Analyzing Tori in Split Groups over Quasifinite Fields via Bruhat-Tits theory

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1 Abstract

In this paper we explore properties of tori in split connected groups over quasifinite fields by utilizing Bruhat-Tits theory. To this end, in a fairly conversational manner, we will recall the essential ideas from Bruhat-Tits theory which are critical in proofs about the aforementioned tori. This paper exhibits a calculation for the degree of the splitting field for unramified and tamely ramified tori in terms of the order of elements in the Weyl group, which are in general well understood. From this, we proceed to the specific case when the tori are anisotropic and the field is Laurent series over the complex numbers. In this setting we realize the building of the torus as a point in the building of the ambient group, and seek to describe this point using only the corresponding weyl group conjugacy class coupled with the degree of an extension field over which the torus splits.

2 An Introduction to Bruhat-Tits theory

The purpose of this section is to introduce basic definitions, examples, and results in Bruhat-Tits theory, enough so that we may understand the proofs presented in section two, which pertain to the degree of splitting fields for unramified and tamely ramified tori, and formulate precisely the work and questions which occupy section three where the tori are assumed to be anisotropic. All illustrations and images of various geometric objects associated to our groups are contained in the Illustrations and Example section at the end of the paper. Please feel free to visit this section while reading to aid comprehension.

Throughout this paper we will use $SL_2$ as an example to illustrate various objects connected to our groups: root system, root subgroups, tori, apartments, parahorics, and the building.

A good introduction to Bruhat-Tits theory for split groups, sometimes called Chevalley groups, may be found in Joseph Rabinoff’s senior thesis at Harvard.
2.1 Apartments

Let $k$ be a field with a nontrivial discrete valuation $\nu$. We assume that $k$ is complete with respect to the metric given by the valuation and its residue field, $f = R/\varpi R$ is complete and perfect; here $R$ is the ring of integers in $k$ and $\varpi$ is a uniformizer. Denote by $p \geq 0$ the characteristic of the residue field. Let $E$ be an either unramified or tamely ramified extension of $k$. We denote by $R_E$ the ring of integers of $E$. Let $F$ be the residue field of $E$. Set $\Gamma = \text{Gal}(K,k)$, and $\theta$ be a topological generator for $\Gamma$.

Let $G$ be a connected split group defined over $k$ and $S$ be a maximal $k$-split $k$-torus. Let $G$ and $S$ be the $k$-rational points of $G$ and $S$ respectively. For example $G$ could be $\text{Sp}_{2n}$, $\text{SO}_n$, $\text{SL}_n$ and $S$ the diagonal matrices intersected with $G$. Of course this makes $G$ equal to $\text{Sp}_{2n}(k)$, $\text{SO}_n(k)$, $\text{SL}_n(k)$ in each case. In this paper we will be interested in the special case when $k$ is quasi finite, meaning that $k$ has a unique extension of degree $n$ and this extension is cyclic. Examples of the include finite fields and Laurent series over algebraically closed fields. For more information one can see Serre’s book on local fields. However, one should keep in mind that everything in this section makes perfect sense without this further assumption on $k$.

We let $A(S) = A(S,k) = X^*(S) \otimes \mathbb{R}$ called the apartment associated to $S$. For those already familiar with Bruhat-Tits theory we can Identify $A(S)$ with the vector space $X^*(S) \otimes \mathbb{R}$ by choosing a chevalley basis. Here $X^*(S)$, $X^*(S)$ denote the Cocharacter and Character groups of $S$, namely $\text{Hom}(\mathbb{G}_m,S)$ and $\text{Hom}(S,\mathbb{G}_m)$ of algebraic homomorphisms.

We now define a map from $X^*(S) \times X^*(S)$ into $\mathbb{Z}$ taking $(x, y)$ to an integer $\langle x, y \rangle$. This integer is given by the following relation:

$\langle x, y \rangle = n$ if and only if $x(y(a)) = a^n$ for all $a \in \mathbb{G}_m$ and $x \in X^*(S)$, $y \in X^*(S)$

This map is nondegenerate and gives rise to a duality between the Character and Cocharacter groups.

