1 Introduction and Definitions

This paper discusses the background behind an REU project conducted at the University of Michigan in the summer of 2011, under the guidance of Dr. Christopher Lyons. The aim of the project is to prove the Tate conjecture, an important conjecture in arithmetic geometry, for a particular class of algebraic surfaces studied by Catanese [1]. I have made an effort to assume a minimum level of background, comparable to level at which I started the REU. For a more in-depth account of this background material, focusing on elliptic curves in particular, consult [3], which was an invaluable reference throughout the project.
Throughout this paper, $K$ denotes a field with fixed algebraic closure $K$. We denote the absolute Galois group $\text{Gal}(\bar{K}/K)$ by $G_K$. We will frequently take $K$ to be the finite field of $q$ elements $\mathbb{F}_q$, an algebraic number field, or the complex numbers.

For any field $K$, define affine $n$-space $\mathbb{A}^n(K)$ to be the set of $n$-tuples $(x_1, \ldots, x_n)$ of elements of $K$. Let $I$ be a prime ideal in the ring of polynomials $\bar{K}[X_1, \ldots, X_n]$ in $n$ variables over $\bar{K}$. Then the set of points $V := \{(x_1, \ldots, x_n) \in \mathbb{A}^n(\bar{K}) : p(x_1, \ldots, x_n) = 0 \ \forall p \in I\}$ is called an affine variety over $\bar{K}$. Conversely, given a variety $V$, we define the ideal $I(V)$ to be the ideal of polynomials over $\bar{K}$ that evaluate to 0 at every point of $V$. If $I(V)$ can be generated by polynomials with coefficients in $K$, we say that $V$ is defined over $K$, and a point $x \in V$ is called a $K$-rational point if the coordinates of $x$ lie in $K$.

The affine coordinate ring $K[V]$ of an affine variety $V$ defined over $K$ is the quotient ring $K[X_1, \ldots, X_n]/I(V/K)$, where $I(V/K) := I(V) \cap K[X_1, \ldots, X_n]$. Elements of this ring may be interpreted as well-defined functions on $V$. The field of fractions of $K[V]$ is denoted $K(V)$ and called the function field of $V$ over $K$, and the dimension of $V$ is the transcendence degree of the field extension $\bar{K}(V)/\bar{K}$. In particular, a curve is a variety of dimension 1, and a surface is a variety of dimension 2.

The best varieties to work with are those which are nonsingular. In a geometric sense, this means roughly that there is a well-defined tangent hyperplane at each point of the variety. Formally, a variety $X \subseteq \mathbb{A}^n$ whose ideal is generated by polynomials $p_1, \ldots, p_m$ is nonsingular at the point $x$ if the $m \times n$ matrix with entries $(\partial f_i/\partial x_j)(x)$ has rank $n - \dim(V)$.

It is often more useful to consider varieties as embedded in projective space, which accounts for points “at infinity”. Projective $n$-space $\mathbb{P}^n(K)$ is the set of equivalence classes of $(n+1)$-tuples in $\mathbb{A}^{n+1}(K) - \{0\}$, where two points $(x_0, \ldots, x_n), (y_0, \ldots, y_n)$ are considered equivalent if there is a constant $\alpha \in K$ such that $y_i = \alpha x_i$ for all $i$. Points in $\mathbb{P}^n(K)$ are written $(x_0 : x_1 : \cdots : x_n)$ to emphasize that the important information is the ratios between the coordinates. It does not make sense to evaluate arbitrary homogeneous polynomials in projective space, but we may consider the zero sets of homogeneous polynomials (polynomials where every monomial has the same total degree), since whether such a polynomial evaluates to 0 is independent of choice of projective coordinates. Thus, a projective variety over $\bar{K}$ is defined to be the zero set of a prime ideal generated by homogeneous polynomials. As above, such a variety is defined over $K$ if the polynomials may be taken to have coefficients in $K$, and the $K$-rational points are those points which may be expressed with coordinates in $K$. 

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Affine space may be embedded in projective space by the map \( (x_1, \ldots, x_n) \mapsto (x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n) \) for any choice of \( i \), and conversely, we can map the set of points in \( \mathbb{P}^n(K) \) with \( x_i \neq 0 \) onto \( \mathbb{A}^n(K) \) by \( (x_0 : x_1 : \cdots : x_n) \mapsto (x_0/x_i, x_1/x_i, \cdots, x_n/x_i) \). To similarly go between affine and projective varieties, we can homogenize polynomials via the map \( p(X_1, \ldots, X_n) \mapsto X_i^{\deg(p)}p(X_0/X_i, \cdots, X_i/X_i, \cdots, X_n/X_i) \), and dehomogenize via the map \( p(X_0, \cdots, X_n) \mapsto p(X_1, \cdots, X_{i-1}, 1, X_i, \cdots, X_n) \). Thus, we can give nonhomogeneous polynomials to describe a projective variety, which is understood to mean the projective variety given by the homogenization of the polynomials. The function field of a projective variety is the function field of an affine variety given by dehomogenization, and similarly the dimension of a projective variety is the dimension of a corresponding affine variety. Also, a projective variety is nonsingular if all of the corresponding affine varieties are nonsingular.

