Computation of equilibrium measures and other quantities of interest for Random Matrices

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Abstract
We have calculated the equilibrium measure of potentials of degree 4 (single and double support), and degree 6 (single support only). We proceed to analyze functions that determines the expected location of the largest eigenvalue of spiked Hermitian matrix models.

1 Introduction
Random matrix theory is the study of the statistical properties of matrices whose entries are random variables. The study of random matrices began for the purpose of modeling scattering resonances of neutrons off heavy nuclei. While there are many types of random matrices in this paper I will restrict attention to a specific type - Hermitian Unitary Ensembles.

Definition 1.1 Let \( \mathcal{H}(n) \) be the set of all \( n \times n \) Hermitian matrices \( (H_{ij} = H_{ji}^*) \) and let \( V(x) \) be a polynomial of even degree i.e.

\[
V(x) = a_{2j}x^{2j} + \cdots + a_0, \quad a_{2j} > 0.
\]

Then a distribution on \( \mathcal{H}(n) \) of the form

\[
Ce^{-\text{Tr}(V(H))},
\]

Where \( H \in \mathcal{H}(n) \) and \( C \) is a normalization constant is called a Unitary Ensemble.

The reason for the requirement of an even polynomial is for the distribution to be well-defined. Also the form \( e^{-\text{Tr}(V(H))} \) is a special case of the requirement that the distribution be invariant under unitary transformations (a natural condition in physics problems) see [1].

The statistical property of interest for this paper is the eigenvalue distribution of \( \mathcal{H}(n) \) in the limit that \( n \) becomes arbitrarily large. But in order to obtain this one must naturally have some notion of the eigenvalue distribution of \( \mathcal{H}(n) \). I will cite the result which can be found again in [1]
Theorem 1.2 The joint distribution of the eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ of a random Hermitian matrix $H \in \mathcal{H}(n)$ from the unitary-invariant ensemble has the density function:

$$
\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^2 \prod_{j=1}^{n} e^{-V(\lambda_j)}.
$$

Where the constant $\hat{Z}_n$ is the normalization constant.

The proof is omitted.

2 Equilibrium Measure

In this section I will describe the means by which we calculate the equilibrium distribution of eigenvalues of $H(n)$ as $n$ becomes arbitrarily large in the unitary ensemble. I will begin by recasting the problem as it is done in [1].

Let us analyze the rescaled probability density function:

$$
\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^2 \prod_{j=1}^{n} e^{-nV(\lambda_j)}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n.
$$

(2)

Notice that the potential has been multiplied by $n$. Now if we take the logarithm of this density function:

$$
\log \left( \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^2 \prod_{j=1}^{n} e^{-nV(\lambda_j)} \right)
$$

(3)

$$
= \sum_{i \neq j} \log |\lambda_i - \lambda_j| - n \sum_{j=1}^{n} V(\lambda_j) - \log \hat{Z}_n.
$$

(4)

Additionally we define a counting measure:

$$
\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}.
$$

(5)

Then result (4) becomes:

$$
-n^2 I(\mu_n) + \text{Constant}.
$$

Where

$$
I(\mu_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log|x-y|^{-1} d\mu_n(x)d\mu_n(y) + \int_{-\infty}^{\infty} V(x)d\mu_n(x).
$$

(6)

The formula (4) provides us the following intuition on the eigenvalues: We can think of it as an energy expression - with the $V(\lambda_j)$ being an external...
attractive potential and the log $|\lambda_j - \lambda_i|$ being a repulsive potential. So then this problem becomes similar to that of putting charged particles on a line. This interpretation has been used in various interesting applications see for example [2].

In this case we can use the interpretation to find a way to the equilibrium measure. In statistical mechanics one can calculate thermodynamical quantities using the canonical ensemble, for which the expression for the probability of the event is:

$$Pr(A) = \sum e^{-\beta E(A)} Z.$$  \hfill (7)

Where $E$ is the energy, the sum is taken over all occurrences of $A$, and $Z$ is the partition function (a normalization factor). What one can see from this is that the highest probability of occurrence is the situation when the energy is low or if the degeneracy of an event is extremely high. Our expression (4) can be seen to be nothing more then the term in the exponential in (7) (without the factor of $\beta$).

After this comparison it becomes plausible that when taking the limit as $n$ becomes arbitrarily large, $I(\mu)$ will approach its minimum possible value (the fact that the minimum value of $I$ is the most probable event and the $n^2$ term gets arbitrarily large justifies this statement).

**Theorem 2.1** For a continuous function $V(x)$ satisfying $\frac{V(x)}{\log(x^2+1)} \to \infty$ as $|x| \to \infty$, there is a unique probability measure $d\mu_c$ such that $I(\mu)$ is at its greatest lower bound. Also $\mu_c$ has compact support. When $V$ is an even degree polynomial with a positive leading coefficient $d\mu_c(x) = \psi_c(x) dx$ where $\psi_c(x)$ is a continuous function that is real analytic in its support.

Now the problem becomes a variational one (we need to solve the minimizer of $I(\mu)$). That is we wish to determine

$$\inf_{\mu \in P} I(\mu) = \inf_{\mu \in P} \left[ -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log |x-y| d\mu(x)d\mu(y) + \int_{-\infty}^{\infty} V(x) d\mu(x) \right],$$

where

$P = \{ \text{Positive Borel measures } \mu \text{ on } \mathbb{R} : \int_{\mathbb{R}} d\mu(x) = 1 \}$.

The way this will be done is by effecting a first order change in a variable $0 \leq t \leq 1$. Consider some $\nu$ that is a probability measure with compact support and $I(\nu)$ exists and is finite. Then let us consider:

$$I(t\nu + (1-t)\mu_c) = I(\mu_c) + t \int_{-\infty}^{\infty} V(x) d(\nu - \mu_c)(x)$$

$$-2t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log |x-y| d(\nu - \mu_c)(x)d\mu_c(y)$$

$$-t^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log |x-y| d(\nu - \mu_c)(x)d(\nu - \mu_c)(y).$$  \hfill (8)
The minimal value of $I$ is achieved when the $O(t)$ terms in (8) are non-negative. Which is equivalent to the statement that for all $\nu$ with compact support such that $I(\nu)$ exists and is finite the following holds:

$$
\int_{-\infty}^{\infty} \left[ -2 \int_{-\infty}^{\infty} \log |x - y| d\mu_c(y) + V(x) \right] d\nu(x) \geq \ell_c.
$$

(9)

Where $\ell_c$ is defined by:

$$
\ell_c = \int_{-\infty}^{\infty} \left[ -2 \int_{-\infty}^{\infty} \log |x - y| d\mu_c(y) + V(x) \right] d\mu_c(x).
$$

(10)

It can be proven from here that if we replace $d\mu_c(x)$ in the previous equations by the $\psi_c(x)dx$ mentioned in the earlier theorem, we obtain the minimizer that we seek:

**Proposition 2.2** Suppose that the minimizer $d\mu_c(x) = \psi_c(x)dx$ for some continuous function $\psi_c$ of compact support. This means that there exists a real constant $\ell_c$ such that:

$$
-2 \int_{-\infty}^{\infty} \log |x - y| \psi_c(y) dy + V(x) \geq \ell_c, \quad x \in \mathbb{R}
$$

(11)

and

$$
-2 \int_{-\infty}^{\infty} \log |x - y| \psi_c(y) dy + V(x) = \ell_c, \quad \text{for } x \text{ such that } \psi_c(x) > 0.
$$

(12)

Conversely, if $\psi_c$ satisfies (11) and (12), then $d\mu_c(x) = \psi_c(x)dx$ is the minimizer of $I$.

