PATTERN AVOIDENCE ON ARCH CONNECTED GRAPHS

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Abstract. In this paper we prove results in graph theory in the area of Wilf equivalencies. We define a group of graphs called Arch Connected Graphs (ACG) and look at properties of pattern avoidance within this group. Let $n$ and $m$ be any positive integers. An $n$-pattern, $p_n$ is a specific ACG on $2n$-vertices. We say that an ACG, $G$, avoids $p$ if $p$ is not contained in $G$. We then count the number of ACGs on $2m$ vertices that avoid $p_n$ and we call this number $a_{p_n}(m)$. We then analyze the relationship that two $n$-patterns, $p_n$ and $q_n$, must satisfy in order for $a_{p_n}(m) = a_{q_n}(m)$ for all $m$. We uncover and prove a variety of relationships and discuss areas for future research.

1. Notation

In this section we will define a group of graphs which we will call Arch Connected Graphs (ACG) and define the terminology that we will use throughout the paper.

Definition. Arch Connected Graph

Let $n$ be a positive integer. An Arch Connected Graph on $2n$-vertices is created by placing $2n$ vertices equally spaced on a line. $n$ arches will then be placed on the line oriented concave down such that each vertex is connected to one arch and each arch is connected to two vertices and no two arches cross.

The following is an example of an ACG on 6, 8, and 10 vertices:

Figure 1
For simplification we will occasionally use parenthesis notation. An open parenthesis will correspond to the beginning of an arch and a close parenthesis will correspond to the end of an arch. For example, Figure 1 will correspond to the following: ( ( ) ) ( ). Figure 2 will be: ( ) ( ) ( ( ) ). Figure 3 will be: ( ( ) ( ) ) ( ( ) ).

Definition. Avoidance

Let \( p_n \) and \( q_m \) be ACG's on \( 2n \)-vertices and \( 2m \)-vertices respectively. Then \( p_n \) avoids \( q_m \) if \( q_m \) cannot be generated by the removal arches and their corresponding vertices from \( p_n \).

Using the examples above we see that Figure 2 avoids Figure 1, however Figure 3 does not void either Figure 1 and Figure 2.

Definition. Let \( p_n \) be an ACG on \( 2n \) vertices. Define \( a_{p_n}(m) \) as the number of ACGs on \( 2m \)-vertices that avoid \( p_n \).

Note that \( m \) does not necessarily have to be bigger than \( n \). That is, if \( m \) is less than \( n \) then \( a_{p_n}(m) \) is the number of all ACGs on \( 2m \)-vertices. If \( n \) equals \( m \) then \( a_{p_n}(m) \) is one less then the number of all ACGs on \( 2m \)-vertices.

Definition. Equivalence

Two ACGs on \( 2n \)-vertices, \( p_n \) and \( q_n \), are equivalent if \( a_{p_n}(m) = a_{q_n}(m) \) for all \( m \). This will be denoted \( p_n \sim q_n \).

In the next section we will discuss conditions on \( p_n \) and \( q_n \) that will cause them to be equivalent.

Definition. Identical

Two ACGs are identical if they have equal number of vertices and the arches on both ACGs are positioned in the same pattern. If \( p \) and \( q \) are identical, it will be denoted \( p \equiv q \).
For example, let $p_4$ and $q_3$ be ACGs on 8 and 6 vertices respectively. Let $q_3$ be $( ( ) ( ) )$. Then $p_4 \equiv ( q_3 )$ if $p_4$ is $( ( ) ) ( )$. Furthermore, it is clear that an ACG $p$ is always equivalent to itself.

**Definition. Mirror**

$p_n$ is the mirror of $q_n$ if $p_n$ is identical to $q_n$ after rotating it $180^\circ$ about the y-axis. The mirror of $p_n$ will be denoted $p'_n$.

For example the following figures are mirrors of each other:

![Figure 4](image1)

![Figure 5](image2)

Furthermore, if $p_n$ is the mirror of $q_n$ then $q_n \equiv p'_n$.

2. Equivalencies

In this section we will discuss conditions under which two ACGs will be equivalent. Let $p_n$ and $q_m$ be ACGs on $2n$ and $2m$ vertices respectively.

**Lemma 1.** The map $\phi : p_n \rightarrow p'_n$ is a bijection.

Proof. By the definition of mirror, $\phi$ is an automorphism. QED.

**Lemma 2.** $q_m$ avoids $p_n$ if and only if $q'_m$ avoids $p'_n$.

Proof. This can be seen easily by contradiction. Assume $q_m$ does not avoid $p_n$. This implies that there are a group of arches with corresponding vertices when removed from $q_m$ would produce $p_n$. Since $q'_m$ and $p'_n$ are simply the reflections of $q_m$ and $p_n$ respectively, we can remove the corresponding arches from $q'_m$ and produce $p'_n$. This implies that $q'_m$ does not avoid $p'_n$. The same argument can be used to show that if $q'_m$ does not avoid $p'_n$ than $q_m$ does not avoid $p_n$. QED.
Theorem 3. $p_n \sim p'_n$.

