Hyperelliptic Hodge Integrals

Peter Troyan

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Abstract
In this paper, we study hyperelliptic Hodge integrals. We seek a formula that allows us to compute as many of these Hodge integrals as possible, and succeed in finding recursions to compute all Hodge integrals that take some specific forms. On the way to this result, we also find a formula to express the first Chern class of the Hodge bundle over our space of admissible covers in terms of the pullbacks of boundary divisors in $\overline{M}_{0,n}$.

1 Introduction
Hodge integrals are a class of intersection numbers on moduli spaces of curves involving the Chern classes $(\lambda_i)$ of the Hodge bundle $E$. For our purposes, we will be concerned with the Hodge bundle restricted to the moduli space of degree-2 admissible covers of nodal rational curves. With this restriction, Hodge integrals will take the general form

$$\int \prod_{i}^{k} \lambda_{i_k}$$

with $\sum \lambda_{i_k} = 2g - 1$ and all $i_k \leq g$, where $g$ is the genus of the curves parametrized by the particular space we are studying on which the $\lambda_{i_k}$’s are defined.

We seek a formula to calculate as many of these Hodge integrals as possible, and prove a result that allows us to reduce computing any Hodge integral which contains a $\lambda_1$ to computing Hodge integrals on spaces of lower genus. In addition, we find nice recursions that allow us to very quickly calculate the values of all Hodge integrals which take a specific form. We are also able to define a generating function for one specific class of these numbers.

The main results are as follows:

**Theorem 1.1** Let $\pi$ be the function that connects $\overline{\text{Adm}}_g$ and $\overline{M}_{0,n}$ $\pi : \overline{\text{Adm}}_g \rightarrow \overline{M}_{0,n}$, and $\pi_*(\lambda_1) \in \overline{M}_{0,n}$ be the push forward of the first Chern class of $E^g$ on $\overline{\text{Adm}}_g$. Then, we can express $\pi_*(\lambda_1)$ as
\[ \pi_*(\lambda_1) = \frac{n^2}{2} \sum_{i=2}^{n/2} \alpha_i \Delta_{i|n-i} \quad (n = 6, 8, 10, \ldots) \]

where

\[ \alpha_i = \begin{cases} \frac{i(n-i)}{16(n-1)} & i = 2, 4, 6, \ldots \\ \frac{(i-1)(n-i-1)}{16(n-1)} & i = 3, 5, 7, \ldots \end{cases} \]

\( \Delta_{i|n-i} \) denotes a boundary divisor with \( i \) points on one twig, and \( n-i \) points on the other.

We can then take any Hodge integral that contains at least one \( \lambda_1 \) and replace it with the pullback of the expression from Theorem 1.1. By doing so, we are able to prove:

**Theorem 1.2** All Hodge integrals that contain at least one \( \lambda_1 \) can be reduced to integrals of lower genus.

After proving these two results, we then use them to look at two specific classes of Hodge integrals, and find explicit recursions to calculate any number in either class:

**Theorem 1.3** Let \( L^g_i = \int \lambda_g \lambda_{g-i}(\lambda_1)^{i-1} \). Then,

\[ L^g_i = \frac{1}{2g+1} \frac{1}{2g+2} \sum_{g_1=1}^{g-1} \sum_{i_1=1}^{g_1-1} g_1 g_2 \left( \frac{2g+2}{2g_1+1} \right) \left( \frac{i-2}{i_1-1} \right) L^g_{i_1} L^g_{i_2} \]

where \( g_2 = g - g_1, i_2 = i - i_1 \)

**Theorem 1.4** Let \( K^g = \int (\lambda_1)^{2g-1} \). Then,

\[ K^g = \frac{1}{2g+1} \frac{1}{2g+2} \sum_{g_1=1}^{g-1} g_1 g_2 \left( \frac{2g+2}{2g_1+1} \right) \left( \frac{2g-2}{2g_1-2} \right) K^{g_1} K^{g_2} \]

where, again, \( g_2 = g - g_1 \).

**Outline of the paper**

The paper is divided into four sections. The second section consists of background in which we briefly introduce both the moduli space \( \overline{M}_{0,n} \) and the moduli space of degree 2 admissible covers. Instead of going through this in detail, many quick examples are given, often by picture, to help illuminate some important properties of these moduli spaces. After that, we recall some important
facts about the Hodge bundle and Chern classes which will be necessary for our results.

Section 3 is a lengthy proof of Theorem 1.1 above. We prove the given expressions for the $\alpha_i$'s by constructing a linear system of equations, inserting our proposed values for the $\alpha$'s, and checking that we get the expected solution vector. This is a very important result, as it allows us to replace one of the $\lambda_1$'s in our Hodge integral with the pullback of the expression from Theorem 1.1, which will be a crucial step in actually calculating many of our Hodge integrals.

In section 4, we use Theorem 1.1 to prove that any Hodge integral with at least one $\lambda_1$ can be reduced to Hodge integrals on curves of lower genus. Then, we give an explicit example to show exactly how this is done in practice. Finally, we apply the method to two larger classes of Hodge integrals, and give explicit recursions to calculate any of the numbers in either class.

2 Background

2.1 The moduli space of $n$ marked points on $\mathbb{P}^1$

$M_{0,n}$ denotes the moduli space of $n$ distinct, marked points on the complex projective line $\mathbb{P}^1$ up to automorphism. The group of automorphisms of $\mathbb{P}^1$ is the group of invertible matrices ($\text{PGL}(2, \mathbb{C})$) modulo a constant $\lambda$. A point $\left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{P}^1$, will be sent to the point:

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} ax + by \\ cx + dy \end{array} \right) \quad (2.1)$$

Since two matrices are considered equivalent if they differ by an overall constant factor $\lambda$, the group is actually 3-dimensional. This means that if we have a copy of $\mathbb{P}^1$ with $n$ distinct, marked points, we can choose to send any 3 points to any other three points we want and still be at the same moduli point of $M_{0,3}$. Usually, this is done by sending $p_1 \mapsto 0$, $p_2 \mapsto 1$, and $p_3 \mapsto \infty$. Once we have done this, the image of the other $n-3$ points is completely determined.