A rational map \( \phi : X \to Y \) between projective varieties in \( \mathbb{P}^n \) is a map given by a sequence of \( n+1 \) rational functions \( f_0, f_1, \cdots, f_n \in K(X) \) such that for each \( x \in X \) where all the coordinate functions are defined, \( (f_0(x) : \cdots : f_n(x)) \in Y \). A rational map is defined over \( K \) if the coordinate functions may be taken in \( K(X) \). A morphism of varieties is a rational map \( \phi = (f_0 : \cdots : f_n) \) where, for each point \( x \in X \), we can choose \( g \in K(X) \) such that \( gf_0, \cdots, gf_n \) are all defined and not all 0 at \( x \). We then let \( \phi(x) = (gf_0(x) : \cdots : gf_n(x)) \) (note that the choice of \( g \) does not matter due to the equivalence relation of projective space).

Elliptic curves, a particularly interesting class of algebraic curves, are nonsingular projective curves which may be described by an equation of the form

\[
Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,
\]

which is the homogenization of the affine equation

\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.
\]

The points of an elliptic curve form an abelian group under an addition law which can be defined purely geometrically, but may also be expressed by rational functions. The identity element of this group is the point at infinity \((0 : 1 : 0)\). An isogeny of elliptic curves is a morphism between elliptic curves which fixes the identity point. It can be shown that any isogeny is also a group homomorphism. Given an isogeny \( \phi : E \to F \) of elliptic curves defined over \( K \), we may define a map of function fields in the opposite direction \( \phi^* : K(F) \to K(E) \) by \( \phi^*f = f \circ \phi \). The field extension \( K(E)/\phi^*K(F) \) will
always be finite, and the degree of the isogeny $\phi$ is defined to be the degree of this extension.

2 The Zeta Function

Given a variety $X$ defined over the finite field $\mathbb{F}_q$, we may consider the number of points of $X$ over each finite extension $\mathbb{F}_{q^r}$. An important way of encoding this information is given by the zeta function of $X$.

**Definition 2.1.** Let $N_r$ be the number of points of $X$ over $\mathbb{F}_{q^r}$. We define the zeta function $Z(X,T)$ to be the formal power series in the variable $T$ given by the formula

$$Z(X,T) := \exp \left( \sum_{r=1}^{\infty} \frac{N_r}{r} \cdot T^r \right).$$

The reason for this choice of definition will become evident in the next few sections, as we see that this function has many useful properties. As a preliminary example, take $X$ to be $n$-dimensional projective space $\mathbb{P}^n$. Over $\mathbb{F}_{q^r}$, the number of points in $\mathbb{P}^n$ is $1 + q^r + q^{2r} + \cdots + q^{nr}$, so the zeta function is

$$Z(\mathbb{P}^n, T) = \exp \left( \sum_{i=0}^{n} \sum_{r=1}^{\infty} \frac{q^{ir}}{r} \cdot T^r \right)$$

$$= \exp \left( \sum_{i=0}^{n} - \log(1 - q^iT) \right)$$

$$= \prod_{i=0}^{n} (1 - q^iT)^{-1}.$$

In particular, $Z(\mathbb{P}^n, T)$ is a rational function of $T$, which turns out to be true for any nonsingular projective variety, along with a number of other remarkable properties.

3 The Weil Conjectures

In 1949, André Weil [5] proposed a significant series of conjectures on the form of the zeta function. Weil was able to prove the conjectures for algebraic curves, and the proof in full generality was completed in 1973 by Pierre Deligne [2]. In this section, we assume that $X$ is a nonsingular projective
variety of dimension \(n\) defined over the finite field \(\mathbb{F}_q\). We then have the following results:

**Theorem 3.1** (Rationality). The zeta function \(Z(X,T)\) is a rational function of \(T\).

**Theorem 3.2** (Functional Equation). There is an integer \(E\), called the Euler characteristic of \(X\), such that the zeta function satisfies the functional equation

\[
Z \left( X, \frac{1}{q^n T} \right) = \pm q^{nE/2} E Z(X,T).
\]

**Theorem 3.3** (Riemann Hypothesis). The zeta function can be factored as

\[
Z(X,T) = \frac{P_1(T) P_3(T) \cdots P_{2n-1}(T)}{P_0(T) P_2(T) \cdots P_{2n}(T)},
\]

such that each \(P_i\) is a polynomial with integer coefficients, \(P_0(T) = 1 - T\), \(P_{2n}(T) = 1 - q^n T\), and each of the polynomials \(P_i\) factors over \(\mathbb{C}\) as

\[
P_i(T) = \prod_j (1 - \alpha_{i,j} T),
\]

where each \(\alpha_{i,j}\) is an algebraic integer of absolute value \(q^{i/2}\).