Proof. We need to show that the number of ACGs on $2m$-vertices that avoid $p_n$ is the same as the number of ACGs on $2m$-vertices that avoid $p'_n$ for all $m$. Let $G_m$ be all the ACGs on $2m$ vertices that avoid $p_n$. By Lemma 2 we see that $\phi(G_m)$ is the group consisting of all ACGs on $2m$ vertices that avoid $p'_n$. By Lemma 1 we see that the $|G_m| = |\phi(G_m)|$. Therefore, by definition of equivalence, $p_n \sim p'_n$. 

\qed

Theorem 4. If $p_n \sim q_n$ then $(p_n) \sim (q_n)$.

Proof. We need to show that the number of ACGs on any given number of vertices that avoid $(p_n)$ is equal to the number that avoid $(q_n)$. We will use a counting argument to prove the theorem. For clarity we will denote $(p_n)$ and $(q_n)$ by $P_{n+1}$ and $Q_{n+1}$, respectively.

To count the number of ACGs on $2m$-vertices that avoid $P_{n+1}$ and $Q_{n+1}$ we will sort the ACGs by the position of the first arch on the left. That is, we will count the number of ACGs that avoid $P_{n+1}$ and $Q_{n+1}$ whose first arch on the left lands on the second vertex from the left. We will than add that to the number of ACGs that avoid $P_{n+1}$ and $Q_{n+1}$ whose first arch on the left lands on the fourth vertex from the left and so on until the first arch lands on the last vertex on the right. A general element, $r_m$, on $2m$-vertices of this form will look like the following:

![Figure 6](image)

Where $x$ is any positive integer, and corresponds to the number of arches inside the first arch. Furthermore, the number of arches to the right of the first arch is $m - x - 1$.

We will now count the number of ACGs of this form that will avoid $P_{n+1}$. When we look at avoidance within the first arch we cannot count the number ACGs on $2x$-vertices that avoid $P_{n+1}$. Instead we must look at the number ACGs on $2x$-vertices that avoid $p_n$. Since there is only one outer arch in $P_{n+1}$, it is necessary for the $2x$-vertices inside the first arch to avoid $p_n$, or else $r_m$ would not avoid $P_{n+1}$. 
For a given $x$ it is clear that the number of ACGs of the form $r_m$ that avoid $P_{n+1}$ is the product of the number of ACGs on $2x$-vertices that avoid $p_n$ and the number of ACGs on $2(m-x-1)$-vertices that avoid $P_{n+1}$. That is the number of ACGs of the form $r_m$ that avoid $P_{n+1}$ is: $a_{p_n}(x) \cdot a_{P_{n+1}}(m-x-1)$.

Therefore for any $x$ the number of ACGs on $2m$-vertices that avoid $P_{n+1}$ is:

$$a_{P_{n+1}}(m) = \sum_{x=0}^{m-1} a_{p_n}(x) \cdot a_{P_{n+1}}(m-x-1)$$

Similarly the number of ACGs on $2m$-vertices that avoid $Q_{n+1}$ is:

$$a_{Q_{n+1}}(m) = \sum_{x=0}^{m-1} a_{q_n}(x) \cdot a_{Q_{n+1}}(m-x-1)$$

We assumed that $p_n \sim q_n$, therefore $a_{p_n}(z) = a_{q_n}(z)$ for all $z$. Therefore the recursion formula that generates $a_{P_{n+1}}(m)$ and $a_{Q_{n+1}}(m)$ is the same. It remains to show that they have the same base cases. As we stated earlier in the paper, if $m$ is less than $n+1$ then the number of ACGs that avoid $P_{n+1}$ and the number of ACGs that avoid $Q_{n+1}$ is equal to the size of the entire set of ACGs on $2m$ vertices. Therefore $a_{P_{n+1}}(m)$ and $a_{Q_{n+1}}(m)$ have the same base cases and $a_{P_{n+1}}(m) = a_{Q_{n+1}}(m)$. By the definition of equivalence we are done.

□

Theorem 5. For any $q_{n-1}$, if $p_n \equiv (q_{n-1})$ then $(p_n) \sim (p_n)$.

Proof. We will use the same technique in this proof as we used to prove the previous theorem. That is we will sort the ACGs that avoid $(p_n)$ and $(p_n)$ by the location of the first arch on the left. For clarity we will denote $(p_n)$ as $\phi_{n+1}$ and $(p_n)$ as $\pi_{n+1}$. Also, $r_m$ will refer to Figure 6, where $x$ is any positive integer.