Because of the equivalence up to automorphism, if we are given two collections of $n$ points on $\mathbb{P}^1$, they may actually correspond to the same point of $M_{0,3}$. In fact, this is always the case when $n = 3$; given any $3$ points, we can always find an automorphism that sends $(p_1, p_2, p_3) \mapsto (0, 1, \infty)$. Therefore, any collection of 3 distinct points is equivalent to any other, and so $M_{0,3}$ is actually just one point.

Now consider $n = 4$. Again, we can find an automorphism $\phi$ that takes the first 3 points to $(0, 1, \infty)$, but now, the image $\phi(p_4)$ depends on what $p_4$ actually is. The isomorphism class will be completely determined by the image of $p_4$, with each choice of $p_4$ corresponding to one point of our moduli space $M_{0,4}$. Since $\phi(p_4)$ can be any element of $\mathbb{P}^1$ except $0, 1, \text{or} \infty$ (the $n$ points must be distinct), $M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$.

As we increase $n$, we can choose each new point to be any element of $\mathbb{P}^1$ with the exception that it must be distinct from all other points, which gives
\[ M_{0,n} = \{(n - 3) \text{ copies of } M_{0,4}\} \setminus \{\text{all diagonals}\}. \]

### 2.2 Compactifying \( M_{0,n} \)

One major problem with the space \( M_{0,n} \) described above is that it is not compact if \( n > 3 \). The space \( \mathbb{P}^1 \) is compact, and so, one idea to compactify \( M_{0,n} \) is to just replace the points we are missing. That is, allow points to coincide (diagonals) and hence, go to a space of \( n \) not necessarily distinct marked points. However, as the following simple example from [1] shows, this will not work.

Consider \( M_{0,4} \) and the families

\[
C_t = (0, 1, \infty, t) \quad \text{and} \quad D_t = (0, \frac{1}{t}, \infty, 1)
\]

For any \( t \neq 0 \), the function \( \phi = \frac{t}{t} \) is an isomorphism between the two families, and so they are the same point of \( M_{0,4} \). However, when \( t = 0 \), we get \( C_0 = (0, 1, \infty, 0) \) and \( D_0 = (0, \infty, \infty, 1) \), which clearly have no isomorphism connecting them. Since these two points are both limit points of equivalent families, we would like them to be the same point in our space. Clearly, just allowing points to coincide is not a good compactification of our space.

As the preceding example shows, we cannot just allow two (or more) points to coincide, but must place further restrictions such as \( p_1 \) and \( p_4 \) coinciding (\( C_0 \) above) is equivalent to \( p_2 \) and \( p_3 \) coinciding (\( D_0 \) above). Clearly, there is no reason to prefer this particular division of points, as we could have easily had \( p_1 = p_2 \) (equivalent to \( p_3 = p_4 \)) or \( p_1 = p_3 \) (equivalent to \( p_2 = p_4 \)). Note that this is very good, because we have exactly three points with which we can replace the three points of \( \mathbb{P}^1 \) that are missing from \( M_{0,4} \).

In fact, the formally correct way to compactify \( M_{0,n} \) is to allow certain rational curves to fill in the gaps left by subtracting off all diagonals. A brief introduction to how this works, based on Kock and Vainsencher, will be given here. For a more complete description, see their book [*Kontsevich’s Formula for Rational Plane Curves*] (7).

**Definition 2.1** A tree of projective lines is a connected curve with the following properties:

1. Each irreducible component (henceforth, called a twig) is isomorphic to \( \mathbb{P}^1 \)
2. The points of intersection of the components are ordinary double points
3. There are no closed circuits

**Definition 2.2** A tree is said to be stable if every twig has at least 3 special points (marks or nodes).

A few pictures will probably help clarify these notions.

In the figures, each line represents a twig, or a copy of \( \mathbb{P}^1 \). The dots are the marked points on \( \mathbb{P}^1 \), and two twigs intersect in a node. Each of the curves 1-4 is stable because each twig has at least three special points (marks or nodes).
Curve 5 is not stable because the twig on the right only has 2 special points (one mark and one node). Curve 6 is not rational because it is a closed circuit.

It can be shown that the space of \( n \)-pointed rational stable curves, labeled \( \overline{M}_{0,n} \), is a smooth moduli space that compactifies \( \overline{M}_{0,3} \).

2.2.1 An Example: \( \overline{M}_{0,4} \)

Rather than go into details about the above point, we will just proceed by an example to help illustrate these notions. Let \( \overline{M}_{0,4} \) be space of 4-pointed rational stable curves. Clearly, any point of \( M_{0,4} \), where all of the 4 points are distinct, is a member of this space. The tree diagram corresponding to a point of \( M_{0,4} \) is shown below. It consists of just one twig with 4 marks.

![Figure 2](image)

Figure 2: The tree diagram corresponding to a point of \( M_{0,4} \), where all points are distinct.

However, in addition to all of these points, \( \overline{M}_{0,4} \) contains three additional points that are not part of \( M_{0,4} \):

![Figure 3](image)

Figure 3: The three 4-pointed rational curves that compactify \( M_{0,4} \).

Besides the curve in Figure 2, where all points are on one twig, the three points in Figure 3 are the only other possible rational stable curves when \( n = 4 \),
because each twig must have at least 3 special points. These three points are exactly the three points we must add to fill in the three gaps in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The best intuitive way to understand what is happening with these boundary divisors is as follows: whenever two points try to come together, a new twig sprouts out and those two points move to this new twig, as in the Figure 4.\(^1\)

![Figure 4](image1.png)

Figure 4: When $p_1$ and $p_4$ attempt to collide, a new twig sprouts out to receive them both.