To see why this is known as the Riemann hypothesis, note that if we make the substitution \(T = q^{-s}\), we have that all the roots of each \(P_i\) have real part \(i/2\). In particular, in the case of curves, all the roots of the zeta function have real part \(1/2\). Further, this substitution makes the functional equation read as

\[
Z(X,n-s) = \pm q^{E-s} E Z(X,s),
\]

which has some similarity to the functional equation for the Riemann zeta function (especially when \(n = 1\)). The next Weil conjecture suggests an important connection with algebraic topology by relating the zeta function to homology. Indeed, the proof of the Weil conjectures required the development of the \(\ell\)-adic cohomology theory for varieties over finite fields, which is discussed in the next section.

**Theorem 3.4** (Comparison). Suppose \(\tilde{X}\) is a variety defined over an algebraic number field \(K\), such that \(X\) is the reduction of \(\tilde{X}\) modulo a prime ideal \(p \subset \mathcal{O}_K\). Then, the degree of each \(P_i\) is the \(i\)th Betti number \(B_i\) of \(\tilde{X}\) viewed as a variety over \(\mathbb{C}\) (i.e. the rank of the \(i\)th singular homology group of the points of \(\tilde{X}\) over \(\mathbb{C}\) viewed as a topological space). The Euler characteristic appearing in the functional equation is equal to the topological Euler characteristic \(\sum_{i=0}^{2n} (-1)^i B_i\).
Consider again the zeta function of projective space calculated above. We already noted that $Z(\mathbb{P}^n, T)$ is a rational function of $T$. Since the Betti numbers of $n$-dimensional complex projective space are 1 for all even dimensions 0 through $2n$ and 0 for all other dimensions, the comparison conjecture suggests that the $P_i$ have degree 1 for even $i$ and 0 for odd $i$. This is indeed the case: we have $P_i = 1 - q^{i/2}T$ for even $i$ and 1 for odd $i$, and the Riemann hypothesis is evident here as well. Finally, the functional equation can be easily verified with $E = n$, so we have confirmed all of the Weil conjectures for this example.

4 Zeta Functions and $\ell$-adic Cohomology

We now discuss the $\ell$-adic cohomology theory used in the proof of the Weil conjectures. Suppose that $X$ is a nonsingular projective variety of dimension $n$ over the field $K$. Then for any prime $\ell$ not equal to the characteristic of $K$, there exist $\ell$-adic cohomology groups $H^i_\ell(X)$ for $0 \leq i \leq 2n$, which are vector spaces over the field $\mathbb{Q}_\ell$ of $\ell$-adic numbers. These cohomology groups share many of the familiar characteristics of singular cohomology:

- **Functoriality:** Given any morphism $\phi : X \to Y$ of varieties, there is a linear map $\phi^* : H^i_\ell(Y) \to H^i_\ell(X)$, such that, for any two morphisms $\phi : X \to Y, \psi : Y \to Z$, we have $(\psi \circ \phi)^* = \phi^* \circ \psi^*$, and if $id : X \to X$ is the identity morphism, $id^*$ is the identity on $H^i_\ell(X)$.

- **Cup Product:** There is a bilinear cup product $\cup : H^i_\ell(X) \times H^j_\ell(X) \to H^{i+j}_\ell(X)$, such that $y \cup x = (-1)^{ij}(x \cup y)$.

- **Poincaré Duality:** $H^{2n}_\ell(X) \cong \mathbb{Q}_\ell$, and the cup product pairing $H^i_\ell(X) \times H^{2n-i}_\ell(X) \to \mathbb{Q}_\ell$ is nondegenerate. Thus, $H^i_\ell(X)$ and $H^{2n-i}_\ell(X)$ may be viewed as dual vector spaces.