### 2.1 General facts needed to calculate $\psi_c(x)$ when $supp(\psi_c) = [-A, A]$

In the case we have $V(x) = -V(-x)$ it is logical to make the ansatz that the support of $\psi_c(x)$ is symmetric on the real line. This can actually be shown as in [1], but we will omit the details. The support of the equilibrium measure can consist of several intervals. For the case of a single interval we use the following:

**Lemma 2.3** For $x \in (-A, A)$,

$$
P.V. \int_{-A}^{A} \frac{\psi_c(y)}{x - y} dy = \frac{V'(x)}{2}.
$$

(13)

Where $P.V.$ denotes the Cauchy Principle Value integral.
Proof. Let \( \log \) denote the principal branch of the logarithmic function. Then
\[
\lim_{\epsilon \to 0} \int_{-A}^{A} \log(x \pm i\epsilon - y)\psi_c(y)dy = \int_{-A}^{A} \log|x - y|\psi_c(y)dy \pm i\pi \int_{x}^{A} \psi_c(y)dy. \tag{14}
\]
For \( y \in (-A,A) \) \( \psi_c(y) \) is analytic, thus the contour in the left hand side of this expression can be modified in a way that \( x \) is not on the contour. In other words we replace the left hand side “plus” expression with a contour lying slightly above \((-A,A)\) called \( \sigma_+ \). Also for the “minus” expression we replace it with a contour slightly below \((-A,A)\) called \( \sigma_- \). Now if we sum these two expressions on the left and divide by two we obtain:
\[
\int_{-A}^{A} \log|x - y|\psi_c(y)dy = \frac{1}{2} \left( \int_{\sigma_+} \log(x - y)\psi_c(y)dy + \int_{\sigma_-} \log(x - y)\psi_c(y)dy \right). \tag{15}
\]
By differentiating this we obtain:
\[
\frac{d}{dx} \int_{-A}^{A} \log|x - y|\psi_c(y)dy = \frac{1}{2} \left( \int_{\sigma_+} \frac{\psi_c(y)}{x - y}dy + \int_{\sigma_-} \frac{\psi_c(y)}{x - y}dy \right). \tag{16}
\]
If we select \( \sigma_{\pm} = [-A, x - \epsilon] \cup [x + \epsilon, A] \cup x + e^{\pm i\theta} : 0 \leq \theta \leq \pi \) and pass the limit as epsilon goes to zero we see that
\[
\frac{d}{dx} \int_{-A}^{A} \log|x - y|\psi_c(y)dy = P.V. \int_{-A}^{A} \frac{\psi_c(y)}{x - y}dy. \tag{17}
\]
This coupled with (12) gives us our result \( \square \).

Also the following are results stated without proof:

- Given a function \( u(x) \) defined on \((-A,A)\) let
  \[
  U(z) = \int_{-A}^{A} \frac{u(y)}{y - z}dy \quad z \in \mathbb{C}\setminus[-A,A] \tag{18}
  \]
  which is the Cauchy transform of \( u(x) \). Then defining \( U_{\pm}(x) \) to be the limit as \( z \) goes to \( x \) from above (or below) the real line respectively the following holds
  \[
  U_{\pm}(x) = P.V. \int_{-A}^{A} \frac{u(y)}{y - x}dy \pm i\pi u(x) \quad x \in (-A,A). \tag{19}
  \]
  Which means
  \[
  U_+(x) + U_-(x) = 2P.V. \int_{-A}^{A} \frac{u(y)}{y - x}dy, \quad x \in (-A,A) \tag{20}
  \]
  and
  \[
  U_+(x) - U_-(x) = 2i\pi u(x), \quad x \in (-A,A). \tag{21}
  \]
• Conversely if we are given a function $U(z)$ analytic in $z \in \mathbb{C} \setminus [-A, A]$ such that it satisfies (21) for a function $u(x)$ which is analytic in $(-A, A)$, in addition to the requirements that $U(z) \to 0$ as $z \to \infty$ and $(z\pm A)U(z) \to 0$ as $z \to \mp A$. Then (18) must hold.

Define:

$$\Psi(z) = \int_{-A}^{A} \frac{\psi_c(y)}{y-z} dy, \quad z \in \mathbb{C} \setminus [-A, A].$$  \hspace{1cm} (22)

So if we expand $\frac{1}{y-z}$ and integrate we get that

$$\Psi(z) = -\frac{1}{z} + O(z^{-2}), \quad z \to \infty,$$  \hspace{1cm} (23)

and also from (21) we have

$$\psi_c(x) = \frac{1}{2\pi i}(\Psi_+(x) - \Psi_-(x)), \quad x \in (-A, A).$$  \hspace{1cm} (24)

Likewise from (20) we have

$$\Psi_+(x) + \Psi_-(x) = 2P.V. \int_{-A}^{A} \frac{\psi_c(y)}{y-x} dy = -V'(x) \quad x \in (-A, A).$$  \hspace{1cm} (25)

Now define

$$R(z) = \sqrt{(z-A)(z+A)}$$  \hspace{1cm} (26)

on the branch such that $R(z)$ is analytic in $z \in \mathbb{C} \setminus [-A, A]$ and is positive for $z = x > A$. Now using our convention when $x \in (-A, A)$ then $R_{\pm}(x)$ stands for the value of $R(z)$ in the limit as we approach above/below the real line respectively. It can be seen easily that $R_{\pm}(x) = -R_{\mp}(x)$. So then

$$\frac{\Psi_+(x)}{R_+(x)} - \frac{\Psi_-(x)}{R_-(x)} = -\frac{V'(x)}{R_+(x)},$$  \hspace{1cm} (27)

which by applying (18) we obtain the result that

$$\Psi(z) = \frac{R(z)}{2\pi i} \int_{-A}^{A} \frac{V'(y)}{R_+(y)(y-z)} dy.$$  \hspace{1cm} (28)

Using $R_+(x) = -R_-(x)$ we rewrite:

$$\Psi(z) = -\frac{R(z)}{4\pi i} \int_{-A}^{A} \frac{V'(y)}{R_+(y)(y-z)} dy + \frac{R(z)}{4\pi i} \int_{-A}^{A} \frac{V'(y)}{R_-(y)(y-z)} dy$$

$$= \frac{R(z)}{4\pi i} \oint_C \frac{V'(y)}{R(y)(y-z)} dy.$$  \hspace{1cm} (29)

Where $C$ is a counter-clockwise oriented contour that contains the interval $[-A, A]$ inside and has the point $y = z$ outside of it. At this point the Residue Theorem can be used to evaluate $\Psi(z)$. In order to find out the constraints on $A$ one must use the (23) to the asymptotic limit of $\Psi(z)$ and match the terms.
2.2 Computation of $\psi_c(x)$ for $V(x) = \frac{g}{4}x^4 + \frac{t}{2}x^2$ Part 1