Also, note that this construction actually solves the problem that arose earlier when we considered the two families $C_t$ and $D_t$, which were the same except at the limit point $t = 0$. To solve that problem, we wanted $p_1$ colliding with $p_4$ to be equivalent to $p_2$ colliding with $p_3$. But, look at what happens when $p_2$ and $p_3$ attempt to collide:

![Figure 5](image2.png)

Figure 5: When $p_2$ and $p_3$ attempt to collide, a new twig sprouts out to receive them both. Notice that the end result is exactly the same as if we had thought of $p_1$ and $p_4$ colliding instead of $p_2$ and $p_3$ (compare with Figure 4).

This is exactly the same as $p_1$ colliding with $p_4$! We have found a good compactification of $M_{0,4}$.

Also, as a note, the three curves presented in Figure 3 can be thought of as two copies of $\mathcal{M}_{0,3}$ attached at one point. But, since $\mathcal{M}_{0,3}$ is just a point, each of those curves is also just a single point, giving exactly the three points needed to replace the three missing points of $M_{0,4}$.

\(^1\)The same happens if more than 2 points come together or if the points are marks, nodes, or some combination of both: all points involved will move to the newly sprouted twig.
2.3 The Boundary

The boundary of $\overline{M}_{0,n}$ will be very important for us. The boundary is defined as the complement of $M_{0,n}$ in $\overline{M}_{0,n}$, and hence consists of all rational stable curves with at least one node. It should be obvious that the dimension of $M_{0,n}$ is $n - 3$ (because we can always fix 3 points at $(0, 1, \infty)$, while the others are free to move in a 1-dimensional space). It is only trivially more difficult to see that the dimension of any boundary strata will be $(n - 3) - (\# \text{ of nodes})$.

A point in the boundary of $\overline{M}_{0,n}$ is a tree of projective lines (twigs) as defined earlier. If we have $\tau$ twigs, then, by the definition of a tree, we will have $\tau - 1$ nodes. Each node counts as two special points (one on each twig that it connects), and so, when we add in the $n$ marks, we have a total of $n + 2(\tau - 1)$ special points to distribute over the $\tau$ twigs. If we place the bare minimum of 3 special points on each twig, then we find that the maximum number of twigs that we can have is $\tau_{max} = n - 2$. Of course, we can also have boundary strata with any number of twigs less than $\tau$, so $2 \leq \tau \leq n - 2$ (the lower bound is 2, not 1, because if $\tau = 1$, we would no longer be on the boundary).

Once we have found all possible arrangements of twigs on the boundary, then, to find all boundary strata, we must distribute the $n$ points in every possible way about those twigs.

As a simple case, consider $n = 5$. We will have boundary strata with $\tau = 2, 3$ twigs to connect in all possible ways. There is clearly only 1 way to connect two twigs, and there is also only one unique way to connect three twigs. The general forms of all possible boundary strata are shown in Figure 6.

![Figure 6: The two possible forms of boundary strata when $n = 5$. To get all possible boundary strata, we must distribute the five distinct points across the marks in all possible ways.](image)

Now that we know the general form of the boundary strata, we must distribute the 5 points in every possible way. For the class on the left, once we pick two points to go on the left twig, the points on the right twig are completely determined, and so there are $\binom{5}{2} = 10$ possibilities. For the boundary strata on the right, we must first pick 2 points for the first twig, then 1 point for the middle twig, but then, the last twig is determined, so at first glance there are $\binom{5}{2} \binom{3}{1} = 30$ ways to do this. However, there is an axis of symmetry involved. Picking $p_1$ and $p_2$ for the first twig is the same as picking $p_1$ and $p_2$ for the last twig, and so we must divide our answer for this class by 2, giving a grand total
of 25 boundary strata on $\overline{\mathcal{M}}_{0,5}$. A few of the individual strata are shown in Figure 7.

![Diagram of boundary strata](image)

**Figure 7**: A few of the 25 possible boundary strata of $\overline{\mathcal{M}}_{0,5}$. The left column is boundary strata of codimension 1 (1 node), and the right column is boundary strata of codimension 2 (2 nodes).

An important characteristic of the boundary strata is their dimension. Clearly, $\overline{\mathcal{M}}_{0,n}$ has dimension $n - 3$. As we move to the boundary and add nodes, the dimension decreases. For example, the strata on the left side of Figure 7 can all be broken down as $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}$, which is 1-dimensional (there is one point on the right hand twig which can move after we have fixed 3 of the points to $(0, 1, \infty)$). On the right side of Figure 7, each twig has exactly 3 special points, and so this will break down as $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3}$, which has dimension 0.

A useful way to characterize boundary strata is with the codimension rather than the dimension (as we will be intersecting cohomology classes later). The codimension of any subset of $\overline{\mathcal{M}}_{0,n}$ is equal to the number of nodes on the $n$-pointed stable rational curve we are looking at. In Figure 7, the left column has codimension 1, and the right hand column has codimension 2.

### 2.3.1 Boundary Divisors

For our purposes, the most important elements of $\overline{\mathcal{M}}_{0,n}$ will be codimension 1 boundary strata, also known as boundary divisors. Boundary divisors represent curves with exactly one node (and therefore two twigs) that have $i$ points on one twig and $n - i$ points on the other. The general form for all boundary divisors when $n = 8$ is shown in the top row of Figure 8. But remember, to count all boundary divisors, we must distribute the points across the marks in all possible
ways, which gives \( \binom{8}{2} + \binom{8}{3} + \binom{8}{4}/2 = 119 \) different boundary divisors in \( \overline{M}_{0,8} \) (the factor of 2 again comes from symmetry). One specific divisor for \( i = 3 \) is also shown in Figure 8.

\[
\begin{align*}
\Delta_{i|n-i} & \text{ will always denote the formal sum of all boundary divisors with any combination of } i \text{ points on one twig and the other } n-i \text{ points on the other twig, and } \\
\Delta_{p_1,\ldots,p_i|p_{i+1},\ldots,p_n} & \text{ will denote the single boundary divisor with the specific points } (p_1,\ldots,p_i) \text{ on one twig and the points } (p_{i+1},\ldots,p_n) \text{ on the other twig.}
\end{align*}
\]

For example, reading across the first row of Figure 8 above, the symmetrized boundary divisors with no points labeled would be referred to as \( \Delta_{2|6}, \Delta_{3|5}, \Delta_{4|4} \).