If $E/k$ is a Galois extension, then $\text{Gal}(E/k)$ acts on $X^*(S,E)$ and $X^*(S,E)$. In fact one can make sense of these definitions with no attention paid to the fact that $S$ is a torus. For any $\sigma \in \text{Gal}(E/k)$ define

$\sigma(\chi) = \sigma(\chi(\sigma(g^{-1})))$

This definition has the effect that the invariants $X^*(S,E)^{\text{Gal}(E/k)}$ are precisely the homomorphisms defined over $k$. If $E/k$ is the algebraic closure, a torus $T$ is called anisotropic if $X^*(T,E)^{\text{Gal}(E/k)} = 1$, that is there are no nontrivial characters of defined over $k$. A similar action and result holds for $X(S,E)$.

An apartment comes equipped with a natural polysimplicial decomposition
given as follows. Let $\Phi = \Phi(G, S)$ denote the set of nontrivial eigencharacters for the action of $S$ on $\mathfrak{g}$ the Lie Algebra of $G$. We assume that $\nu$ is surjective, and let $\Psi = \Psi(G, S, \nu)$ denote the corresponding set of affine roots, that is

$$\Psi = \{ \gamma + n | \gamma \in \Phi, n \in \mathbb{Z} \}$$

Each $\psi = \gamma + n \in \Psi$ defines an affine function on $\mathcal{A}$ by

$$\psi(\lambda \otimes r) = r \langle \lambda, \gamma \rangle + n$$

From this we define the hyperplanes

$$H_\psi = \{ x \in \mathcal{A} | \psi(x) = 0 \} \subset \mathcal{A}$$

for each $\psi \in \Psi$. These hyperplanes give us the familiar polysimplicial decomposition of $\mathcal{A}$. We usually call a polysimplex occurring in this decomposition a facet and the maximal facets are called acolone.

Finally, just as the Weyl group $W = N_G(S)/S$ acts transitively on (spherical) chambers, the extended affine Weyl group $\overline{W} = N_G(S)/S(R)$ acts transitively on acolones.

### 2.2 Objects attached to Facets

To each Facet in $\mathcal{A}$ we attach several objects of various types including: those that live in $G$, $\mathfrak{g}$, and still others which are best described over the residue field $f$. In this section we will define these objects and display any necessary facts useful for our work to follow.

For each root $\gamma \in \Phi$ we can define a subgroup isomorphic to $(k, +)$ called a root subgroup. These are found as follows: Let $B$ be a borel subgroup with Levi decomposition $B = UT$ as a semidirect product, here $U$ is the unipotent radical in $B$. We also have the opposite Borel, $B^- = U^-T$. It is in fact true that $U$ and $U^-$ are maximal unipotent subgroups of $G$. The roots subgroups are minimal proper subgroups of $U$ and $U^-$ which are normalized by $T$. These are connected unipotent groups of dimension one, hence isomorphic to $\mathbb{G}_a$. The action of $T$ on these root subgroups by conjugation gives a homomorphism from $T \to Aut(\mathbb{G}_a)$. However the only algebraic automorphisms of $\mathbb{G}_a$ are scaling by units. Thus $Aut(\mathbb{G}_a) \cong \mathbb{G}_m$, and each one dimensional unipotent subgroup determines an element of $X^*(S)$, called the roots. For each $\alpha \in \Phi$ let $U_\alpha$ be the root subgroup, similarly in the Lie Algebra we have $\mathfrak{g}_\alpha$.

