(i\) and \(j\) are two or more apart, since the transpositions only deal with the \(i^{th}, (i+1)^{th}\) and \(j^{th}, (j+1)^{th}\) positions respectively, the two transpositions effect two completely different sets of objects and therefore may be switched. The *identity operation* deletes two transpositions that are the same if they are positioned next to each other. Clearly, two transpositions positioned next to each other switch two objects and make the same switch back, so their appearance is redundant.

\[
1215232 \sim 1212532 \quad \text{(commuting operation)} \\
\sim 2122532 \quad \text{(braid operation)} \\
\sim 21532 \quad \text{(identity operation)}
\]

Equivalent reduced words of course have the same length. Therefore, the identity operation can never be used on a reduced word, as it would imply that the word was actually not reduced. In addition, given one reduced word, there exists a finite path to any other equivalent word using the braid relation and the commuting relation.

It is known that the longest reduced words for any symmetric group will be those that completely reverse the order of the objects; that is, the words that map to the permutation that sends the first, second, \(\ldots\), \(n^{th}\) words to the \(n^{th}, (n-1)^{th}, \ldots\), first positions respectively. We know that the length of the longest reduced words in \(S_n\) is \(\binom{n}{2}\). Equation 2, derived by Richard P. Stanley [Sta], provides the total number of reduced longest words for a Symmetric group of degree \(n\).

\[
r(w_0) = \binom{n}{2}/1^{n-1}3^{n-2}5^{n-3}\ldots(2n-3)!
\]

### 1.4 Caldero Toric Degenerations

Caldero [C] defines toric degenerations of Schubert varieties and homogeneous spaces \(G/P\) for each reduced word \(w_0\) for the longest Weyl element \(w_0\). Later his results were extended to spherical varieties by Alexeev and Brion [AB].

Suppose that \(G\) is a simply connected semisimple complex algebraic group. Fix a maximal torus \(T\) and a Borel subgroup \(B\) such that \(T \subset B \subset G\). Let \(W\) the Weyl group of \(G\) relative to \(T\). Let \(P\) be the weight lattice and let \(Q\) be the root lattice. Let \(P^+ \subset P\) be the semigroup of dominant
weights. For all $\lambda$ in $P^+$, let $L_{\lambda}$ be the line bundle on $G/B$ corresponding to $\lambda$; that is, $V_{\lambda} := H^0(G/B, L_{\lambda})$ is the irreducible representation with highest weight $\lambda$. Then, the direct sum of all irreducible representations $R := \oplus_{\lambda \in P^+} H^0(G/B, L_{\lambda})$ has a natural structure of a $P^+$-graded $\mathbb{C}$-algebra. There is a natural action of $G$ on $R$ which of course restricts to an action of the torus $T$.

Given the irreducible representation $V_{\lambda}$, and a weight $\beta$ of the torus $T$, let $V_{\lambda}(\beta)$ denote the subspace of $V_{\lambda}$ of weight $\beta$.

There is a special basis $B$ for $R$ called the canonical basis which restricts to a basis for each $V_{\lambda}(\beta)$. Let $B^*$ denote the dual basis to $B$; this is called the dual canonical basis. The dual canonical basis $B^*$ has a special parametrization called the string parametrization (see §3 of [BZ]) which depends on the reduced word $\tilde{w}_0$.

Each dual canonical basis vector $b \in V_{\lambda}$ has a string parameter $\phi \in \mathbb{N}^\ell$, where $\ell$ is the length of the longest Weyl element. The dual canonical basis restricted to $V_{\lambda}$ can be indexed as $b_{\lambda, \phi}$ as $\phi$ ranges over all the string parameters for the dual canonical basis elements in $V_{\lambda}$.

If we multiply two such basis elements, a fundamental problem is to rewrite the product as a linear combination of dual canonical basis elements. The dual canonical basis has the following multiplicative property (see [AB]):

$$b_{\lambda, \phi} b_{\nu, \psi} = b_{\lambda + \nu, \phi + \psi} + \sum_{\gamma} c^{\gamma}_{\lambda, \phi, \nu, \psi} b_{\lambda + \nu, \gamma},$$

(3)

where $c^{\gamma}_{\lambda, \phi, \nu, \psi} = 0$ unless $\gamma$ lexicographically precedes $\phi + \psi$.