Notation: As boundary divisors will be fairly important, we need to introduce some notation that will make them easier to work with. \( \Delta_{i|n-i} \) will always denote the formal sum of all boundary divisors with any combination of \( i \) points on one twig and the other \( n-i \) points on the other twig, and \( \Delta_{p_1,\ldots,p_i|p_{i+1},\ldots,p_n} \) will denote the single boundary divisor with the specific points \( (p_1,\ldots,p_i) \) on one twig and the points \( (p_{i+1},\ldots,p_n) \) on the other twig.

Figure 8: The top row shows the generic boundary divisors for \( \overline{M}_{0,8} \), with no labels on the points. The bottom row shows one specific boundary divisor when \( i = 3 \) (middle of the top row).

It is actually fairly easy to count the exact number of boundary divisors we have in \( \overline{M}_{0,n} \) for any \( n \) by simply generalizing the combinatorial calculations we did earlier. Let \( i \) denote the number of points on one twig of the boundary divisor. We must have at least 2 marks on each twig (since there is one node), and so \( i \) can take on any value between \( 2 \leq i \leq n - 2 \). The number of specific boundary divisors for any value of \( i \) is simply \( \binom{n}{i} \), choosing \( i \) of the \( n \) points to go on one curve. But, since putting \( i \) points on one twig by default puts \( n-i \) points on the other twig, these two values in the summation will actually give us the same divisors, and so we must divide the entire expression by 2. Putting this all together yields

\[
\frac{1}{2} \sum_{i=2}^{n-2} \binom{n}{i}
\]

for the total number of boundary divisors on \( \overline{M}_{0,n} \). As you can see, the number of boundary divisors grows very rapidly.
2.4 Intersections on the Boundary

Intersection will be the most important operation on boundary strata of $\overline{M}_{0,n}$. If we think of the strata on the boundary as representing cohomology classes in $\overline{M}_{0,n}$ of codimension $k$ (where $k$ is the number of nodes), then two strata of codimension $k$ and $j$ should give an intersection of codimension $k + j$ (i.e., a strata with $k + j$ nodes). Sometimes the two strata we are intersecting will be transversal and everything will work out very nicely; other times, however, one stratum will actually be fully contained in the other, and in this case, the intersection will not be of the right codimension. To deal with this, we will have to introduce the notion of $\psi$ classes.

2.4.1 Transversal Intersections

To find the intersection of two curves that intersect transversally, we must simply find all stable curves that are contained in both of our original strata. As an example, consider the following intersection:

![Figure 9: An example of an intersection in $\overline{M}_{0,7}$.](image)

The most intuitive way to understand this is to think of the curve on the right as a special case of both of the curves on the left, where certain points have “collided” with each other. This is not hard to see for curve II: when $p_5$ approaches the node of the twig it is on, a new twig will sprout out to receive $p_5$ and the node, just as we described earlier in the compactification of $\overline{M}_{0,4}$, giving exactly curve III.

However, it may be slightly more complicated to see exactly how to obtain III from I, and so the steps for this are sketched in Figure 10.

To obtain curve III from curve I, we first switch around $p_5$ and $p_3$ (this is merely for pictorial convenience, as each twig actually represents a copy of $\mathbb{P}^1$). Then, we look at the limit as $p_5$ approaches the node labeled $\star$. Just as we described before, when $p_5 \to \star$, a new twig sprouts, and both $p_5$ and $\star$ lie on this new twig, giving the third curve in Figure 10. Now, we allow the points $p_3$, $p_4$, and the node labeled (■) to all come together at the same time. Another new twig sprouts out to receive these three points, giving finally the curve labeled III.

A few points must be emphasized about these intersections:
• When looking for such intersections, we must look at all possible collisions between points (both marks and nodes).

• A twig with exactly three points cannot have any of the points collide, since we can always find an automorphism to send the three points back to $(0, 1, \infty)$. If we ever have exactly three marked points on one twig, those three points must always remain on the same twig.

• Points cannot ‘jump over’ nodes. This will be important later on, and so to make this clear, consider again curve II from Figure 10. Notice that there is a node ($\star$) between $p_5$ and $p_3$, for example (any of the points on the two twigs would work just as well, since any two points on one twig are on an equal footing).

The curves above are not contained in II because they would involve $p_5$ and/or $p_3$ jumping over $\star$. Essentially, this means that the ‘order’ of the points
must be preserved between twigs. If we have a bunch of points on one twig (e.g., $p_1, p_2,$ and $p_5$ above), then the points from a different twig ($p_3$) cannot move ‘in between’ them. However, the order of points that are on the same twig is irrelevant, and two points that initially started on the same twig can always be interchanged in the end (e.g., in III in Figure 9, we could switch $p_5$ with $p_1$ or $p_2$ and still get a curve that is contained in II, although it would no longer be contained in I and hence not in the intersection of I and II). For a more rigorous derivation of these rules, see [7].

2.4.2 Psi Classes

Sometimes, when we intersect curves as above, the intersection will actually be of the wrong codimension. To deal with this, we now introduce the notion of $\psi$ classes. For a more complete discussion of $\psi$ classes than that found below, see [6].