$k$ carries a filtration of subgroups

$$k \supset \ldots \supset \varpi^{-2} R \supset \varpi^{-1} R \supset R \supset \varpi R \supset \varpi^2 R \ldots$$
which we will use to define the corresponding natural filtration in $U_\alpha$ indexed by $\{\alpha + n | n \in \mathbb{Z}\}$. That is

$$U_{\alpha + n} = G(\varpi^n R) \cap U_\alpha$$

and similarly

$$g_{\alpha + n} = G(\varpi^n R) \cap g_\alpha$$

We now have the pieces in place to define the objects we are interested in, namely the parahorics. For $x \in A$, we define $G_x$, the parahoric subgroup attached to $x$, by

$$G_x = \langle S(R), U_\psi \rangle_{\psi \in \Psi; \psi(x) \geq 0}$$

That is, $G_x$ is generated by $S(R)$ and the subgroups $U_\psi$ such that $\psi(x) \geq 0$. Since a facet in $A$ is determined by the intersection of hyperplanes defined above, we have that $G_x = G_y$ if $x$ and $y$ are in the same facet. Hence, If $F$ is any Facet in $A$, the notation $G_F$ is well defined. Note that if $x$ is the origin in $A$, then $G_x = G(R)$. As always we can perscribe a similar object to the Lie Algebra, namely $g_x$.

The parahoric subgroup $G_F$, for some facet $F$ in $A$, always contains a normal subgroup denoted by $G_F^+$ called pro-unipotent radical. It has the powerful property that $G_F^+ / G_F^+$ is the group of $f$-rational points of a connected $f$-group $G_F$. In the case where $k = \mathbb{Q}_p$ is a usual p-adic field, this quotient group is finite. For $x \in A$ we define $G_x^+$ to be:

$$G_x^+ = (S(R)^+, U_\psi)_{\psi \in \Psi; \psi(x) > 0}$$

here

$$S(R)^+ = \{a \in A(R) | \nu(\chi(a) - 1) > 0 \forall \chi \in X^*(A)\}$$

As before the notation $G_F^+$ makes sense and if $F_1$ and $F_2$ are two facets in $A$ such that $F_1$ is contained in the closure of $F_2$, then we have

$$G_{F_1}^+ < G_{F_2}^+ < G_{F_2}^+ < G_{F_1}$$

and $G_{F_2}^+ / G_{F_1}^+$ is a parabolic subgroup of $G_{F_1}(f) = G_{F_1}^+ / G_{F_1}$ with unipotent radical isomorphic to $G_{F_2}^+ / G_{F_1}^+$ and levi factor isomorphic to $G_{F_2}(f)$. It follows that if $F_2$ is an aclove then $G_{F_2}^+ / G_{F_1}^+$ is a borel subgroup of $G_{F_1}(f)$. Below is a table of the parahorics, pro-unipotent radicals, and $f$-groups ascotiated to various facets in $SL_2$.

<table>
<thead>
<tr>
<th>$F$</th>
<th>$G_F$</th>
<th>$G_F^+$</th>
<th>$G_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{-1}$</td>
<td>$\left( \frac{R}{\varpi R} \right)$</td>
<td>$1 + \left( \frac{\varpi R}{\varpi^2 R} \right)$</td>
<td>$SL_2(f)$</td>
</tr>
<tr>
<td>$F_{-1}$</td>
<td>$\left( \frac{R}{\varpi R} \right)$</td>
<td>$1 + \left( \frac{\varpi R}{\varpi^2 R} \right)$</td>
<td>$GL_1(f)$</td>
</tr>
<tr>
<td>$o$</td>
<td>$SL_2(R)$</td>
<td>$1 + \left( \frac{\varpi R}{\varpi^2 R} \right)$</td>
<td>$SL_2(f)$</td>
</tr>
<tr>
<td>$C_o$</td>
<td>$\left( \frac{R}{\varpi R} \right)$</td>
<td>$1 + \left( \frac{\varpi R}{\varpi^2 R} \right)$</td>
<td>$GL_1(f)$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$\left( \frac{R}{\varpi R} \right)$</td>
<td>$1 + \left( \frac{\varpi R}{\varpi^2 R} \right)$</td>
<td>$SL_2(f)$</td>
</tr>
</tbody>
</table>
Here we see a surprising relation between subgroups in $G$ defined over $k$ and those defined over the residue field. These ideas have been used to prove very powerful results in the representation theory of p-adic groups, where the correspondence above is between p-adic groups and finite groups of Lie type, which are well understood. For this, see papers by Allan Moy and Gopal Prasad.