For this selection of $V(x)$ the result I will derive can be found in [3]. I will begin with the case in which the support of $\psi_c(x)$ is a single interval and then find the parameter regime for which it holds. Then I will move on to the case that the support is two intervals (and this will be all possible cases).

$$\{ x \in \mathbb{R} : \psi_c(x) > 0 \} = (-A, A)$$

We will find out the conditions on $g$ and $t$ for which this is correct. Following the guide in the previous section (Using the same definitions of $\Psi$ and $R(z)$) we have that:

$$\Psi(z) = \frac{R(z)}{4\pi i} \oint_C \frac{(gy^3 + ty)}{R(y)(y - z)} dy$$

(30)

Where $C$ is a counter-clockwise oriented contour that contains the interval $[-A, A]$ inside and has the point $y = z$ outside of it. Now I can use the Residue Theorem to evaluate this.

$$\Psi(z) = \frac{R(z)}{4\pi i} \left( \frac{-2\pi i(gz^3 + tz)}{R(z)} + \lim_{r \to \infty} \oint_{|y| = r} \frac{gy^3 + ty}{R(y)(y - z)} dy \right)$$

(31)

The second integral in this expression can be evaluated by doing a change of variables to $y = \frac{\zeta}{z}$

$$\lim_{r \to \infty} \oint_{|y| = r} \frac{gy^3 + ty}{R(y)(y - z)} = \lim_{r \to \infty} \oint_{|\zeta| = \rho} \frac{gd\zeta}{\zeta^5(1 - z\zeta)\sqrt{1 - A^2\zeta^2}} + \lim_{r \to \infty} \oint_{|\zeta| = \rho} \frac{td\zeta}{\zeta\sqrt{1 - A^2\zeta^2}(1 - z\zeta)}$$

(32)

Each of these integrals can be evaluated using the Residue Theorem.

$$\lim_{r \to \infty} \oint_{|y| = r} \frac{gy^3 + ty}{R(y)(y - z)} = i\pi(g(A^2 + 2z^2) + 2t)$$

(33)

So then

$$\Psi(z) = -\frac{1}{2}(gz^3 + tz) + \frac{R(z)}{4}(g(A^2 + 2z^2) + 2t)$$

(34)

$$= -\frac{1}{2}(gz^3 + tz) + \frac{z}{4}\sqrt{1 - \frac{A^2}{z^2}}(g(A^2 + 2z^2) + 2t)$$

(35)

Now using the fact that

$$\sqrt{1 - \frac{A^2}{z^2}} = 1 - \sum_{k=1}^{\infty} \left( \prod_{j=1}^{k} \frac{2j - 1}{2j} \right) \left( \frac{A}{z} \right)^{2k} \frac{1}{2k - 1}$$

We can see that

$$\Psi(z) = -\frac{3gA^4}{16z} - \frac{tA^2}{4z} + O(z^{-2})$$

(36)
Which when compared with (23) tells us that
\[ 3gA^4 + 4tA^2 - 16 = 0 \]  
(37)
And so A which must be positive must be:
\[ A = \sqrt{-2t + \sqrt{4t^2 + 48g}} \]
(38)
As for the measure itself we can obtain it using (24)
\[
\psi_c(x) = \frac{1}{2\pi i} \left( \Psi_+(x) - \Psi_-(x) \right)
\]
\[
= \frac{1}{4\pi i} \left( R(x)(g(A^2 + 2x^2) + 2t) \right)
\]
\[
= \frac{\sqrt{A^2 - x^2}}{\pi} \left( \frac{gA^2}{4} + \frac{gx^2}{2} + \frac{t}{2} \right)
\]
(39)
Now since the measure must be positive for the domain \([-A, A]\) this means at its minimum \(\psi_c(x)\) has to be greater than or equal to zero. Which means that:
\[
\frac{t}{2} + \frac{gA^2}{4} = \frac{t + \sqrt{t^2 + 3g}}{3} \geq 0
\]
(40)
And so we can come to the conclusion that \(\psi_c(x)\) has a single interval support when:
\[ t \geq -2\sqrt{g} \]
(41)

2.3 Generalizing the computation technique when \(\text{supp}(\psi_c) = [-A, -B] \cup [B, A]\)

The general technique outlined in section 2.1 still holds with a few modifications. The first is that in equation (13) the left hand side is replaced by:
\[
P.V. \left[ \int_{-A}^{-B} \frac{\psi_c(y)}{x-y} dy + \int_{B}^{A} \frac{\psi_c(y)}{x-y} dy \right]
\]
(42)
Likewise the integrals in (18),(19),and (20) are replaced by:
\[
\int_{-A}^{-B} \frac{u(y)}{y-x} dy + \int_{B}^{A} \frac{u(y)}{y-x} dy
\]
(43)
And \(\Psi(z)\) as one might guess is now redefined to be:
\[
\Psi(z) = \int_{-A}^{-B} \frac{\psi_c(y)}{y-z} dy + \int_{A}^{B} \frac{\psi_c(y)}{y-z} dy, \quad z \in \mathbb{C} \setminus [-A, -B] \cup [B, A].
\]
(44)
The only other deviation is that \(R(z)\) now must be redefined to be:
\[
R(z) = \sqrt{(z-A)(z+B)(z-A)(z-B)}
\]
(45)
All of the other tools we used in section 2.1 still hold.
2.4 Computation of $\psi_c(x)$ for $V(x) = \frac{g}{4}x^4 + \frac{t}{2}x^2$ Part 2

Using our method for the case of two interval support we obtain:

$$
\Psi(z) = \frac{R(z)}{4\pi i} \left( \oint_{C_1} \frac{gy^3 + ty}{R(y)(y - z)} \, dy + \oint_{C_2} \frac{gy^3 + ty}{R(y)(y - z)} \, dy \right). \tag{46}
$$

Where $C_{1,2}$ are both counter-clockwise oriented contours with $y = z$ located outside both. $C_1$ contains $[-A, -B]$ and $C_2$ contains $[B, A]$. Now using the Residue Theorem:

$$
\Psi(z) = \frac{R(z)}{4\pi i} \left( \frac{-2\pi i (gy^3 + ty)}{R(y)} + \lim_{r \to \infty} \oint_{|y| = r} \frac{gy^3 + ty}{R(y)(y - z)} \, dy \right), \tag{47}
$$

and the second integral on the right hand side can be rewritten with a change of variables to $y = \frac{1}{\zeta}$:

$$
\lim_{r \to \infty} \oint_{|y| = r} \frac{gy^3 + ty}{R(y)(y - z)} \, dy = \lim_{\rho \to 0} \oint_{|\zeta| = \rho} \frac{gd\zeta}{\zeta^2 \sqrt{1 - \zeta^2} \sqrt{1 - \zeta^2 A^2} \sqrt{1 - \zeta^2 B^2} (1 - \zeta z)} + \lim_{\rho \to 0} \oint_{|\zeta| = \rho} \frac{td\zeta}{\sqrt{1 - \zeta^2 A^2} \sqrt{1 - \zeta^2 B^2} (1 - \zeta z)}. \tag{48}
$$