- **Lefschetz Trace Formula:** Suppose $\phi : X \to X$ is a morphism of varieties. Then, we define two subvarieties of $X \times X$: the diagonal $\Delta$, consisting of points $(x, x)$ for $x \in X$, and the graph $\Gamma_\phi$ of $\phi$, consisting of points of the form $(x, \phi(x))$. The intersection number $\Delta \cdot \Gamma_\phi$ then counts (with a suitable notion of multiplicity) the number of fixed points of $\phi$. The Lefschetz trace formula for $\ell$-adic cohomology then reads:

$$\Delta \cdot \Gamma_\phi = \sum_{i=0}^{2n} (-1)^i \text{tr}(\phi^* : H^i_\ell(X) \to H^i_\ell(X)).$$
The Lefschetz formula is the key to understanding the Weil conjectures. Note that the number of points $N_r$ over $\mathbb{F}_q^r$ is same as the number of fixed points of the Frobenius morphism $F : X \to X$ which maps $(x_0 : x_1 : \cdots : x_n) \mapsto (x_0^q : x_1^q : \cdots : x_n^q)$. Thus, letting $F_r$ be the matrix of $F^*$ on the $i$th cohomology group $H^i_\ell(X)$, the Lefschetz trace formula gives

$$N_r = \sum_{i=0}^{2n} (-1)^i \text{tr}(F_i^r),$$

and thus the zeta function of $X$ is

$$Z(X, T) = \exp \left( \sum_{i=0}^{2n} (-1)^i \sum_{r=1}^{\infty} \left( \text{tr}(F_i^r) \frac{T^r}{r} \right) \right).$$

It is a simple linear algebra exercise to show that

$$\sum_{r=1}^{\infty} \left( \text{tr}(F_i^r) \frac{T^r}{r} \right) = -\log(\det(1 - F_i T))$$

as formal power series. Thus, we have that the zeta function can be written as

$$Z(X, T) = \prod_{i=0}^{2n} \det(1 - F_i T)^{(-1)^{i+1}},$$

where the terms of this product are the $P_i$ appearing in the Weil conjectures. In particular, the degree of each $P_i$ (which corresponds to the Betti number $B_i$) is the dimension of the cohomology group $H^i_\ell(X)$, and the Poincaré duality between $H^i_\ell(X)$ and $H^{2n-i}_\ell(X)$ gives rise to the duality between $P_i$ and $P_{2n-i}$ suggested in the Weil functional equation.

## 5 The Tate Conjecture

Suppose that the field $K$ is finitely generated over its prime subfield. An additional feature of $\ell$-adic cohomology that does not arise in the theory of singular cohomology is that the Galois group $G_K$ acts on the $\ell$-adic cohomology groups, giving a representation $G_K \to \text{Aut}_{\mathbb{Q}_\ell}(H^i_\ell(X))$ for each $i$. The Tate conjecture, proposed by John Tate in 1963[4] and still a major unsolved problem, characterizes the cohomology classes which are fixed by a finite-index subgroup of $G_K$ (with a modification discussed below).

To state the Tate conjecture, we need to consider one other representation of $G_K$, known as the $\ell$-adic cyclotomic character. To define this
representation, we first define the \( \ell \)-adic Tate module of roots of unity in the algebraic closure \( \bar{K} \).

**Definition 5.1.** Let \( \mu_{\ell^n} \) be the group of \( \ell^n \)th roots of unity in \( \bar{K} \). These groups form an inverse system with the maps \( \mu_{\ell^{n+1}} \rightarrow \mu_{\ell^n} \) given by raising to the \( \ell \)th power. We may then define the \( \ell \)-adic Tate module \( T_\ell(\mu) \) as the inverse limit \( \varprojlim_n \mu_{\ell^n} \).

Since we have group isomorphisms \( \mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z} \), we see that \( T_\ell(\mu) \) is isomorphic to the group \( \mathbb{Z}_\ell \) of \( \ell \)-adic integers. Additionally, the Galois group \( G_K \) acts on \( T_\ell(\mu) \), giving a representation \( G_K \rightarrow \text{Aut}(\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^\times \). The embedding \( \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times \) then gives a one-dimensional representation of \( G_K \) over \( \mathbb{Q}_\ell \), which is called the \( \ell \)-adic cyclotomic character of \( G_K \) and denoted by \( \mu_\ell(1) \). We may then define \( \mathbb{Q}_\ell(k) \) for integers \( k \geq 0 \) as the \( k \)-fold tensor product of \( \mathbb{Q}_\ell(1) \). Finally, for any finite-dimensional representation \( V \) of \( G_K \) over \( \mathbb{Q}_\ell \), we define the \( k \)-fold Tate twist \( V(k) \) of \( V \) to be the representation \( V \otimes \mathbb{Q}_\ell(k) \).