As always, similar operations can be carried out in the Lie algebra $g$. Specifically if $F$ is a facet in $A$ we form the lattice $g^+_F$ so that $g_F / g^+_F$ is $L_F(f)$, defined to be $Lie(G_F(f))$, the Lie algebra of $G_F(f)$. We will readily use all of these object in the group as well as its Lie Algebra.

We now record some useful results before moving on to defining the building.

1. If $n \in N$, the normalizer of $T$ in $G$, then $nG_x n^{-1} = G_{nx}$ for all $x \in A$.

Here $N$ acts on the apartment via its image in the affine weyl group.

2. $G_x \cap T = T(R)$ for any $x \in A$.

3. $\bigcap_{x \in A} G_x = T(R)$.

4. If $F$ is an alcove, then the affine weyl group is generated by the reflections over the hyperplanes bordering $F$.

5. If $G_x$ and $G_y$ are any two parahoric subgroups, the $G = G_x N G_y$, called the affine Bruhat decomposition.

6. If $\Omega \subset A$ is any nonempty subset then $\bigcap_{x \in \Omega} (G_x \cdot N) = G_\Omega \cdot N$ and $\bigcap_{x \in \Omega} (N \cdot G_x) = N \cdot G_\Omega$.

Finally, any introduction of Bruhat-Tits theory would be incomplete without at least a brief discussion of Moy-Prasad lattices. These objects will not be critically utilized in this paper, but have proven to be invaluable to the representation theory of the type of groups covered in this paper.

**Definition 1.** For any $x \in A$ and positive real number $r$, we define

$$G_{x,r} = < T(1 + \varpi^r), U_\alpha (\varpi^{-(\alpha(x) - r)}) >_{\alpha \in \Phi}$$

$$G^+_{x,r} = < T(1 + \varpi^{r+1}), U_\alpha (\varpi^{-(\alpha(x) - r)}) >_{\alpha \in \Phi}$$

Here are several facts about Moy-Prasad lattices which are useful.

1. $G_{x,r} \subset G_{x,s}$ when $r > s$.

2. $G_x = G_{x,0}$ and $G^+_x = G^+_{x,0}$.

3. $G^+_{x,r}$ is normal in $G_{x,r}$ and the quotient is abelian for $r > 0$.

4. For any $x \in A$, there exists $\epsilon > 0$ such that $G^+_{x,r} = G_{x,r+\epsilon}$.

5. For any $x \in A$ the $G_{x,r}$ form a neighborhood basis of the identity in $G$ of open subgroups.
2.3 The Bruhat-Tits building

We now define the full Bruhat-Tits building for split connected groups and list some of its basic properties, including the action of Galois groups and its relation to Tori. As this paper is only meant to include a short introduction to the theory, most proofs and details will be left out. Please see the table of contents for a full treatment of the following material.

**Definition 2.** We put the following equivalence relation on the set $G \times A$:

namely set $(g,x) \sim (h,y)$ if there exists $n \in \mathbb{N}$ such that $n \cdot x = y$ and $g^{-1}hn \in G$. From this define the Bruhat-Tits building to be $\mathcal{B} = \mathcal{B}(\mathbb{G},k) = \mathcal{B}(G) = G \times A/\sim$. We write $[g,x]$ for the class of $(g,x)$. Define the action of $G$ on $\mathcal{B}(G)$ to be $g \cdot [h,x] = [gh,x]$.

In order to make certain definitions fit, we need to make use of several minor results.

1. The map $x \mapsto [g,x]$ is an injection of $A$ into $\mathcal{B}(G)$.

2. For $x \in A$, $G_x = \text{stab}_G(x)$

3. If $g \in G$, $x \in A$, and $gx \in A$, then $gG_xg^{-1} = G_{gx}$. Here we are implicitly identifying $A$ with its image $1 \times A$ in $\mathcal{B}(G)$.