$\psi_i$ (the $\psi$ class at the $i^{th}$ marked point) is a codimension 1 cohomology class on $\overline{M}_{0,n}$. To actually define $\psi$ classes on $\overline{M}_{0,n}$, we need a universal family over it. To do so, we will use the forgetful morphism $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ that forgets the extra mark $p_{n+1}$. Then, $\psi_i$ will be the first Chern class of the cotangent line bundle $L$ at the $i^{th}$ point:

$$\psi_i := c_1(L_i)$$

We will need to be able to compute $\psi_i$ on any $\overline{M}_{0,n}$. Fortunately, any $\psi_i$ on $\overline{M}_{0,n+1}$ can be found very easily from the pullback $\pi_{n+1}^*(\psi_i)$ [6]:

$$\psi_i = \pi_{n+1}^*(\psi_i) + \Delta_{p_1, p_{n+1}|p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n}$$

(2.2)

A few quick examples will make clear how to calculate $\psi$ classes. First, $\overline{M}_{0,3}$ is just a point, and so $\psi_{p_1}, \psi_{p_2}, \psi_{p_3} = 0$. Moving to $\overline{M}_{0,4}$, the first term in (2.2) is still 0, and so the only contribution to $\psi_{p_1}$, for example, is

$$\psi_{p_1} = \Delta_{p_1, p_4|p_2, p_3}$$

One should note that, as described earlier, $\Delta_{p_1, p_4|p_2, p_3}$ is actually equivalent to $\Delta_{p_1, p_3|p_2, p_4}$ and $\Delta_{p_1, p_2|p_3, p_4}$. Relations like this will also hold when we have more marked points, so that we must always remember that a given $\psi$ class can actually be represented by many different geometric loci.

Following the same procedure for $n = 5$, we get

$$\psi_{p_1} = \frac{\pi_{n+1}^*(\psi_{p_1})}{\text{term of (2.2)}}$$

We can continue in a similar fashion to calculate $\psi$ classes on $\overline{M}_{0,n}$ for any $n$. 

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2.4.3 Non-transversal intersections

The reason for the above discussion on $\psi$ classes is that they will be necessary for us to compute non-transversal intersections, like the one shown in Figure 12. When intersecting cohomology classes, the codimension of the intersection should equal the sum of the codimensions of the classes we are intersecting. In Figure 12, however, IV is actually fully contained in V, which means that we would get an intersection of codimension 2, not codimension 3, as it should be.

To deal with this problem, we use Euler classes. Specifically, given a space $X$, a subspace $Y \subseteq X$, and a further subspace $Z \subseteq Y$, the intersection of $Y$ and $Z$ is defined to be the Euler class of the normal bundle to $Y$ in $X$, restricted to $Z$:

$$Y.Z = e((N_{Y\mid X})|Z)$$

In our case, this becomes the sum of the $-\psi$ classes at the node common to both curves, restricted to the curve of higher codimension [4].

For example, in Figure 12, the intersection would be calculated as the sum of the $-\psi$ classes at $\star$ of the curve V.

From Figure 13, we can see that $IV.V = -\psi\Box - \psi\Diamond$. What this means exactly is that we break up our curve V at the node common to both curves ($\star$), remembering that this node is a special point on our two new curves ($\Box, \Diamond$). Then,

Figure 12: A non-transversal intersection. IV is fully contained in V, which makes the intersection of codimension 2, not 3, as it should be.

Figure 13: The curve V can be broken down into the two pieces shown above, where $\Diamond$ and $\Box$ are glued together to form the node $\star$. 
just as before, we calculate \( \psi \blacklozenge \), which is a \( \psi \)-class on \( \overline{M}_{0,3} \), and so \( \psi \blacklozenge = 0 \). \( \psi_\lozenge \) is a \( \psi \)-class on \( \overline{M}_{0,4} \), and so it is the one point shown in Figure 14.

![Figure 14: \( \psi_\lozenge \)](image)

But then, remembering that \( \lozenge \) is actually glued to \( \blacklozenge \), we get finally that the intersection IV.V is the one point shown in Figure 15.

![Figure 15: The intersection IV.V](image)

### 2.5 The Space of Admissible Covers

The space that we are actually interested in studying is not \( \overline{M}_{0,n} \), but a very closely related space, the space of degree-2 admissible covers of nodal rational curves, branched over \( n \) points, which we will denote \( \text{Adm}_g \). (For a complete discussion of ramified covers, see [3].) The \( g \) will be the genus of the cover, and will be related to the number of marked points of \( \overline{M}_{0,n} \) by \( n = 2g + 2 \). A point in the moduli space \( \overline{M}_{0,n} \) is a copy of the complex projective line (i.e., a sphere) with \( n \) points marked.

![Figure 16: A point in \( \overline{M}_{0,4} \) (a sphere with 4 marked points).](image)
A point in $\overline{Adm}_1$ will be a 2:1 cover of the sphere shown in Figure 16, ramified over the 4 marked points. If we imagine cutting along lines connecting the marked points, as shown in Figure 17, we would have a cover that consists of two separate pieces, each of which covers the marked copy of $\mathbb{P}^1$.

![Figure 17: The same point of $\overline{M}_{0,4}$ (bottom sphere) and its 2:1 cover in $\overline{Adm}_1$ (top 2 cylinders), with the marked points and the lines connecting them cut out. The point $x$ is not a ramification point, so that every open neighborhood around it has 2 preimages, one on each cylinder, as shown.](image)

But then, putting these lines (and marked points) back in causes the two cylinders to be glued together along the indicated lines, giving the cover to be a torus ($g = 1$) (see Figure 18). Each point on $\mathbb{P}^1$ has 2 preimages on the torus, except for the 4 marked points, which only have one preimage.

![Figure 18: After gluing the two cylinders along the lines shown in Figure 17, we get that the cover of a point of $\overline{M}_{0,4}$ is a torus. Again, the point $x$ is not a ramification point, and so any neighborhood of $x$ will have two preimages on the torus.](image)

In order for this to work, $n$ must be even, and so, from now on, we will only consider spaces with an even number of marked points. Every time we add 2
points, we must make one additional cut, which will give one more hole when we glue back together. This means that, in general, a cover of a point in $\mathcal{M}_{0,n}$ will be a surface of genus $g = \frac{n-2}{2}$ with $n$ marked points. Since the dimension of $\mathcal{M}_{0,n}$ is $n - 3$, this implies that the dimension of $\overline{\text{Adm}}_g$ is $2g - 1$.