The Tate conjecture concerns the classes in \( H^d_\ell(X)(d) \), for \( 0 \leq d \leq n \), which are fixed by a finite-index subgroup of \( G_K \). In particular, it asserts that these classes arise algebraically in the following sense. Define the algebraic cycle group \( Z^d(X_{\bar{K}}) \) to be the free abelian group generated by the subvarieties of \( X \) (viewed as a variety over \( \bar{K} \)) of codimension \( d \). Then, there is a homomorphism of abelian groups \( c : Z^d(X_{\bar{K}}) \rightarrow H^d_\ell(X)(d) \), which commutes with the action of \( G_K \) (note the Tate twist).

To motivate the map \( c \), it is helpful to consider an analogy with singular cohomology of nonsingular projective varieties over \( \mathbb{C} \). In this case, a nonsingular subvariety of codimension \( d \) corresponds to a submanifold of (real) codimension \( 2d \). This gives a homology class in \( H_{2n-2d}(X) \), through Poincaré duality may be identified with a cohomology class in \( H^{2d}(X) \), giving a map \( Z^d(X_{\mathbb{C}}) \rightarrow H^{2d}(X) \).

We are now ready to state the Tate conjecture. Let \( H^{2d}_{\text{alg}} := c(Z^d(X_{\bar{K}})) \otimes \mathbb{Q}_\ell \) be the subspace of \( H^{2d}_\ell(X)(d) \) generated by the algebraic cycles in \( Z^d(X_{\bar{K}}) \) via the map \( c \). By the Hilbert Basis Theorem, any subvariety of \( X \) over \( K \) is defined over some finite extension \( L/K \), so any cycle class in \( Z^d(X_{\bar{K}}) \) is fixed by a finite-index subgroup of \( G_K \). Denote the set of all classes in \( H^{2d}_\ell(X)(d) \) fixed by a finite-index subgroup of \( G_K \) by \( H^{2d}_{\text{Tate}} \). Then, since the map \( c \) commutes with the action of \( G_K \), we have \( H^{2d}_{\text{alg}} \subseteq H^{2d}_{\text{Tate}} \). The Tate conjecture asserts that, in fact, \( H^{2d}_{\text{alg}} = H^{2d}_{\text{Tate}} \).

Returning to the case where \( K \) is the finite field \( \mathbb{F}_q \), we now discuss how to use the zeta function \( Z(X,T) \) to determine the dimension of \( H^{2d}_{\text{Tate}} \).
First, note that the $q$-th power Frobenius automorphism $\sigma$ is a topological generator for $G_{F_q}$ (i.e. the subgroup generated by $\sigma$ is dense in the Krull topology). This is because any element of $G_{F_q}$ acts on any particular finite extension $F_{q^r}$ as some power of $\sigma$. It is known that the representation of $G_{F_q}$ on $H^i_\ell(X)$ is continuous, so the action of $G_{F_q}$ is determined completely by the action of $\sigma$. If we let Frob be the inverse of $\sigma$, it is clear that the action of $G_{F_q}$ is determined completely by the action of Frob as well. This is important because the map $F^*: H^i_\ell(X) \to H^i_\ell(X)$ arising from the morphism $F: X \to X$ of varieties, used in our discussion of the Weil conjectures, turns out to be equal to the image of Frob in the representation $G_K \to \text{Aut}_{Q_\ell}(H^i_\ell(X))$ occurring in the Tate conjecture. This leads to the following result relating the zeta function to $H^{2d}_{\text{Tate}}$:

**Theorem 5.2.** The dimension of $H^{2d}_{\text{Tate}}$ is equal to the number of roots of the polynomial $P_{2d}$ in the denominator of $Z(X,T)$ (counted with multiplicity) which are $q^{-d}$ times a root of unity.

**Proof.** Recalling that, in the factorization of $Z(X,T)$, we have $P_i = \det(1 - F_i T)$, we see that the roots of each $P_i$ are the reciprocals of the eigenvalues of $F^*$ on $H^i_\ell(X)$, or, equivalently, the reciprocals of the eigenvalues of Frob on $H^i_\ell(X)$. A class in $H^{2d}_{\text{Tate}}$ is fixed (after incorporating the Tate twist) by a finite-index subgroup of $G_{F_q}$, and thus fixed by a finite power of Frob. The dimension of this subspace is thus exactly the number of eigenvalues of Frob on $H^{2d}_\ell(X)$ (counted with multiplicity) which are roots of unity (again, after accounting for the Tate twist).