4. If $gA \subset A$ or $gA \supset A$, then $gA = A$

**Definition 3.** If $g \in G$ and $x \in A$, then we define $G_{g,x} = gG_xg^{-1}$ and $G^{+}_{g,x} = gG^+_xg^{-1}$

Note that the above results in the list make this definition well defined.

It is a fact for algebraic groups that maximal $k$-split $k$-tori are conjugate. If $gSg^{-1} < G$ is a maximal $k$-split torus, then we define the apartment $A(gSg^{-1})$ associated with $gSg^{-1}$ to be $g \cdot A$. This gives us a way of defining apartments for all $k$-split tori. Furthermore, this definition makes sense since $gA = hA$ if and only if $gSg^{-1} = hSh^{-1}$

In summary, we have defined the apartment of $gSg^{-1}$ to be the standard apartment of $gGg^{-1}$, and defined the parahorics of this apartment to be the conjugates of the standard parahorics, $gG_xg^{-1}$ for $x \in A$. We note also that if $[g,x] \in \mathcal{B}(G)$ is any point then $[g,x] \in g \cdot A = A(gTg^{-1})$. So the building is the union of all the apartments. Like with apartments and parahoric subgroups, we can define arbitrary facets to be conjugates of facets in the standard apartment.

A corollary of the affine Bruhat decomposition is the following useful fact: Given any two points $x, y \in \mathcal{B}(G)$, there exists an apartment $A(gSg^{-1})$ containing them both for some $g \in G$. This allows us to define a metric on $\mathcal{B}(G)$ on which $G$ acts by isometries.
**Definition 4.** Suppose \( x, y \in B(G) \), and \( A(gSg^{-1}) \) is an apartment containing both \( x \) and \( y \). Define \( d(x, y) \) to be the norm of the vector \( g^{-1} \cdot x - g^{-1} \cdot y \in A \).

Having given \( B(G) \) a natural topology and described its basic structure as a gluing of apartments, we now turn to how the building behaves under field extensions. This is the basis for our later work.

The Bruhat-Tits building \( B(G, E) \) of \( G(E) \) exists for any algebraic extension \( E/k \) and is functorial in \( E \). For a complete introduction to this material see papers by Bruhat and Tits.

If \( E \) is a Galois extension of \( k \), there is a natural action, by simplicial isometries, of the Galois group \( \text{Gal}(E/k) \) on the building \( B(G, E) \). The action is given as follows. Suppose \( x \in B(G, E) \). By our earlier discussion we can choose an apartment \( A(T) \) containing \( x \), \( T \) a maximal \( E \)-split torus in \( G(E) \), and write \( x = \sum \lambda_i \otimes r_i \). Then we define

\[
\sigma(x) = \sum \sigma(\lambda_i) \otimes r_i
\]

where \( \text{Gal}(E/k) \) acts on \( X^*(T) \) as discussed in section one. The Galois fixed points, \( B(G, E)_{\text{Gal}(E/k)} \subset B(G, E) \), is a convex subset that contains \( B(G, k) \). It is known, due to Prasad and Rousseau separately, that if \( E/k \) is an unramified extension or tamely ramified extension, that

\[
B(G, E)_{\text{Gal}(E/k)} = B(G, k)
\]

Suppose now that \( x \in B(G, E) \) is \( \text{Gal}(E/k) \) fixed, which we will be interested in when this fixed point coresponds to an anisotropic torus. In this case the parahoric subgroup \( G_x = G(k)_x = G(E)_{x}^{\text{Gal}(E/k)} \), and the pronipotent radical of \( G_x \) is \( G_x^+ = (G(E)_{x}^+)^{\text{Gal}(E/k)} \). The analogous fact is true of their Lie Algebras.

It is no accident that we assume our extensions are unramified or tamely ramified in order to utilize this fact.