The second integral is a holomorphic function at $\zeta = 0$ so it evaluates to zero, the first one can be calculated simply, it is $2\pi izg$. This gives us:

$$
\Psi(z) = \frac{1}{2} \left( -\frac{gz^3 + tz}{2} + \frac{R(z)zg}{2} \right) = \frac{1}{2} \left( -gz^3 - tz + gz^3 \sqrt{1 - \frac{A^2}{z^2}} \sqrt{1 - \frac{B^2}{z^2}} \right). \tag{49}
$$

Now using the expansion in (2.2) for both square root terms we pick out terms of order higher than $z^{-2}$:

$$
\Psi(z) = \frac{1}{2} \left( -zt - \frac{g z(A^2 + B^2)}{2} - \frac{g(A^2 - B^2)^2}{8z} \right) + O(z^{-2}). \tag{50}
$$

So applying (23) we get the system of equations:

$$
-zt - \frac{g z(A^2 + B^2)}{2} = 0, \tag{52}
$$

$$
-\frac{g(A^2 - B^2)^2}{16} = -1. \tag{53}
$$

Which implies (taking into account that $A^2 > B^2$) that

$$
A^2 + B^2 = -\frac{2t}{g}, \tag{54}
$$

9
\[ A^2 - B^2 = \frac{4}{\sqrt{g}} \]  

(55)

Giving us the final result that:

\[ A = \sqrt{\frac{\sqrt{g} - t}{g}}, \quad B = \sqrt{\frac{-\sqrt{g} - t}{g}} \quad t < -2\sqrt{g}. \]  

(56)

And then using (24) we find that

\[ \psi_c(x) = \frac{g|x|}{2 \pi} \sqrt{(A^2 - x^2)(B^2 - x^2)}. \]  

(57)

3 Calculating \( \psi_c(x) \) when \( V(x) = \frac{x^6}{6} + \frac{ax^4}{4} + \frac{bx^2}{2} \) in the case that \( \text{supp}(\psi_c(x)) = [-A, A] \)

In this section I will discuss the equilibrium measure calculation for the potential that is most important for this paper. I will restrict interest to the single interval support case. Using the same technique in section 2.1 we get:

\[ \Psi(z) = \frac{R(z)}{4 \pi i} \int_C \frac{y^5 + ay^3 + by}{R(y)(y - z)} dy. \]  

(58)

Where again \( C \) is a counter-clockwise oriented contour with \( y = z \) on the outside and the interval \([ -A, A ] \) contained in it. Evaluating the expression using the Residue Theorem gives:

\[ \Psi(z) = \frac{R(z)}{4 \pi i} \left( \frac{-2\pi i(z^5 + az^3 + bz)}{R(z)} + \lim_{r \to \infty} \int_{|y|=r} \frac{(y^5 + ay^3 + by)dy}{R(y)(y - z)} \right). \]  

(59)

I will evaluate each term in the second integral using the change of variables \( y = \frac{1}{\zeta} \)

\[
\begin{align*}
\lim_{r \to \infty} \int_{|y|=r} \frac{y^5 dy}{R(y)(y - z)} &= \lim_{\rho \to 0} \int_{|\zeta|=\rho} \frac{d\zeta}{\zeta^5 \sqrt{1 - \zeta^2 A^2}(1 - \zeta)} \\
&= 2\pi i \left( z^4 + \frac{z^2 A^2}{2} + \frac{3 A^4}{8} \right),
\end{align*}
\]

\[
\begin{align*}
\lim_{r \to \infty} \int_{|y|=r} \frac{ay^3 dy}{R(y)(y - z)} &= \lim_{\rho \to 0} \int_{|\zeta|=\rho} \frac{d\zeta}{\zeta^3 \sqrt{1 - \zeta^2 A^2}(1 - \zeta)} \\
&= a\pi i (A^2 + 2z^2),
\end{align*}
\]

\[
\begin{align*}
\lim_{r \to \infty} \int_{|y|=r} \frac{bydy}{R(y)(y - z)} &= \lim_{\rho \to 0} \int_{|\zeta|=\rho} \frac{d\zeta}{\zeta \sqrt{1 - \zeta^2 A^2}(1 - \zeta)} \\
&= 2\pi ib.
\end{align*}
\]

Which put together gives us:

\[ \Psi(z) = -\frac{1}{2}(z^5 + az^3 + bz) + \frac{R(z)}{4} \left( 2z^4 + z^2 A^2 + \frac{3 A^4}{4} + aA^2 + 2az^2 + 2b \right). \]  

(60)
Upon expanding $R(z)$ using (2.2) all terms higher than $O(z^{-1})$ cancel and we are left with:

$$\Psi(z) = -\frac{(5A^6 + 6aA^4 + 8bA^2)}{32z} + O(z^{-2}). \quad (61)$$

Applying (23) we get the condition on $A$:

$$5A^6 + 6aA^4 + 8bA^2 - 32 = 0. \quad (62)$$

This is a cubic equation in $A^2$. The place where we will find solutions is in those parameter regimes for which this cubic equation has one real positive root (I will discuss the problem of finding this region in the next section). Using the value of $\Psi(z)$ from above we can obtain the equilibrium measure:

$$\psi_c(x) = \sqrt{\frac{A^2 - x^2}{\pi}} \left( \frac{x^4}{2} + x^2 \left( \frac{A^2}{4} + \frac{a}{2} \right) + \frac{3A^4}{16} + \frac{aA^2}{4} + \frac{b}{2} \right). \quad (63)$$

### 3.1 Regions in the $a$-$b$ plane where the support of $\psi_c$ is a single interval

The cubic equation in $A^2$ we obtained in the previous section:

$$5A^6 + 6aA^4 + 8bA^2 - 32 = 0$$

must only have one real positive solution for $A^2$ in order for the support to be a single interval. In order to find this region I will resort to using an elementary method called Sturm’s Theorem (I use [5] as my reference). Before I can state this theorem, I must first define:

**Definition 3.1** Let $f$ be a polynomial with simple roots and let $I$ be a given interval. Define $f_0 = f$ and $f_1 = f'$. We use the division algorithm on $f_0$ and $f_1$ and call the quotients from successive division $q_0, q_1, \cdots q_n$ and the remainders $-f_2, -f_3, \cdots -f_{n+2}$ i.e.

- $f_0 = q_0f_1 - f_2$
- $f_1 = q_1f_2 - f_3$
- $\vdots$

Let $s$ be the smallest number such that $f_s$ has the same sign throughout $I$. Then the sequence $\{f_i\}_{i=s}$ is called a **Sturm Chain**.

Given an interval $I$ we will be able to use Sturm’s Theorem to find out how many zeros of $f$ lie in $I$:

**Theorem 3.2** The number of real roots of an algebraic equation with real coefficients whose real roots are simple over an interval $I$ (the endpoints of which are not roots) is equal to the difference between the number of sign changes of the Sturm Chain evaluated at both endpoints of $I$. 