To consider the effect of the Tate twist, we must determine the image of Frob in the representation $Q_\ell(d)$. Note that the image of $\sigma = \text{Frob}^{-1}$ in $Q_\ell(1)$ is $q$ (viewed as an element of $\mathbb{Z}_\ell^*$), since $\sigma$ sends each root of unity to its $q$th power. Thus the image of Frob in $Q_\ell(1)$ is $q^{-1}$. Since $Q_\ell(d)$ is the $d$-fold tensor product of $Q_\ell(1)$, the image of Frob in $Q_\ell(d)$ is then $q^{-d}$.

Therefore, the eigenvalues of Frob on $H^{2d}_\ell(X)$ without the Tate twist (i.e. the reciprocals of the roots of $P_{2d}$) are exactly $q^d$ times the eigenvalues of Frob on $H^{2d}_\ell(X)(d)$ with the Tate twist. We know that the dimension of $H^{2d}_{\text{Tate}}$ is the number of eigenvalues of $H^{2d}_\ell(X)(d)$ which are roots of unity, and this is then equal to the number of roots of $P_{2d}$ which are $q^{-d}$ times a root of unity. 

Since we have $H^{2d}_{\text{alg}} \subseteq H^{2d}_{\text{Tate}}$, the number of roots of $P_{2d}$ of this form provides an upper bound on the dimension of $H^{2d}_{\text{alg}}$. In particular, the Picard number $\rho(X)$ of an algebraic surface $X$, an important geometric invariant
which may be defined as \( \dim(H^2_{\text{alg}}) \), is bounded above by the number of roots of \( P_2 \) which are \( q^{-1} \) times a root of unity.

## 6 Catanese Surfaces

We now turn our attention to the class of surfaces studied in this REU project. These surfaces, which we will refer to as Catanese surfaces, were described by Catanese in [1]. Rather than defining these surfaces explicitly, we describe some of their useful characteristics which may help in a proof of the Tate conjecture for the surfaces.

Take \( X \) to be a Catanese surface, which is a smooth projective surface defined over \( \mathbb{C} \). To study the Tate conjecture, we need \( X \) to be defined over a finitely generated field, so we will assume further that \( X \) is defined over an algebraic number field \( K \). An important feature of the surface \( X \) is that there exists an elliptic curve \( E \), and a surjective morphism \( p : X \to E \), which gives a fibration of \( X \) into a family of curves of genus 2. This means, that, with the exception of finitely many singular fibers, the fibers \( p^{-1}(e) \), for each point \( e \in E \), will be nonsingular curves of genus 2. Moreover, this fibration has one other important characteristic: the induced map on the first \( \ell \)-adic cohomology groups, \( p^* : H^1_\ell(E) \to H^1_\ell(X) \), is an isomorphism. Since the \( \ell \)-adic cohomology of elliptic curves is well understood, this means that we can easily determine all the \( \ell \)-adic cohomology of \( X \) except for the important case of \( H^2_\ell(X)(1) \). Thus, the object of our investigation of Catanese surfaces, with respect to the Tate conjecture, is to understand the representation \( H^2_\ell(X)(1) \).

Unfortunately, it is difficult to determine explicit equations for Catanese surfaces, or for the fibers \( p^{-1}(e) \), making it difficult to understand their \( \ell \)-adic cohomology. To make the situation easier, we instead consider a specific double cover \( Y \) of a Catanese surface \( X \). To obtain the double cover \( Y \), we consider the elliptic curve \( E \), and suppose that we have another elliptic curve \( F \) with a degree 2 isogeny \( \phi : F \to E \). Then we can form the fiber product of \( X \) and \( F \) over \( E \), giving a surface \( Y \), and morphisms \( \Phi : Y \to X \) and \( r : Y \to F \) such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi} & X \\
\downarrow r & & \downarrow p \\
F & \xrightarrow{\phi} & E
\end{array}
\]

commutes. The map \( r : Y \to F \) has the same useful properties as \( p : X \to E \):
the fibers $r^{-1}(f)$ are nonsingular curves of genus 2 for all but finitely many points $f \in F$, and the induced map $r^* : H^1_1(F) \to H^1_1(Y)$ is an isomorphism.

Fortunately, it is possible to find explicit equations for the fibers $r^{-1}(f)$, making it easier to study the characteristics of $Y$ relevant to the Tate conjecture. Specifically, given elliptic curves $E$ and $F$ with degree 2 isogeny $\phi : F \to E$, we can define 7 polynomials $\psi_0, \psi_1, \cdots, \psi_6$ in the variable $Z$, whose coefficients are rational functions defined on $F$. Then, for all points $a = (a_0 : \cdots : a_6) \in \mathbb{P}^6$ lying outside a proper subvariety of $\mathbb{P}^6$, there is a nonsingular Catanese surface $X(a)$, with double cover $Y(a)$ such that the fibers $r^{-1}(f)$ for $f \notin \ker(\phi)$ are described by the affine equations

$$W^2 = a_0\psi_0 + a_1\psi_1 + \cdots + a_6\psi_6.$$ 

There is also a separate equation for the fibers $r^{-1}(f)$ for $f \in \ker(\phi)$, and together these equations give all the necessary information about $Y(a)$ that we need to attempt a proof of the Tate conjecture for almost all Catanese surfaces.