Finally in this section we will discuss how to think about \( B(T) \subset B(G) \), when \( T \) is not necessarily split over \( k \). Suppose \( T \) is split over \( E \). Define

\[
T(E)_c = \{ t \in T(E) ; \nu(\chi(t)) = 0 \forall \chi \in (X)^*(T) \}
\]

From this, the work of Bruhat and Tits implies the natural embedding

\[
B(T) = B(T(E))^{\text{Gal}(E/k)} = A(T(E))^{\text{Gal}(E/k)}
\]

\[
(B(G, E))^{\tau(E)_c}^{\text{Gal}(E/k)} = B(G, E)^{\tau(E)_c \times \text{Gal}(E/k)}
\]

\[
\subset B(G)
\]

Now that the ground work has been laid, we can move on to apply Bruhat-Tits theory in calculations about the degree of splitting fields for unramified and tamely ramified tori.
3 Analyzing Unramified and Tamely Ramified Tori via Bruhat-Tits Theory

Suppose $T$ is a maximal $k$-torus such that $T$ is split over an unramified or tamely ramified Galois extension $E/k$, that is $T(E) \cong \mathbb{G}_m^n$ as algebraic groups for some natural number $n$. Here $\mathbb{G}_m$ denotes the multiplicative group of $E$. Recall that we assume $k$ to be a quasifinite field. Thus $\text{Gal}(E/k)$ is a cyclic group with generator $\sigma = \text{Gal}(E/k)$. We describe an element in the Weyl group associated to the $G(E)$-conjugacy class of $T = T(E)$.

As $T$ and $S = S(E)$ are split over $E$, and all maximal $E$-split tori are conjugate, there exists $g \in G(E)$ such that $gT = S$. Here $gT$ is shorthand notation for $gT := \{ h \in G(E) | \exists t \in T, h = gtg^{-1} \}$

Since $S$ and $T$ are $k$-tori, they are $\text{Gal}(E/k)$ stable: namely $\sigma(T) = T$ and $\sigma(S) = S$. This gives us

$\sigma(g^{-1})gT = \sigma(g^{-1}) (gT) = \sigma(g^{-1}) S = \sigma(g^{-1})gT = \sigma(g^{-1}) S = \sigma(T) = T$

hence $\sigma(g^{-1})g$ is an element of the normalizer of $T$, $N_{G(E)}(T)$, and similarly $g\sigma(g^{-1})$ is an element of the normalizer of $S$, $N_{G(E)}(S)$.

We pause for a moment to make a few calculations in $SL_2(\mathbb{C}(t))$. Up to conjugacy there are two maximal tori:

$S = \{ (a \quad 0) \quad \frac{0}{a}^{-1} | a \in k - \{0\} \}$

$T = \{ (\begin{array}{cc} a & b \\
 b & a \end{array}) | a^2 - b^2 t = 1 \}$

$g = \left( \begin{array}{cc} 1 & -1/(2\sqrt{t}) \\
 1/2 & 1 \end{array} \right) \in SL_2(\mathbb{C}(t))[\sqrt{t}]$ $\text{Gal}(\mathbb{C}(t))[\sqrt{t}/\mathbb{C}(t)) \cong \{ \pm 1 \} \cong \mathbb{W}$

We notice that the Weyl group and Galois group coincide, which we do not expect to hold in general. However, in this section we will show that the Galois group can be realized as a subgroup of $W$.

This lays the foundation for a theorem of Debacker’s: If $K$ is a fixed maximal unramified extension of $l$, then we have a map conjugacy classes of maximal tori in $G(K)$ and $\sigma$-conjugacy classes in $W$. To see a full discussion and proof of this fact consider Debacker’s paper in the Michigan Math Journal. For this paper, we will simply ascotiate to $T$, an element $\sigma(g^{-1})gT(E)$ of the Weyl group. We now exhibit a surprising relation between the order of this element in the Weyl group and the degree of the extension $E/k$. 

8
Theorem 1. Let $k$ be a complete quasifinite field with nontrivial discrete valuation, and $G$ a connected split $k$-group with $k$-split $k$-torus $S$. Suppose $T$ is a $k$-torus split over an unramified or tamely ramified extension $E/k$, and $g \in G(E)$ is such that $\sigma(T)(E) = S(E)$. Then $|E : k| = |(\sigma(g^{-1})g)T(E)|$.