11
I will clarify this with an example:

**Ex:** Evaluating the number of real roots of \(x^5 - 3x - 1 = 6\) from \(x = -2\) to \(x = 2\). The Sturm Chain is given by:

\[
\begin{align*}
    f_0 &= x^5 - 3x - 1 \\
    f_1 &= 5x^4 - 3 \\
    f_2 &= 12x + 5 \\
    f_3 &= 1.
\end{align*}
\]

Now letting \(Z(x)\) denote the number of sign changes in the Sturm Chain at \(x\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f_0)</th>
<th>(f_1)</th>
<th>(f_2)</th>
<th>(f_3)</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-27(-)</td>
<td>77(+)</td>
<td>-19(-)</td>
<td>1(+)</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>25(+)</td>
<td>77(+)</td>
<td>29(+)</td>
<td>1(+)</td>
<td>0</td>
</tr>
</tbody>
</table>

Hence there are three roots of \(f_0\) from \(x = -2\) to \(x = 2\).

With this method available one can apply it to the condition from the previous section, defining \(f_0 = 5x^3 + 6ax^2 + 8bx - 32\) and selecting the interval to be \((0, \infty)\). Notice that we are treating the variable \(x\) as the value \(A^2\) and we are finding positive roots of this equation. Negative roots would lead to complex values for \(A\) which is not what we seek. Evaluating the Sturm chain yields:

\[
\begin{align*}
    f_0 &= 5x^3 + 6ax^2 + 8bx - 32 \tag{64} \\
    f_1 &= 15x^2 + 12ax + 8b \tag{65} \\
    f_2 &= \left(\frac{8a^2}{5} - \frac{16b}{3}\right)x + \frac{16ab}{15} + 32 \tag{66} \\
    f_3 &= \frac{20(9a^2b^2 - 40b^3 + 108a^3 - 540ab - 2700)}{(3a^2 - 10b)^2} \tag{67}
\end{align*}
\]

Which when evaluated yields:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f_0)</th>
<th>(f_1)</th>
<th>(f_2)</th>
<th>(f_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-)</td>
<td>(\text{sign}(b))</td>
<td>(\text{sign}(30 + ab))</td>
<td>(\text{sign}(9a^2b^2 - 40b^3 + 108a^3 - 540ab - 2700))</td>
</tr>
<tr>
<td>(\infty)</td>
<td>(\pm)</td>
<td>(\text{sign}\left(\frac{3a^2}{10} - b\right))</td>
<td>(\text{sign}(9a^2b^2 - 40b^3 + 108a^3 - 540ab - 2700))</td>
<td></td>
</tr>
</tbody>
</table>

Now I will search for regions with only one root.
This is a graph of each region we will need to search in. It turns out that all of the regions from 1 to 13 have one zero, and region 14 has three zeros. We will spend all of our time analyzing one zero potentials in the fourth quadrant of the a-b plane.

4 Spiked Random Matrix Model

Now that I have discussed the computation of equilibrium measures when the potential $V(x)$ is an even degree polynomial I will describe what happens when the random matrix is effected by an external source. My reference for this will be [4]. What is meant by “spiked random matrix” is that we take an $n \times n$ diagonal matrix of rank one with real elements called $A(n)$ and apply it on (1) as follows:

$$Ce^{-Tr(V(H))-A(n)}H \text{ diag}(A(n)) = (a, 0, ..., 0).$$

(68)

Our concern in this paper is what happens to the position of the largest eigenvalue of the distribution in the limit that $n$ gets arbitrarily large. It turns out in the case that $V(x)$ is convex (i.e. $V''(x) > 0$) to the right of the support of the equilibrium measure we can say that the largest eigenvalue $\lambda_{max}$ goes as:

$$\lambda_{max} = e, \quad a \leq \frac{V'(e)}{2}$$

(69)

$$= \xi_0(a), \quad a > \frac{V'(e)}{2}.$$ 

(70)
Where \( e \) is the value of \( \lambda_{\text{max}} \) of the distribution when \( a = 0 \) and \( \xi_0(x) \) is some continuous increasing function. However this may not hold for non-convex potentials as discussed in [4]. The goal of this section is to find an explicit example of a potential for which the above results do not hold. We find such an example in the class of polynomial \( V \) of degree 6.

In order to do this a few auxiliary functions must be defined. The first is something that we have seen before:

\[
g(z) = \int_{\text{supp}(\psi_c(x))} \log(z - s)\Psi(s)ds, \quad z \in \mathbb{C}\backslash(-\infty,e) \quad (71)
\]

The next two depend on the previous definition:

\[
G(z; a) = g(z) - V(z) + az \quad (72)
\]

\[
H(z; a) = -g(z) + az + \ell_c \quad (73)
\]

Where \( \ell_c \) is the same as it is in (12). I will define a function:

**Definition 4.1** For \( a \in (0, \frac{V'(e)}{2}) \), define \( c = c(a) \) as the unique point in \((e, \infty)\) satisfying

\[
g'(c(a)) = a. \quad (74)
\]

For \( a \geq \frac{V'(e)}{2} \), define \( c(a) = e \)

I will also define a set:

\[
\mathcal{A}_V = \{ a \in (0, \infty) \mid \exists x \in (c(a), \infty) : G(x; a) > H(c(a); a) \}. \quad (75)
\]

Now the idea is that the value of \( a \) for which \( \lambda_{\text{max}} \) begins to shift is the infimum of the set \( \mathcal{A}_V \). The purpose of the rest of this paper is to find a potential to apply this technique to (one that is non-convex to the right of support of the equilibrium measure).

### 4.1 Two necessary integrals

In the next few sections I will compute \( g(z) \) for a few cases of \( V \) but first I must evaluate two integrals. The first is:

\[
J_k = \int_A^{-A} \frac{x^{2k}(A^2 - x^2)^{\frac{3}{2}}}{\pi} dx. \quad (76)
\]

We can evaluate this by using a trigonometric substitution \( x = A\sin(\theta) \) which gives us

\[
J_k = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A^{2k+4} \sin^{2k} \theta (1 - \sin^2 \theta)^2 d\theta \quad (77)
\]

now integrating by parts we obtain:

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k} \theta d\theta = \frac{2k - 1}{2k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2(k-1)} \theta d\theta, \quad (78)
\]
and substituting this into the expression before we can obtain:

\[ J_k = \frac{3 \prod_{j=1}^{k} 2j - 1}{\prod_{j=1}^{k+1} 2j} \text{, } J_0 = \frac{3}{8} A^4. \]  

(79)

The second integral we will need is:

\[ I_k = \int_{-A}^{A} \frac{x^{2k} \sqrt{A^2 - x^2} \log|x|}{\pi} dx, \]  

(80)

integrating this by parts using \( u = x^{2k-1} \log|x| \) and \( dv = \frac{x \sqrt{A^2 - x^2}}{\pi} \)

\[ I_k = \frac{1}{3\pi} \left[ \int_{-A}^{A} x^{2(k-1)}(A^2 - x^2)^{\frac{3}{2}} + x^{2(k-1)}(A^2 - x^2)^{\frac{3}{2}}(2k - 1) \log|x| \right] \]  

(81)

\[ = \frac{1}{3\pi} \int_{-A}^{A} x^{2(k-1)}(A^2 - x^2)^{\frac{3}{2}} + A^2(2k - 1)I_{k-1} - \frac{(2k - 1)}{3} I_k \]  