7 An Approach to Proving the Tate Conjecture for Catanese Surfaces

Due to some special properties of Catanese surfaces, it is known that, to prove the Tate conjecture for almost all such surfaces, all we need to prove is that at least one surface in this family has geometric Picard number 2. Further, if the double cover $Y$ of a Catanese surface $X$ has $\rho(Y) = 2$, then we have $\rho(X) = 2$ as well, so we can approach the problem by seeking a double cover $Y$ with $\rho(Y) = 2$. It is also known that $\rho(Y) \geq 2$, so it suffices to prove that $\rho(Y) \leq 2$.

While computing Picard numbers in general can be difficult, Theorem 5.2 showed that there is a relatively straightforward way to compute an upper bound for the Picard number of a surface over a finite field using its zeta function. Thus, rather than compute $\rho(Y)$ directly, we first reduce to a finite field. In particular, if $Y$ is defined over a number field $K$, we can clear denominators so that the defining polynomials of $Y$ lie in the ring of integers $O_K$, and then reduce modulo a prime ideal $\mathfrak{p}$ to obtain the surface $\bar{Y}$. Provided that the reduction $\bar{Y}$ is smooth, we have $\rho(Y) \leq \rho(\bar{Y})$, so it suffices to find a reduction $\bar{Y}$ with $\rho(\bar{Y}) \leq 2$. Recall that $\rho(\bar{Y})$ is bounded above by the number of roots of the polynomial $P_2$ in the denominator of the zeta function $Z(\bar{Y}, T)$ which are $q^{-1}$ times a root of unity, where $q := \#(O_K/\mathfrak{p})$. 

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It remains to actually compute the zeta function. It is known that the factor $P_2$ in the denominator of $Z(\bar{Y}, T)$ has degree 22. Further, we know from the Weil conjectures that $P_0 = 1 - T$ and $P_3 = 1 - qT$. Finally, for $P_1$ and $P_3$, we recall the isomorphism in cohomology $r^* : H^1_\ell(F) \to H^1_\ell(Y)$. It turns out that the reduction modulo $p$, $\bar{r}^* : H^1_\ell(\bar{F}) \to H^1_\ell(\bar{Y})$, is not only an isomorphism of $\mathbb{Q}_\ell$-vector spaces, but also of $G_{\mathcal{O}_K/p}$-representations. Thus, the $P_1$ occurring in $Z(\bar{Y}, T)$ is the same as the $P_1$ in $Z(\bar{F}, T)$, so if we know the zeta function of $\bar{F}$, we know $P_1$. Also, by the Weil functional equation (or by the Poincaré duality used in its proof), we see that $P_3$ is related to $P_1$ by $P_3(T) = P_1(qT)$. Thus, all that is left is to compute the coefficients of $P_2$. By the Weil conjectures, we may factor $P_2$ as

$$P_2(T) = \prod_{j=1}^{22} (1 - \alpha_{2,j} T),$$

where the $\alpha_{2,j}$ are algebraic integers of absolute value $q$. In order to determine these factors, we may count the number of points of $\bar{Y}$ over the first several finite extensions of $\mathbb{F}_q = \mathcal{O}_K/p$. Counting the number of points $N_r$ over $\mathbb{F}_{q^r}$ determines the sum of the $r$th powers of the $\alpha_{2,j}$, since we have

$$Z(\bar{Y}, T) = \prod_{i=0}^4 P_i(T)^{(-1)^{i+1}}$$

$$= \prod_{i=0}^4 \left( \prod_{j=1}^{B_i} (1 - \alpha_{i,j} T) \right)^{(-1)^{i+1}}$$

$$= \exp \left( \sum_{i=0}^4 (-1)^i \left( \sum_{j=1}^{B_i} - \log(1 - \alpha_{i,j} T) \right) \right)$$

$$= \exp \left( \sum_{i=0}^4 (-1)^i \left( \sum_{j=1}^{B_i} \sum_{r=1}^{\infty} \alpha_{i,j}^r \frac{T^r}{r} \right) \right)$$

$$= \exp \left( \sum_{r=1}^{\infty} \left( \sum_{i=1}^{4} (-1)^i \sum_{j=1}^{B_i} \alpha_{i,j}^r \frac{T^r}{r} \right) \right)$$

$$= \exp \left( \sum_{r=1}^{\infty} \left( \sum_{i=1}^{\infty} N_r \frac{T^r}{r} \right) \right),$$