Since the boundary divisors (codimension 1 strata) of $\mathcal{M}_{0,n}$ are the most important boundary strata, we now explain how to construct a curve in the boundary of $\overline{\text{Adm}}_g$. The easiest way to do this is to simply break the boundary divisor into its constituent twigs (keeping track of the points that are glued to form the node), find the covers of the individual twigs, and then reattach the covers at the points that were originally attached at the node in $\mathcal{M}_{0,n}$.

To make this more precise, recall that a boundary divisor with $i$ points on one twig and $n - i$ points on the other can be thought of as the product $\mathcal{M}_{0,i} \times \mathcal{M}_{0,n-i}$. We can then make a similar construction with the cover of that curve in $\overline{\text{Adm}}_g$, by breaking up the two twigs into two distinct curves, one in $\mathcal{M}_{0,i}$ and one in $\mathcal{M}_{0,n-i}$. We then pullback to find the cover of each of these curves, one of which will live in $\overline{\text{Adm}}_{g_1}$ and the other in $\overline{\text{Adm}}_{g_2}$, where $g_1, g_2 < g$.

The big issue that arises here is what to do with the node. There are two distinct possibilities. Recall that the number of marked points on any twig must always be even. But, if $i$ is odd, that would actually mean that when we break up our boundary divisor, both of our twigs will have an odd number of marked points. We solve this problem by letting the node on each twig be the last marked point. Then, we construct the cover of each half of the boundary divisor as usual, remembering which point is the node, and finally, we reattach the two points on each half of the cover that originally came from the node, and we have the cover of our boundary divisor, which lives in the space $\overline{\text{Adm}}_{g_1} \times \overline{\text{Adm}}_{g_2}$ (where $g_1, g_2$ are related to $i, n - i$ as described before).

Now, if $i$ is even, we are actually able to form proper covers of each twig without the addition of the node as a special point. However, when we break up our boundary divisor into its constituent twigs, we must still add an extra point to each twig to keep track of the node. The difference here is that when we pullback to the space $\overline{\text{Adm}}_{g_1}$ (or $\overline{\text{Adm}}_{g_2}$), since the node is not a branch point on the twig, its preimage will not be a ramification point on the cover, so it will have two preimages.

A remark on notation: We will denote the space of covers of a boundary divisor whose node is a branch point by $\overline{\text{Adm}}_{g_1} \times \overline{\text{Adm}}_{g_2}$. If the node is not a branch point, we will denote the space of covers by $\overline{\text{Adm}}_{g_1,*} \times \overline{\text{Adm}}_{g_2,*}$, where the $*$ serves to remind us that the node actually has two preimages.

As an example of this, consider the boundary divisor of $\mathcal{M}_{0,6}$ shown in Figure 19, with the covers of each twig drawn above.

The left twig has only 2 marks, and so the point $*$ (which is the point that is glued to form the node) is not a ramification point and has two preimages. With only 2 ramification points, the left twig has a cover of genus 0. The right twig then has 4 marks, and so $\blacksquare$ is also not a ramification point, giving two preimages to glue to the 2 preimages of $*$. Since the right twig has 4 ramification points, it will be a curve of genus 1. Upon gluing, the cover of the boundary
Figure 19: The cover of the boundary divisor shown is a genus 0 cover attached to a genus 1 cover at 2 points. The gluing at two points adds an extra hole, giving the complete cover genus 2.

divisor becomes the genus 2 curve shown in the figure: a genus 0 curve attached to a genus 1 curve at 2 points, living in the space $\overline{\text{Adm}}_{0,\ast} \times \overline{\text{Adm}}_{1,\ast}$ (note that the second hole comes from the gluing of $\ast$ and $\blacksquare$, giving a total $g = 2$).

Figure 20 below shows what happens if the number of marks on each twig is odd. Now, the points which come from the node ($\ast$ and $\blacksquare$) are ramification points, and so have only one preimage. The individual twigs both have 4 ramification points, giving each the genus 1 cover shown. Gluing $\ast$ to $\blacksquare$ gives the cover of the boundary divisor to be the genus 2 curve shown: two genus 1 curves attached at exactly 1 point ($\in \overline{\text{Adm}}_{1} \times \overline{\text{Adm}}_{1}$).

Later on, we will be dealing with covers of strata of higher codimension. However, the covers will behave in exactly the same way as they did for boundary divisors. The space $\overline{\text{Adm}}_{g}$ will restrict to the boundary as $\overline{\text{Adm}}_{g_{1}} \times \overline{\text{Adm}}_{g_{2}} \cdots \times \overline{\text{Adm}}_{g_{k}}$ (where here we have not denoted by $\ast$ whether the nodes are ramified or not, since this will depend on specifically how the marks are distributed).

Any boundary divisor of $\overline{M}_{0,n}$ will have a cover of genus $g = \frac{n-2}{2}$, as expected, and the cover will break down in one of the two ways illustrated above. To see this, let $i$ be the number of marks on one twig ($2 \leq i \leq (n-2)$) with $n-i$ marks on the second twig. Then, consider the two possible cases:

1. $i$ is even

Since $i$ is even, $n-i$ is even as well, and the two twigs both have an even number of marks. Therefore, the node is not a ramification point, and we have as our cover a curve of $g_{1} = \frac{i-2}{2}$ attached at two points to a curve of $g_{2} = \frac{n-i-2}{2}$. The attaching at two points gives one extra hole, and so, after gluing, the entire cover has genus $g = \frac{n-2}{2}$. 
Figure 20: The cover of this boundary divisor is two genus 1 covers attached at one point, once again giving the complete cover genus 2.

\[ g = g_1 + g_2 + 1 = \frac{(i - 2) + (n - i - 2)}{2} + 1 = \frac{n - 2}{2} \]

2. \( i \) is odd

Again, both \( i \) and \( n - i \) are odd, so the node now is a ramification point on both twigs, giving the two twigs \( i + 1 \) and \( n - i + 1 \) ramification points, respectively. This means that our cover is a curve of genus \( g_1 = \frac{i+1-2}{2} \) attached at one point to a curve of genus \( g_2 = \frac{n-i+1-2}{2} \). Here, the attaching at only one point does not give an extra hole, and so, the genus of the entire cover will be

\[ g = g_1 + g_2 = \frac{(i + 1 - 2) + (n - i + 1 - 2)}{2} = \frac{n - 2}{2} \]

As expected, in both cases, we get a curve of genus \( \frac{n-2}{2} \), although the genera \((g_1 \text{ and } g_2)\) of the individual curves that are attached at the node depend on whether \( i \) is even or odd. This will have very important consequences later on.