(82)

\[ = \frac{J_{k-1}}{2(k+1)} + \frac{A^2(2k - 1)I_{k-1}}{2(k+1)}, \text{ } I_0 = \frac{A^2}{2} \log \left[ \frac{A}{2} \right] - \frac{A^2}{4}. \]  

(83)

To calculate \( I_0 \) we needed to use the fact that:

\[ \int_{0}^{\frac{\pi}{2}} \log|\sin \theta|d\theta = \int_{0}^{\frac{\pi}{2}} \log \left[ \frac{2 \sin \theta}{2 \cos \theta} \right] \]  

(84)

\[ = \pi \log \sqrt{2} + \int_{0}^{\frac{\pi}{2}} \left( \log \left[ \sin \frac{\theta}{2} \right] + \log \left[ \cos \frac{\theta}{2} \right] \right) d\theta. \]  

(85)

Now using the periodicity of \( \sin \) and \( \cos \) we obtain:

\[ \int_{0}^{\frac{\pi}{2}} \log|\sin \theta|d\theta = \pi \log \sqrt{2} + 2 \int_{0}^{\frac{\pi}{2}} \log|\sin \theta| d\theta. \]  

(86)

Giving us the final result:

\[ \int_{0}^{\frac{\pi}{2}} \log|\sin \theta|d\theta = -\pi \log \sqrt{2}. \]  

(87)

4.2 Computing \( g(z) \) for \( V(x) = \frac{gA^4}{4} + \frac{tx^2}{2} \)

(only when \( \text{supp}(\psi_c) = [-A, A] \))

Given the definition of \( g(z) \) we can see that

\[ \frac{dg}{dz} = -\Psi(z). \]  

(88)

We have already calculated \( \Psi(z) \) in this paper so integrating both sides we obtain:

\[ g(z) = \int \left[ \frac{1}{2} (gz^2 + tz) - \frac{R(z)}{4} (g(A^2 + 2z^2)) - \frac{tR(z)}{2} \right] dz. \]  

(89)
This yields:
\[ g(z) = \frac{gz^4}{8} + \frac{tz^2}{2} - \frac{gA^2}{4} \int R(z)dz - \frac{g}{2} \int z^2 R(z)dz - t \int \frac{R(z)}{2}dz + C. \quad (90) \]

Where \( C \) is a constant we will determine. Now using the trigonometric substitution \( z = A \sec \theta \) and integrating by parts we obtain:

\[
\int R(z)dz = \frac{z\sqrt{z^2 - A^2} - A^2 \log \left| \frac{z + \sqrt{z^2 - A^2}}{A} \right|}{2} \]

and

\[
\int z^2 R(z)dz = \frac{z^3\sqrt{z^2 - A^2} - A^2z\sqrt{z^2 - A^2} - A^4 \log \left| \frac{z + \sqrt{z^2 - A^2}}{A} \right|}{8}.
\]

From which we obtain:

\[
g(z) = \frac{gz^4}{8} + \frac{tz^2}{4} - \left( \frac{gA^2}{4} + \frac{t}{2} \right) \left( \frac{z\sqrt{z^2 - A^2} - A^2 \log \left| \frac{z + \sqrt{z^2 - A^2}}{A} \right|}{2} \right) + C. \quad (91)
\]

Now all that remains is to find the value of \( C \). In order to do this we take the limit of \( g(z) \) as \( z \downarrow 0 \) which we call \( g_+(0) \):

\[
g_+(0) = \int_{-A}^{A} \psi_c(x) \log |x|dx + i\pi \int_{0}^{A} \psi_c(x)dx. \quad (92)
\]

Using the value of \( \psi_c \) that we calculated earlier:

\[
g_+(0) = \left( \frac{gA^2}{4} + \frac{t}{2} \right) I_0 + \frac{gI_1}{2} + i \int_{0}^{A} \sqrt{A^2 - x^2} \left( \frac{gA^2}{4} + \frac{gx^2}{2} + \frac{t}{2} \right) dx
\]

\[
= \left( \frac{gA^2}{4} + \frac{t}{2} \right) \left( \frac{A^2}{2} \log \left[ \frac{A}{2} \right] - \frac{A^2}{4} + i\pi A^2 \right)
\]

\[
+ g \left( \frac{A^4}{32} + \frac{A^4 \log \left[ \frac{4}{A} \right]}{8} + i\pi A^4 \right). \quad (93)
\]

Now using the expression in (92) we can obtain another expression for \( g_+(0) \):

\[
g_+(0) = \frac{3gA^4i\pi}{32} + \frac{tA^2i\pi}{8} + C. \quad (94)
\]

Setting the two expressions equal we solve for \( C \):

\[
C = \left( \frac{3gA^4}{16} + \frac{A^2t}{4} \right) \log \left[ \frac{A}{2} \right] - \frac{3gA^4}{64} - \frac{A^2t}{8}. \quad (95)
\]

All together we have:

\[
g(z) = \frac{gz^4}{8} + \frac{tz^2}{4} - \left( \frac{gA^2}{16} + \frac{t}{4} \right) \left( z\sqrt{z^2 - A^2} - \frac{gA^2}{4} \right)
\]

\[
+ \left( \frac{3A^4g}{16} + \frac{tA^2}{4} \right) \left[ \log \left[ \frac{z + \sqrt{z^2 - A^2}}{2} \right] - \frac{3gA^4}{64} - \frac{A^2t}{8} \right]. \quad (96)
\]
4.3 Main Case: Computing \( g(z) \) for \( V(x) = \frac{x^6}{6} + \frac{ax^4}{4} + \frac{bx^2}{2} \) and \( \text{supp}(\psi_c) = [-A, A] \)

I will re-use the fact from the previous section:

\[
d\frac{g}{dz} = -\Psi(z). \tag{97}
\]

Since we already have the value of \( \Psi(z) \) we can say:

\[
d\frac{g}{dz} = \frac{1}{2}(z^5 + az^3 + bz) - \frac{z^4R(z)}{2} - \left( \frac{A^2}{4} + \frac{a}{2} \right)z^2R(z) - \left( \frac{3A^6}{16} + \frac{aA^4}{4} + \frac{b}{2} \right)R(z). \tag{98}
\]

We integrate both sides, and using the same substitution in the previous section we evaluate:

\[
\int z^4R(z)dz = z^5\sqrt{z^2-A^2} - \frac{A^2z^3\sqrt{z^2-A^2}}{24} - \frac{A^4z\sqrt{z^2-A^2}}{16} - \frac{A^6\log \left| \frac{z+\sqrt{z^2-A^2}}{2} \right|}{16}.
\]

Using the other integrals in the previous section:

\[
g(z) = \frac{z^6}{12} + \frac{az^4}{8} + \frac{bz^2}{4} - \frac{z^5\sqrt{z^2-A^2}}{12} - \left( \frac{A^2}{24} + \frac{a}{8} \right)z^3\sqrt{z^2-A^2} - \left( \frac{A^4}{32} + \frac{aA^2}{16} + \frac{b}{4} \right)z\sqrt{z^2-A^2} + \left( \frac{5A^6}{32} + \frac{3aA^4}{16} + \frac{bA^2}{4} \right)\log \left| \frac{z+\sqrt{z^2-A^2}}{A} \right| + C. \tag{99}
\]