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so that
\[ N_r = \sum_{i=1}^{4} (-1)^i \sum_{j=1}^{B_i} \alpha_{i,j}^r. \]

Thus, since we know all the \( \alpha_{i,j} \) except the ones in \( P_2 \), each \( N_r \) allows us to determine the sum of the \( r \)th powers of the \( \alpha_{2,j} \). Then, using Newton’s identities, which relate the coefficients of a polynomial (which are the elementary symmetric polynomials in the roots) to the power sums of its roots, the first 11 values of \( N_r \) give us 12 coefficients of the polynomial \( P_2 \). If we write
\[ P_2(T) = \prod_{j=1}^{22} (1 - \alpha_{2,j} T) = \sum_{i=0}^{22} c_i T^i, \]
the first 11 \( N_r \) determine:
\[
\begin{align*}
   c_0 &= 1, \\
   c_1 &= -\left( \alpha_{2,1} + \alpha_{2,2} + \cdots + \alpha_{2,22} \right), \\
   c_2 &= \alpha_{2,1}\alpha_{2,2} + \alpha_{2,1}\alpha_{2,3} + \cdots + \alpha_{2,21}\alpha_{2,22}, \\
   \vdots \\
   c_{11} &= -\left( \alpha_{2,1}\alpha_{2,2} \cdots \alpha_{2,11} + \cdots + \alpha_{2,12} \cdots \alpha_{2,22} \right).
\end{align*}
\]

At this point, we may take advantage of the Weil functional equation to compute the remaining coefficients (up to a choice between two possibilities). By the functional equation, the map \( \alpha \mapsto q^2/\alpha \) is a permutation of the \( \alpha_{2,j} \). In particular, this implies that the product
\[
\left( \prod_{j=1}^{22} \alpha_{2,j} \right)^2 = \prod_{j=1}^{22} \left( \alpha_{2,j} \cdot \frac{q^2}{\alpha_{2,j}} \right) = q^{44},
\]
so \( \prod_{j=1}^{22} \alpha_{2,j} = \pm q^{22} \).

Using this fact, it is easy to see that the remaining coefficients satisfy the formula \( c_i = \pm q^{2i-22} c_{22-i} \), where the sign is the same in all cases, depending on whether the product of the \( \alpha_{2,j} \) is \( q^{22} \) or \( -q^{22} \). Thus, counting points over the first 11 extensions of \( \mathbb{F}_q \) allows us to narrow down the zeta function \( Z(Y, T) \) to two possibilities, and generally the Weil Riemann hypothesis can eliminate one of these as invalid (if absolutely necessary, the 12th extension will determine the sign, since it determines the 12th coefficient without requiring a sign choice).
In summary, the above discussion describes the following algorithm for determining if a particular Catanese surface has Picard number 2, and thus the Tate conjecture holds for almost all such surfaces:

1. Given an elliptic curve $F$ over a number field $K$ with a specified 2-torsion point $C$, and a point $a \in \mathbb{P}^6(K)$, we can find equations for the fibers of the double cover $Y(a)$ of the Catanese surface $X(a)$ over each point of $F$.

2. Reducing modulo a prime ideal $p \subseteq \mathcal{O}_K$, we obtain equations for the fibers of the reduction $\bar{Y}$, which allows us to count the number of points of $\bar{Y}$ over each finite extension of $F_q = \mathcal{O}_K/p$.

3. Using what we know about the zeta function $Z(\bar{Y}, T)$, and applying Newton’s identities as described above, we can use the point counts over $F_{q^r}$ for $1 \leq r \leq 11$ to give two possible zeta functions, and we can determine which is the correct one by checking against the Weil conjectures or the 12th point count.

4. Having computed $Z(\bar{Y}, T)$, we can factor the polynomial $P_2$ in the denominator. The number of roots of $P_2$ which are $1/q$ times a root of unity gives an upper bound on the Picard number $\rho(Y)$. If this upper bound happens to be 2, this gives a proof of the Tate conjecture for almost all Catanese surfaces.

To carry out this algorithm, we have implemented a program in the Magma environment to count the necessary points, using the equations for the fibers of $\bar{Y}$ above each point of $\bar{F}$. As of the time of writing, the program has not yet produced a conclusive result, but hopefully, we will find a surface of Picard number 2 in the near future and prove a significant special case of the Tate conjecture.

References