Proof. Let $S$ and $T$ be the $E$-rational points of $S$ and $T$ respectively, and $n = \sigma(g^{-1})g$. We will utilize Bruhat-Tits theory to prove this result.

Let us first figure out how the Galois group acts on the apartment $A(T, E) \subset B(G, E)$. Let $\lambda \in X_*(T, E)$. Then $\sigma$ implies there exists $\tau \in X_*(S, E)$ such that $\lambda = \sigma^{-1} \tau$. Then

$$\sigma(\lambda) = \sigma(\sigma^{-1}) \sigma(\tau) = \sigma(\sigma^{-1}) \tau = \sigma(g^{-1}g) \tau = \sigma(g^{-1})g \tau = \sigma(g^{-1})g \lambda \equiv n \lambda$$

as $Gal(E/k)$ acts trivially on the cocharacters of $S$, since it is split over $k$. Our calculation shows $Gal(E/k)$ has the adjoint action of $n = \sigma(g^{-1})g$ on $X_*(T, E)$. However, since $Gal(E/k)$ acts on $A(T, E)$, via $\sigma(x) = \sum_i \sigma(\lambda_i) \otimes r_i$, for $x = \sum_i \lambda_i \otimes r_i$, we know that $Gal(E/k)$ acts on $A(T, E)$ via

$$\sigma(x) = \sum_i (n\lambda) \otimes r_i$$

Inductively

$$\sigma^m(\lambda) = n^m \lambda$$

and

$$\sigma^m(\lambda) = \sigma^{m-1}(\sigma(\lambda)) = \sigma^{m-1}(\sigma(n)\sigma(\lambda)) = \sigma^{m-1}(\sigma^2(g^{-1})\sigma(g)\sigma(s^{-1})g)\lambda = \sigma^{m-1}(\sigma^2(g^{-1})g)\lambda \cdots = \sigma^m(g^{-1})g \lambda$$

Now we can begin to relate the order of sigma and the order of $nT(E)$. Suppose $|E : k| = m$, that is $\sigma$ has order $m$, then

$$\lambda = \sigma^m(\lambda) = n^m \lambda$$

for all $\lambda \in X_*(T, E)$. This is so if and only if $n^m \in T(E)$. So if $c$ is the order of $nT(E)$ in the weyl group, $c$ must divide $m$.

On the other hand since $n^c \in T(E)$,

$$\sigma^c(\lambda) = n^c \lambda = \lambda$$

So the work above implies the subgroup of $Gal(E/k)$ generated by $\sigma^c, < \sigma^c >$, fixes the apartment $A(T, E)$. Let

$$E^{< \sigma^c>} = \{ e \in E | \sigma^c(e) = e \}$$
be the fixed field of \(<\sigma^c>\). Simple Galois theory implies
\[<\sigma^c> = \text{Gal}(E/E^{\sigma^c})\]
and the work of Rousseau and Prasad implies
\[X_*(T, E^{\sigma^c}) \otimes \mathbb{Z} \mathbb{R} = A(T, E) = A(T, E) = X_*(T, E) = X_*(T, E) \otimes \mathbb{Z} \mathbb{R}\]
Which implies that
\[E^{\sigma^c} = E\]
and follows from an equivalence of categories: Tori over \(k\) split over \(E\) and \(\mathbb{Z}[\text{Gal}(E/k)]\)-modules which are free and finite rank as \(\mathbb{Z}\)-modules.

Hence \(<\sigma^c> = 1\), and finally that \(m\) must divide \(c\). \(\blacksquare\)

This theorem greatly aids our understanding of tori in quasifinite fields. Specifically, in \(k = \mathbb{C}((t))\) where there is a bijective correspondence between conjugacy classes of maximal tori and conjugacy classes in the weyl group, we can compute splitting fields for tori without a full description of the tori themselves. For a full description of the conjugacy classes in the weyl group see Carter’s work. Soon we will be interested in the case where the tori are assumed to be anisotropic. Let us make a full description of anisotropic tori in \(SL_n(\mathbb{C}((t)))\) by using the theorem.