Caldero [C] shows that if we pick a large enough integer $N >> 0$, and define a filtration of $R$ by setting the filtration level of the basis vector $b_{\lambda, \phi}$ to be equal to $N^{\ell-1}t_1 + N^{\ell-2}t_2 + \cdots + Nt_{\ell-1} + t_\ell$, where $\phi = (t_1, \ldots, t_\ell)$, then the associated graded ring $\text{gr}(R)$ to this filtration is isomorphic to the semigroup algebra $\mathbb{C}[S]$ where $S$ is the semigroup of all pairs $(\lambda, \phi)$ under addition, where $\phi$ is the string parameter of some dual canonical basis vector in $V_{\lambda}$. In fact, for $G(3,6)$, it is not hard to show that $N = 10$ suffices. That is, the above relation (3) degenerates into the binomial relation

$$\bar{b}_{\lambda, \phi} \bar{b}_{\nu, \psi} = \bar{b}_{\lambda + \nu, \phi + \psi}.$$

(4)

### 1.5 Computing string parameters in $V_{\omega_k}$

The string parametrization is defined in the following way. Let $U = U_q(G)$ denote the quantum universal enveloping algebra of $G$; it is an algebra over $\mathbb{C}(q)$, graded by the root lattice $Q$. It contains special generators $E_1, \ldots, E_r$ where $r = \text{rank}(G)$ in degrees $\alpha_1, \ldots, \alpha_r$ where $\alpha_1, \ldots, \alpha_r$ are the simple roots. The representation $V_{\lambda}$ of $G$ is a $U$-module for any value of $q$. The elements $E_i$ take $V_{\lambda}(\beta)$ to $V_{\lambda}(\beta + \alpha_i)$. In particular the $E_i$’s are nilpotent operators on $V_{\lambda}$ since $V_{\lambda}$ has only a finite number of weight spaces.

The string parameter of an element of $V_{\lambda}$ is defined as follows in [BZ]. Given a nonzero $v \in V_{\lambda}$, let $c_i(v)$ be equal to the maximum natural number $t$ such that $E^t_i(v) \neq 0$. Suppose that the reduced word $\tilde{w}_0 = i_1 i_2 \ldots i_\ell$. 

4
Given $v \in V^\lambda$, let $\phi(v) = (t_1, t_2, \ldots, t_\ell)$, where
\[
t_1 = c_1(v), \quad t_2 = c_2(E_{i_1}^{t_1}(v)), \quad \ldots, \quad t_\ell = c_\ell(E_{i_\ell-1}^{t_{\ell-1}} \cdots E_{i_1}^{t_1}(v)).
\]

In the case that $\lambda = \pi_k$ is a fundamental weight, the string parameters are very easy to compute for the dual canonical basis vectors, which are necessarily equal to the Plücker coordinates $[j_1, j_2, \ldots, j_k]$. Here, the operator $E_i$ acting on $[j_1, \ldots, j_k]$ has the effect of replacing the index $i + 1$ with $i$, and yields zero if there is no index equal to $i + 1$ among the indices $[j_1, \ldots, j_k]$. Additionally, if two $i$’s are introduced this way, this gives zero. Otherwise, $E_i$ takes $[j_1, \ldots, j_k]$ to another Plücker coordinate, which no longer contains the index $i + 1$. Hence, $E_i^2(b) = 0$ for any $b = [j_1, \ldots, j_k]$. Therefore, the string parameter for any Plücker coordinate is a sequence of 0’s and 1’s. The string parameter is all 0’s for the highest weight vector $[1, 2, \ldots, k]$. One can compute string parameters by computing at any value of $q$; in particular if we take $q = 0$ we get the crystal operators and the crystal graph which shows the action of lowering operators $F_i$ on the dual canonical basis, see Figure 1.