2.6 The Hodge Bundle and Chern Classes

The Hodge bundle (denoted \( E \)) is a complex vector bundle of rank \( g \) on \( \text{Adm}_g \) [1]. Its Chern classes are called \( \lambda \)-classes. We will not go into very many details about the Hodge bundle or Chern classes beyond recalling certain properties which will be important for our work. For a more complete reference on Chern classes, see [4].
• The Hodge bundle $E^g$ is a complex rank $g$ vector bundle that lives over $\text{Adm}_g$, assigning to each point of $\text{Adm}_g$ a $g$ (complex) dimensional vector space.

• The Chern classes $c_j(E)$ are called lambda classes ($c_j(E) = \lambda_j$). $\lambda_j$ is a characteristic class of codimension $j$.

• $\lambda_0 = 1$ always, and $\lambda_j = 0$ for all $j > g$.

• There is a cousin bundle $E^\nu$, called the dual bundle, whose Chern classes are $c_j(E^\nu) = (-1)^j \lambda_j$.

• When we restrict $E^g$ to boundary divisors, the Hodge bundle will break down in one of two possible ways, again depending on whether $i$ (which still represents the number of points on one twig of the boundary divisor as above) is even or odd. This restriction will give some very useful relations, shown below.

1. If $i$ is even, our cover becomes that defined in section 1.5, and we again define $g_1$ and $g_2$ as we did there. Then, $g = g_1 + g_2 + 1$ and $E^g$ restricted to those covers that break down as described above become

$E^g_{g_1+g_2+1=g} = E^{g_1} \oplus E^{g_2} \oplus O$

where $O$ is the trivial line bundle (with only the 0-th Chern class, which is equal to 1). Using the Whitney formula, we can write the lambda classes on the left in terms of lambda classes on the right, giving the following relation:

\[
\lambda \text{ classes from } E^{g_1} \lambda \text{ classes from } E^{g_2} \\
1+\lambda_1+\lambda_2+\cdots+\lambda_g = (1 + \lambda^L_1 + \lambda^L_2 + \cdots + \lambda^L_{g_1})(1 + \lambda^R_1 + \lambda^R_2 + \cdots + \lambda^R_{g_2})(1)
\]

The superscripts $L$ and $R$ serve to denote on which twig (left or right) of the boundary divisor the corresponding $\lambda$-class is defined, because in general the covers of the two twigs will have different genera and hence will have different $\lambda$-classes. The left twig has genus $g_1$, and so will only have nonzero $\lambda$-classes up to codimension $g_1$, and similarly for right twig. To use the above relation, we simply multiply out the right side, noting that the codimensions (subscripts) add, and equate the term with the term of equivalent codimension on the left side. For example, we would get the following relation between $\lambda_3$ on the Hodge bundle $(E^g)$ restricted to a boundary divisor and the $\lambda$-classes on the Hodge bundles of lower rank $(E^{g_1}, E^{g_2})$ (assuming both $g_1, g_2 \geq 2$)

$\lambda_3 = \lambda_1^L \lambda_2^R + \lambda_2^L \lambda_1^R$
since we must collect all terms with total codimension 3. Also, in this case (i even), note that the maximum value \(g_1 + g_2\) can take is \(g - 1\), and so there are no terms with total codimension \(g\) on the right hand side of the above equation. This implies that \(\lambda_g = 0\) whenever we restrict the Hodge bundle to a boundary divisor with an even number of marks on each twig.

2. If \(i\) is odd, then (see section 1.5), when we restrict \(E^g\) to a divisor with \(i\) marked points, we have the relation \(g = g_1 + g_2\) and the Hodge bundle will break down as \(E^g = E^{g_1} \oplus E^{g_2}\) (with no trivial line bundle this time). This means that we will have the same relation as in the \(i\) even case, but now, \(\max(g_1 + g_2) = g\), so here, \(\lambda_g \neq 0\), but instead

\[
\lambda_g = \lambda_{g_1} \lambda_{g_2}
\]

- One final piece that will be important for us is Mumford’s Relation [8]. Very succinctly, Mumford’s relation states

\[
c_{tot}(E \oplus E') = 1
\]

Put in a more explicit form, this says that

\[
(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots \pm \lambda_g) = 1
\]

Since the only factor on the right hand side is a 1, when we multiply out the left side, the sum of all binomials that have the same total codimension must equal 0. Specifically, this says that

\[
\lambda_g^2 = 0
\]
\[
2\lambda_g \lambda_{g-2} - \lambda_{g-1}^2 = 0
\]
\[
\vdots
\]
\[
2\lambda_2 - \lambda_1^2 = 0
\]

All of the above relations will be very useful for simplifying the computations involved in achieving our main goal, calculating Hodge integrals of the form

\[
\int_{\text{Adm}_g} \prod_{i=1}^{k} \lambda_{i_k}
\]

3 Expressing \(\lambda_1\) in terms of the pullback of boundary divisors

Let \(\pi\) be the function that connects the space of degree 2 admissible covers to the space \(\overline{M}_{0,n}\), as in the following diagram:
\[ \overline{Adm}_g \]
\[
\begin{array}{c}
\pi \\
\downarrow \\
\overline{M}_{0,n}
\end{array}
\]
Note that the degree of this map \( \pi \) is actually 1/2, because when we pullback a point in \( \overline{M}_{0,n} \) using \( \pi^* \), the cover that lives in \( \overline{Adm}_g \) actually has one non-trivial automorphism.