Where again \( C \) is a constant. Now to obtain \( C \) we again evaluate \( g_+(0) \), starting with the expression we just obtained:

\[
g_+(0) = \left( \frac{5A^6}{32} + \frac{3aA^4}{16} + \frac{bA^2}{4} \right)\frac{i\pi}{2} + C. \tag{100}
\]

Using the original definition:

\[
g_+(0) = \int_{-A}^{A} \log |x|\psi_c(x)dx + i\pi \int_{0}^{A} \psi_c(x)dx. \tag{101}
\]

First, of all:

\[
i\pi \int_{0}^{A} \psi_c(x)dx = i\pi \left( \frac{5A^6}{64} + \frac{3aA^4}{32} + \frac{bA^2}{8} \right).
\]
And
\[
\int_{-A}^{A} \log|x|\psi_c(x)dx = \frac{I_2}{2} + \left( \frac{A^2}{4} + \frac{a}{2} \right) I_1 + \left( \frac{3A^4}{16} + \frac{aA^2}{4} + \frac{b}{2} \right) I_0
\]
\[
= -\frac{5A^6}{192} - \frac{3A^4a}{64} - \frac{A^2b}{8}
+ \left( \frac{5A^6}{32} + \frac{3aA^4}{16} + \frac{A^2b}{4} \right) \log \left[ \frac{A}{2} \right].
\]

So adding these two together and comparing it to the first expression for \( g(z) \) shows us that
\[
C = -\frac{5A^6}{192} - \frac{3A^4a}{64} - \frac{A^2b}{8}
+ \left( \frac{5A^6}{32} + \frac{3aA^4}{16} + \frac{A^2b}{4} \right) \log \left[ \frac{A}{2} \right]. \tag{102}
\]

Altogether we obtain:
\[
g(z) = \frac{z^6}{12} + \frac{az^4}{8} + \frac{bz^2}{4} - \frac{z^5\sqrt{z^2-A^2}}{12} - \left( \frac{A^2}{24} + \frac{a}{8} \right) z^3\sqrt{z^2-A^2}
- \left( \frac{A^4}{32} + \frac{aA^2}{16} + \frac{b}{4} \right) z\sqrt{z^2-A^2} - \frac{5A^6}{192} - \frac{3A^4a}{64} - \frac{A^2b}{8}
+ \left( \frac{5A^6}{32} + \frac{3aA^4}{16} + \frac{bA^2}{4} \right) \log \left[ \frac{z + \sqrt{z^2-A^2}}{2} \right]. \tag{103}
\]

4.4 Finding the Potential

In this section I will demonstrate how to find a potential that is not convex to the right of its support numerically. The first program is simply the function definition of the potential called \( V.m \):

```matlab
function Potential=V(x,a,b)
    Potential= ((x^6)/6)+(a*(x^4)/4)+b*(x^2)/2
%For support (-A,A)
```

This second program is the definition of the function \( g(z) \) and the program is called \( g.m \):

```matlab
function Potential=g(x,a,b)
%This function gives the value of g(z) associated with
%V(x) = (1/6)*(x^6)+(a/4)*(x^4)+(b/2)*(x^2).
%The function takes in as arguments the value of z for which g
%needs to be evaluated, the parameters a, and b, and finally the
%value of A for which
%A^6+6*a*A^4+8*b*A^2-32=0.
```
function extra = g(z,a,b,A)
extra=((z^6)/12)+(a*(z^4)/8)-(3/64)*a*(A^4)
 + (b*(z^2)/4)-(z^5)*sqrt((z^2)-(A^2))*(1/12)
-(((A^2)/24)+(a/8))*(z^3)*sqrt((z^2)-(A^2))
-((a/16)*(A^2)+(1/32)*(A^4)+(b/4))*z*sqrt((z^2)-(A^2))
 +((5/32)*(A^6)+(3/16)*a*(A^4)+(b/4)*(A^2))*log((z/2)
        +(sqrt((z^2)-(A^2))/2))-(5/192)*(A^6)-(A^2)*(b/8);

The next file finds the values of \(a\) and \(b\) that we were seeking it is called regionsearch.m.

%Clear the screen and clear all variables
clc
clear all
close all

%Initialize a value of \(b\) that lies close to the region where %the cubic in \(A^2\) has three solutions.
b=30;

%This is a corresponding value of \(a\) that lies in the region %we select it close to the boundary so that the search is %accurate
a = -2*sqrt(b) -.001;

%Initiate a vector \(K\) that will hold the valid values of \(a\) %for which our conditions are met (if we fix \(b\) and search %\(a\) this line is valid) Also we only store 10000 values.
K = zeros(10000,1);

%However if we let both \(a\) and \(b\) vary (which is not done %in this program)
%K=zeros(10000,2)

%j is the loop that we increment to find the values of \(a\)
j=1;

%This while loop is to to search in both regions \(a\) and \(b\) %the code as it stands searches only through \(a\) with \(b\) fixed) %notice that we limit it up to \(b=50\), this upper limit is
% arbitrary it is kept only to allow the loop to terminate.
% Also notice the condition K(1)==0; this condition is in
% place so that the code will stop once the first valid value
% of a and b is found

% while K(1) == 0 && b < 50
% This while loop ensures that a and b are in the region
% where there is one root for the cubic in A^2
while (9*(a^2)*(b^2)-40*(b^3)+108*(a^3)-540*(a*b)-2700)<0
 && (a^2)-4*b >0

% The next six lines solve for A
r = roots([5,6*a,8*b,-32]);
for i =1:length(r)
    if imag(r(i))==0
        A = sqrt(r(i));
    end
end
% This line calculate the first positive nonzero local
% minima of the potential
convex=sqrt((-a+sqrt((a^2)-4*b))/2);
% If this value is greater than A than we have the
% condition that the potential is not convex outside
% the interval of support. Therefore we can store this
% value and increment by j.
if convex>A
    K(j) = a;
    j = j+1;
end

% Now we increment a and continue the search.
% a = a-.05;
end

% The next 3 lines are for looping through b. The incrementation
% is 1.
% b = b+1;
% a = -2*sqrt(b) -.001;
% end

% This is just to display the proper value of b when called on.
% b=b-1;
%The rest of this program loops through the K values and plots %the potential. This part is omitted for the sake of brevity.

Below are two results found (the vertical lines are the values of ±A. The plot on the top is the first result found, and the plot on the bottom is a more clear example of a potential that is not convex outside its interval of support. The first plot is at b=30, a=-10.955; the second plot is at b=30, a=-11.405.

The next few programs searched out various other examples of potentials and functional plots of G, and H. This first program called Gprime.m calculates the value of the function $G'(z)$. 

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%Given the value of z,a,b,A, and ac (in the cited paper it is referred to as a) this program calculates the derivative of G at z.

function DerivG = Gprime(z,a,b,A,ac)
%Value of the derivative of the auxiliary function G
DerivG=-(1/2)*((z^5)+a*(z^3)+b*z)-(((z^4)/2)
 +(((A^2)/4)+(a/2))*(z^2)+ac
 +((3/16)*A^6+(a/4)*A^2+(b/2))*sqrt((z^2)-(A^2));
end

This next program is a modification of the program above it is called regionsearch2.m.