We will now count the number of ACGs that avoid $\phi_{n+1}$. The arches on the $2x$-vertices within the first arch must avoid $\phi_{n+1}$. This is because $\phi_{n+1}$ has two exposed arches and as a result the first arch is not relevant to this particular avoidance. However the arches on the $2(m-x-1)$-vertices to the right of the first arch must avoid $p_n$. It is clear by examination that if these arches do not avoid $p_n$ then $r_m$ will not avoid $\phi_{n+1}$. Therefore for a specific $x$ the number of ACGs of the form $r_m$ that avoid $\phi_{n+1}$ is: $a_{\phi_{n+1}}(x) \cdot a_{p_n}(m-x-1)$. Furthermore, if we generalize $x$ the number of ACGs that avoid $\phi_{n+1}$ is:
\[
a_{\phi_{n+1}}(m) = \sum_{x=0}^{m-1} a_{\phi_{n+1}}(x) \cdot a_{p_n}(m - x - 1)
\]

We will now count the number of ACGs that avoid \(\pi_{n+1}\). Since there is only one outer arch on \(\pi_{n+1}\), it is necessary for the arches on the \(2x\)-vertices within the first arch to avoid \(p_n\) in order for \(r_m\) to avoid \(\pi_{n+1}\). Similarly, the arches on the \(2(m - x - 1)\)-vertices to the right of the first arch must avoid \(\pi_{n+1}\) in order for \(r_m\) to avoid \(\pi_{n+1}\). Therefore for a specific \(x\) the number of ACGs of of the form \(r_m\) that avoid \(\pi_{n+1}\) is:

\[
a_{\phi_{n+1}}(m) = \sum_{x=0}^{m-1} a_{\phi_{n+1}}(x) \cdot a_{p_n}(m - x - 1)
\]

If we expand the summation for \(a_{\phi_{n+1}}(m)\) and \(a_{\phi_{n+1}}(m)\) we get:

\[
a_{\phi_{n+1}}(m) = a_{\phi_{n+1}}(0) \cdot a_{p_n}(m-1) + a_{\phi_{n+1}}(1) \cdot a_{p_n}(m-2) + \cdots + a_{\phi_{n+1}}(m-1) \cdot a_{p_n}(0)
\]

\[
a_{\pi_{n+1}}(m) = a_{p_n}(0) \cdot a_{\pi_{n+1}}(m-1) + \cdots + a_{p_n}(m-2) \cdot a_{\pi_{n+1}}(1) + a_{p_n}(m-1) \cdot a_{\pi_{n+1}}(0)
\]

By inspection we see that the two recursions are the same, we are just summing the terms in opposite order. Therefore to finish the proof we need only show that the two recursions have the same base cases.

As we stated earlier in the paper, if \(m\) is less than \(n+1\) then the number of ACGs that avoid \(\phi_{n+1}\) and the number of ACGs that avoid \(\pi_{n+1}\) is equal to the size of the entire set of ACGs on \(2m\) vertices. Therefore \(a_{\phi_{n+1}}(m)\) and \(a_{\pi_{n+1}}(m)\) have the same base cases. This implies \(a_{\phi_{n+1}}(m) = a_{\pi_{n+1}}(m)\), so by the definition of equivalent we are done.

\[
\square
\]

### 3. Further Research

In the previous section we have proved a variety of conditions under which two ACG’s will be equivalent. However, after inspecting the data in Appendix A, we see that these are not the only conditions.

Let's look at the ACG’s on 8-vertices that are in the equivalence class with \( ( ( ( ( ) ) ) ) \). We see that all of the ACGs in this class can be shown to be equivalent by the relations we showed in the previous section except for the ACG \( ( ( ) ( ( ) ) ) \), which we will call \(p_8\). It is clear that \(p_8\) cannot be shown to be equivalent to any other ACG
in this class by the theorems shown in the previous section. However after using a counting argument similar to the one we used to prove our theorems, we see that the recursion that generates $a_{p_8}(m)$ is the same as the recursion that generates $a_{Q_8}(m)$ where $Q$ is any other ACG in this class.

In future research we will try to prove generally why $p_8$ is in this class and look for other equivalences in larger ACGs. It would also be interesting to investigate if there is a more general theorem that would incorporate all the previous theorems, as well as proving if the conditions we have shown are the only conditions that would allow two ACGs to be equivalent.

In further research we could also examine pattern avoidances when we take the two ends of an ACG and connect them to form a circle. We would examine not only pattern avoidance on the circles but also how they are related to their corresponding patterns on lines.

**APPENDIX A. Data**

The following are all the ACGs on 4, 6, and 8 vertices sorted into equivalence classes. Underneath each equivalence class is a formula for $a_p(m)$ for any ACG, $p$, in the given class, if known.
\[ a_p(m) = 2^{m-1} \]

\[ a_p(m) = 1 + \frac{(m-1)m}{2} \]

\[ a_p(m) = 3a_p(m-1) - a_p(m-2) \]