![Figure 1: The crystal graph for the representation $V_{\pi_k}$.](image)

See Table 1 for an example of the computation of string parameters.
1.6 Relating Caldero’s filtration to initial ideals

Suppose that \( R_m, m \in \mathbb{N} \), is a filtration of an algebra \( R \) which is increasing and exhaustive (i.e. \( R_m \subset R_{m+1} \) and \( \cup_m R_m = R \), and \( R_m R_n \subset R_{m+n} \) for all \( m, n \)), let \( \text{gr}(R) = \oplus_m R_m / R_{m-1} \) be the associated graded algebra (define \( R_{-1} = 0 \)). Suppose that \( \text{gr}(R) \) is finitely generated by homogeneous elements \( y_1, \ldots, y_s \). Let \( x_1, \ldots, x_s \) be lifts of the respective \( y_1, \ldots, y_s \) to \( R \). That is, if \( \deg(y_i) = d_i \) then \( x_i \in R_{d_i} \) and \( y_i \) is the image of \( x_i \) under the projection \( R_{d_i} \to R_{d_i} / R_{d_i-1} \). Then the \( x_i \)'s generate \( R \). Let \( X_1, \ldots, X_s \) denote formal variables, and let \( S = \mathbb{C}[X_1, \ldots, X_s] \). We have surjective maps \( \pi : S \to R \) and \( \pi' : S \to \text{gr}(R) \) where \( \pi(X_i) = x_i \) and \( \pi'(X_i) = y_i \). We define the filtration level of the variable \( X_i \) to be equal to \( d_i \). This defines a filtration of \( S \), i.e. take \( S_n \) to be all polynomials of filtration level \( \leq n \). Now \( R_n \) is the image of \( S_n \).

\[
0 \to \ker(\pi) \to S \to R \to 0,
\]

\[
0 \to \ker(\pi') \to S \to \text{gr}(R) \to 0.
\]

Let \( w \) be the weighting on the variables \( X_i \) defined by \( w(X_i) = d_i \). Then, it is easy to check that \( \ker(\pi') = \text{in}_w(\ker(\pi)) \).

Now, if \( w \) is a weight realizing one of Caldero’s toric degenerations of the coordinate ring of the Grassmannian, then the initial ideal is generated by binomials and hence the corresponding cone in the tropical image is maximal. Now, the relations in the semigroup algebra are all of the form \( m_1 - m_2 \) where \( m_1 \) and \( m_2 \) are monomials with coefficient 1. Thus, the sum of the coefficients for any element of \( \text{in}_w(I) \) is equal to zero, and thus they cannot all be positive. This shows why Caldero’s initial ideals are all positive.

The tricky part in the direct comparison of Caldero’s degenerations with the positive Tropical Grassmannian is that the associated graded algebra to the filtration of Caldero may contain generators in degree greater than one. However we may simply remove the generators having degree greater than one. Still, the resulting restricted initial ideal (to degree one generators) will be positive (it is also generated by binomials \( m_1 - m_2 \) of opposite sign).

2 Methods

The procedure for mapping one reduced word to its respective cone within \( \text{Trop}^+(G(3, 6)) \) by hand is a lengthy process, so, since there are 292,854 longest reduced words we wish to examine (see equation (2)), the use of a computer program was necessary. Using the idea that all reduced words equivalent words have a finite set of braid and/or commuting operation applications separating them (see section 1.3), it was possible to generate all of the words from just one known word. The program had two lists, list one, full of words left to check, and list two, with words that had
already been checked. The program took a word from list one and, after confirming that the word was not in list two, would analyze the word. All words that could be generated after one braid action or one commuting action from this word were placed in list one, and the generating word was placed in list two. This program was effective in preventing duplication of words by checking the word from list one against list two before operations took place. The end result was the full list of words contained in list two. The list was known to contain reduced words by only moving between equivalent reduced words (i.e. by using the braid and commuting actions) and confirmed to contain all reduced words by having the correct number of elements.

The next step in the process was to compute string parameterizations of all the Plucker coordinates for a given reduced word, as described in section 1.5. See Table 1 for an example calculation when the reduced word is equal to 121321432154321. Given the string parameter $\phi = (t_1, \ldots, t_{15})$ of a Plucker coordinate $ijk$, we may take the weight to be $-(t_110^{14} + t_210^{13} + \cdots + t_{15})$; in other words, simply read the string parameter as a number in base 10, but negated.