%Clear the screen and clear all variables
clc
clear all
close all

%Initialize the value of a and b to lie in the region where the cubic in A^2 has three solutions.
b=10;
a = -2*sqrt(b) -.001;

%Fineness of search for maximum of the G function
dx = .001;

%Interval length beyond A to search for the maximum
interval = 6;

%L will store the solutions that we find, I set the length at 10000
L = zeros(10000,1);
%G will store the values of the function G at each point
G = zeros(length(0:dx:interval),1);
%K will store the values of the function g at each point
K = zeros(length(G),1);
%Y will store the values of the function V at each point
Y = zeros(length(G),1);
%DG will store the value of the function Gprime at each point
DG = zeros(length(G),1);
%This while loop is for the same purpose as in the previous program
j=1;
while L(1) == 0 && b <30
while (9*(a^2)*(b^2)-40*(b^3)+108*(a^3)-540*(a*b)-2700)<0
&& (a^2)-4*b >0
%The next six lines solve for A
r = roots([5,6*a,8*b,-32]);
for i =1:length(r)
    if imag(r(i))==0
        A = sqrt(r(i));
    end
end
end
%Define X
X = A:dx:A+interval;
X = X';
%Note the value ac actually corresponds to the value
%of a described in the paper (I apologize for the
%unfortunate choice of notation).
%We set it to the value of g'(e) = V'(e)/2
ac= ((A^5)+a*(A^3)+b*(A))*(1/2);
%Find G and DG
for k=1:length(X)
    K(k) = g(X(k),a,b,A);
    Y(k) = V(X(k),a,b);
    DG(k) = Gprime(X(k),a,b,A,ac);
end
%Calculate G
G = K - Y + ac*X;
%Find the local maxima
for i=1:length(DG)-1
    if DG(i)>=0 && DG(i+1)<=0
        peak = i;
    end
end
convex=sqrt((-a+sqrt((a^2)-4*b))/2);
if convex>A && G(peak)>G(1)
    L(j) = a;
    j = j+1;
end
a = a-.05;
end
b = b*.001;
a = -2*sqrt(b) -.001;
The results of this program are selections of a and b for which the second peak of the function \( G(z) \) is greater than the value of \( G(A) \). Here is one such result:

The next program is called `acriticalcalc.m` calculates the critical value of \( ac \) (called \( a \) in the paper) such that the two local maxima are equal (to some degree of error). The program assumes that the second maxima has a smaller value than the first. It also assumes that increasing \( ac \) will result in the second maxima reaching the same value as the first. This is not a general scenario, rather it is one suited for the cases we examined.

```matlab
clc
clear all
close all

% Take in values of a and b
a = input('Please insert the value for a > ');
b = input('Please insert the value for b > ');

% The fineness of our search and the interval beyond A to search
dx = .0001;
interval = 4;

%Solve for A
r = roots([5,6*a,8*b,-32]);
for i =1:length(r)
    if imag(r(i))==0
        A = sqrt(r(i));
    end
end

%Value of g'(e) = V'(e)/2
gprimee= ((A^5)+a*(A^3)+b*(A))*(1/2);

%The values we are looking at to find the max of G.
X = A:dx:A+interval;
X=X';

G = zeros(length(X),1);
K = zeros(length(X),1);
Y = zeros(length(X),1);
DG = zeros(length(X),1);

%initiate ac
ac = gprimee;

for k=1:length(X)
    K(k) = g(X(k),a,b,A);
    Y(k) = V(X(k),a,b);
    DG(k) = Gprime(X(k),a,b,A,ac);
end

%Initiate the peak so that G(peak)<G(1);
peak2=2;
G(peak2)=-1;
peak1 = 1;

while G(peak2)<G(peak1)

    G = K - Y + ac*X;

    %Now we search for zeros of DG to find maximum of G
    %This code will compare the two maximums of G and adjust
    %ac until the second peak of G is smaller than the first.
    %at this point we have found the value of ac we are looking for.
    l = 0;

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for i=1:length(DG)-1
    if DG(i)>=0 && DG(i+1)<=0 && (l == 0)
        peak1 = i;
        l = l+1;
    end
    if DG(i)>=0 && DG(i+1)<=0 && (l==1)
        peak2 = i;
    end
end
if G(peak2)<G(peak1)
    DG = DG - ac*ones(length(DG),1);
    ac = ac+.0001;
    DG = DG + ac*ones(length(DG),1);
end
end

The result for the case that $a = -11.4055$ and $b = 30$ is that the critical value is $ac = 7.5838$.

This final program stores the location of the maximum of $G(z)$ for given values of ac. It is called `xmaximizerplotter.m`.

```matlab
clc
clear all
close all

%Take in values of a and b
a = input('Please insert the value for a > ');
b = input('Please insert the value for b > ');

%The fineness of our search and the interval beyond A to search
dx = .0001;
interval = 4;

%Solve for A
r = roots([5,6*a,8*b,-32]);
for i =1:length(r)
    if imag(r(i))==0
        A = sqrt(r(i));
    end
end

%Value of g'(e) = V'(e)/2
gprimee= ((A^5)+a*(A^3)+b*(A))*(1/2);

%The values we are looking at to find the max of G.
X = A:dx:A+interval;
X=X';
```
G = zeros(length(X),1);
K = zeros(length(X),1);
Y = zeros(length(X),1);
DG = zeros(length(X),1);

%initiate ac
ac = gprimee;

for k=1:length(X)
    K(k) = g(X(k),a,b,A);
    Y(k) = V(X(k),a,b);
    DG(k) = Gprime(X(k),a,b,A,ac);
end

%Array to store the max position of G.
MAX = zeros(length(gprimee:.01:10),1);

for k=1:length(gprimee:.01:10)
    G = K - Y + ac*X;

    %Now we search for zeros of DG to find maximum of G
    %Note this will only give us the peak we are looking for
    %Because it will give us the largest i that satisfies this
    %which is the second peak.
    l = 0;
    for i=1:length(DG)-1
        if DG(i) &gt;= 0 &amp;&amp; DG(i+1) &lt;= 0 &amp;&amp; (l == 0)
            peak1 = i;
            l = l+1;
        end
        if DG(i) &gt;= 0 &amp;&amp; DG(i+1) &lt;= 0 &amp;&amp; (l==1)
            peak2 = i;
        end
    end
    if G(peak2) &lt;= G(peak1)
        MAX(k) = X(peak1);
    end
    if G(peak1) &lt;= G(peak2)
        MAX(k) = X(peak2);
    end
    DG = DG - ac*ones(length(DG),1);
    ac = ac+.01;
    DG = DG + ac*ones(length(DG),1);
end
Using the vector MAX in this program we were able to construct a plot of the x-position of the maximum of \( G(z) \) versus the value of \( ac \).

The figure is unlabeled due to the fact that its qualitative features are most important. In particular the discontinuity is of note because it occurs at the critical value of \( ac \).

References