**Corollary 1.1.** The anisotropic torus, up to conjugacy, in \(SL_n(\mathbb{C}((t)))\) splits over a degree \(n\) extension. As algebraic extensions of \(\mathbb{C}((t))\) are unique up to degree, the tori split over \(\mathbb{C}((t))[t^{1/n}]\).

**Proof.** As mentioned above in \(k = \mathbb{C}((t))\) where there is a bijective correspondence between conjugacy classes of maximal tori and conjugacy classes in the weyl group. This map restricts a bijective correspondence between anisotropic conjugacy classes in the weyl group \(W \cong S_n\), and \(SL_n(\mathbb{C}((t)))\)-conjugacy classes of minisotropic tori. Here anisotropic classes are ones such that no element of the class intersects a parabolic subgroup of \(W\), and minisotropic tori are anisotropic tori.

However, the only anisotropic conjugacy class in \(S_n\) are the \(n\)-cycles. Hence, by the theorem, the splitting field for the torus is of degree \(n\), as the order of its corresponding element in the weyl group is \(n\). \(\blacksquare\)

We note that the corollary is merely a special case of a more general result. Namely, the coxeter class in the weyl group always corresponds to a minisotropic torus, and the order of coxeter elements are well known.

Before moving on, and staying with \(k = \mathbb{C}((t))\), we will record the splitting fields for anisotropic tori in the groups \(SL_n\), \(G_2\), and \(D_4\).
Anisotropic Tori and Splitting Fields

<table>
<thead>
<tr>
<th>Group</th>
<th>Torus</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL_n</td>
<td>A_n</td>
<td>n</td>
</tr>
<tr>
<td>G_2</td>
<td>G_2</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>A_2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>A_1 + A_1</td>
<td>3</td>
</tr>
<tr>
<td>D_4</td>
<td>D_4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>D_4(a_1)</td>
<td>2</td>
</tr>
</tbody>
</table>

4 A look ahead: Points in the building and Anisotropic Tori

If T is an anisotropic torus split over E/k, then we may realize B(T, k) ⊂ B(G, k) as a point. I spent quite sometime trying to identify this point using only its corresponding class in the weyl group along with the degree of the splitting field. I have successfully predicted the anisotropic tori in SL_n, G_2, and D_4 and have strong reasons to believe these methods will work in general.

The method is as follows: We can, via Bruhat-Tits theory, assume that

\[ B(T, k) = A(S, k) \cap B(T, E) \]

From here one identifies B(T, k) as the fixed point in A(S, k) of \( w_T = w_0 + t_{[E:k]} \in W \), where \( w_0 \) is a representative of the conjugacy class corresponding to T and \( t_{[E:k]} \) is a translation depending on the degree of the splitting field over k. The author will be much obliged for a proof or insight in the general case.

It gives me great pleasure to thank my advisor Stephen Debacker for his patience and insight. I would also like to thank Gopal Prasad for his many useful comments.

5 Examples and Illustrations

\[ \text{Here each conjugacy class of anisotropic tori is given by Carter’s notation for the corresponding conjugacy class in the weyl group.} \]
Figure 1: Apartment of $SL_2$
and Settings/Wade Hindes/My Documents/My Pictures/Apartment.PNG

Figure 2: Aclove in $G_2$
and Settings/Wade Hindes/Desktop/Facet1.PNG

Figure 3: Aclove in $SL_3$
and Settings/Wade Hindes/My Documents/My Pictures/Facet2.PNG
Figure 4: A love in $Sp_4$

and Settings/Wade Hindes/My Documents/My Pictures/Facet3.PNG

Figure 5: Apartment of $SL_2(Q_2)$

and Settings/Wade Hindes/My Documents/My Pictures/wire36.jpg

References