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Table 1: Method for finding initial weight of a reduced word

Recall that the positive tropical $G(3, 6)$ is a 10 dimensional polyhedral complex, isomorphic to $P \times \mathbb{R}^6$ where $P \subset \mathbb{R}^{20}$ is the image under $\mathbb{R}^4$ of the map defined in Table 2. The $\mathbb{R}^6$ component relates to the action of the six dimensional torus which acts on the positive tropical $G(3, 6)$. Our
weight $w$ does not lie in $P$, so we normalize $w$ using the torus action so that it does (this is a linear map which is simple to compute so we leave out the exact calculation). This allows us to compute the four parameters $y_1, y_2, y_3, y_4$.

The cones in $P$ can be identified by noting which linear form is minimized among the 10 weights associated to the Plücker coordinates $135, 136, 235, 236, 245, 246, 256, 346, 356$ (see Table 2). There are 48 such cones as computed in [SpW], and we arbitrarily labeled these as cones $1, 2, \ldots, 48$.

<table>
<thead>
<tr>
<th>Plücker Coordinate</th>
<th>Equation from Web</th>
<th>Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
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<td>Normalization</td>
</tr>
<tr>
<td>124</td>
<td>0</td>
<td>Normalization</td>
</tr>
<tr>
<td>125</td>
<td>0</td>
<td>Normalization</td>
</tr>
<tr>
<td>126</td>
<td>0</td>
<td>Normalization</td>
</tr>
<tr>
<td>134</td>
<td>$\min(0, y_1)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>135</td>
<td>$\min(0, y_1, y_2)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>136</td>
<td>$y_1$</td>
<td>Calculating $\vec{y}$</td>
</tr>
<tr>
<td>145</td>
<td>$\min(y_1, y_2)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>146</td>
<td>$y_2$</td>
<td>Calculating $\vec{y}$</td>
</tr>
<tr>
<td>156</td>
<td>$y_3$</td>
<td>Calculating $\vec{y}$</td>
</tr>
<tr>
<td>234</td>
<td>$\min(0, y_1, y_3)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>235</td>
<td>$\min(0, y_1, y_3, y_2 + y_3 - y_1, y_4 - y_1)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>236</td>
<td>$\min(y_1, y_3)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>245</td>
<td>$\min(y_1, y_3, y_2 + y_3 - y_1, y_4 - y_1)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>246</td>
<td>$\min(y_2, y_2 + y_3 - y_1, y_4 - y_1)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>256</td>
<td>$\min(y_2, y_2 + y_3 - y_1, y_4 - y_1)$</td>
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</tr>
<tr>
<td>345</td>
<td>$y_3$</td>
<td>Calculating $\vec{y}$</td>
</tr>
<tr>
<td>346</td>
<td>$\min(y_3, y_2 + y_3 - y_1, y_4 - y_1)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>356</td>
<td>$\min(y_2 + y_3 - y_1, y_4 - y_1, y_4 - y_1)$</td>
<td>Cone Mapping</td>
</tr>
<tr>
<td>456</td>
<td>$y_4$</td>
<td>Calculating $\vec{y}$</td>
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</table>

Table 2: Equations Given by Web Method for the 20 Plücker Coordinates
3 Results

The cones were arbitrarily labelled for each vertex. Tables 3 and 4 display the frequency with which each cone had a reduced word mapped to it.

<table>
<thead>
<tr>
<th>Cone Number</th>
<th>Number of Words</th>
<th>Example Word</th>
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<td>1</td>
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<td>243423123454321</td>
</tr>
<tr>
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<td>25456</td>
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</tr>
<tr>
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<tr>
<td>4</td>
<td>0</td>
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<td>212321432154321</td>
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<tr>
<td>6</td>
<td>24989</td>
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<tr>
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<td>121343231254321</td>
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<tr>
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<td>143234123254321</td>
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<td>14323412354321</td>
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<tr>
<td>24</td>
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</table>

Table 3: Frequency with which each cone has a reduced word mapped to it: first 24 cones
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<th>Example Word</th>
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<td>48</td>
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</tr>
</tbody>
</table>

Table 4: Frequency with which each cone has a reduced word mapped to it: last 24 cones
References