The space \( \overline{Adm}_g \) has associated \( \lambda \)-classes which carry all of the properties given in section 1.6. These \( \lambda \)-classes can be pushed down to the space \( \overline{M}_{0,n} \). The goal of the next theorem is to establish a connection between \( \pi^*(\lambda_1) \) and boundary divisors on \( \overline{M}_{0,n} \). As we will see, this will allow us to calculate Hodge integrals that contain at least one \( \lambda_1 \).

**Theorem 3.1** Let \( \pi_*(\lambda_1) \in \overline{M}_{0,n} \) be the push forward of the first Chern class of \( \mathbb{E}^g \) on \( \overline{Adm}_g \). Then (using the notation for boundary divisors introduced in Section 1.3.1), we can express \( \pi_*(\lambda_1) \) as

\[
\pi_*(\lambda_1) = \frac{n}{2} \sum_{i=2}^{n/2} \alpha_i \Delta_{i|n-i} \quad (n = 6, 8, 10, \ldots)
\]

where

\[
\alpha_i = \begin{cases}
\frac{i(n-i)}{16(n-1)} & \text{if } i = 2, 4, 6, \ldots \\
\frac{(i-1)(n-i-1)}{16(n-1)} & \text{if } i = 3, 5, 7, \ldots
\end{cases}
\]

The proof of this theorem will be rather lengthy.

**Proof:** We know from [5] that, for any \( \lambda_1 \in \overline{Adm}_g \), we can express \( \pi_*(\lambda_1) \) of in terms of symmetrized boundary divisors of \( \overline{M}_{0,n} \):

\[
\pi_*(\lambda_1) = \sum_{i=2}^{n/2} \alpha_i \Delta_{i|n-i} \quad (3.1)
\]

What the above equation says is that the coefficient \( \alpha_i \) does not depend on which \( i \) points we put on one twig of the boundary divisor \( \Delta_{i|n-i} \); every boundary divisor with \( i \) points on one twig will have the same coefficient \( \alpha_i \). So, in order to find the \( \alpha_i \)'s, we need only intersect both sides of this equation with \( n/2 - 1 \) curves \( (C_1, C_2, \ldots, C_{n/2-1}) \) in \( \overline{M}_{0,n} \), which will give a system of \( n/2 - 1 \) equations in \( n/2 - 1 \) unknowns \( (\alpha_2, \ldots, \alpha_{n/2}) \). With a good choice of the curves \( C_j \), the matrix for the system will be very predictable, allowing us to easily calculate the \( \alpha_i \)'s.

To accomplish this, we will choose the curve \( C_j \) to be a chain of \( n-3 \) twigs, each with three special points (marks or nodes) except for the \( j \)th twig, which will have 4 special points (see Figure 21).
Note: Throughout this paper, in diagrams of a chain of twigs such as below, ‘\ldots’ will always indicate an arbitrary number of twigs connected in a similar chain-like fashion, each with exactly 1 marked point.

![Diagram of a chain of twigs](image)

Figure 21: On the curve $C_j$, the $j^{th}$ twig has 4 special points (2 marks and 2 nodes).

**Lemma 3.1** When we intersect both sides of (3.1) with the curves $C_j$ described above, we will get a matrix equation that takes the form

$$
\begin{pmatrix}
\Delta_2|n-2 & \Delta_3|n-3 & \Delta_4|n-4 & \Delta_5|n-5 & \Delta_6|n-6 & \Delta_7|n-7 & \cdots & \Delta_{n/2}|n/2 \\
C_1 & 3 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
C_2 & 0 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\
C_3 & 1 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\
C_4 & 1 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
C_{n-2} & 1 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
C_{n-1} & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_2 \\
\alpha_3 \\
\vdots \\
\alpha_{n-2} \\
\end{pmatrix}
= 
\begin{pmatrix}
1/4 \\
0 \\
\vdots \\
1/4 \\
0 \\
\end{pmatrix}
$$

Once we have established the form of the matrix above, it will be a simple matter to insert our values for $\alpha_i$ and complete the proof by checking the multiplication gives the expected solution vector. However, establishing the form of the large matrix (call it $A$) will require some work.

The $(C_j, \Delta_{i|n-i})$ entry of $A$ is just the number of points in the intersection $(C_j) \cdot (\Delta_{i|n-i})$, and so we will prove that $A$ takes the form above by showing every point in this intersection. However, we must remember that $\Delta_{i|n-i}$ represents all boundary divisors with any combination of $i$ points on one twig, and so, when calculating these intersections, we must find the intersection of $C_j$ with every boundary divisor with $i$ points on one twig; fortunately, the vast majority of these intersections will be 0.

The matrix $A$ above is very regular, with almost every row having a 1 as the first element, a 2 on the diagonal, −1’s on either side of the diagonal, and 0’s everywhere else. This holds true for every row except the first, second, and last, and so we must consider these cases separately.

**Row 1** In the first row, the first element $A_{11} = 3$ and the second element $A_{12} = -1$. All other elements $A_{ij} = 0$. 

22
• $A_{11} = (C_1) \cdot (\Delta_{2|n-2}) = 3$

One point of the intersection is shown in Figure 22, which happens when $p_3$ "crashes" into the node and a new twig is sprouted. Of course, there is democracy among the points $p_1$, $p_2$, and $p_3$, so we could also have $p_1$ or $p_2$ move to the newly sprouted twig instead of $p_3$, which gives two more points of intersection. Therefore, $A_{11} = 3$.

![Figure 22: One point of the intersection $C_1 \cdot \Delta_{2|n-2}$](image)

• $A_{12} = (C_1) \cdot (\Delta_{3|n-3}) = -1$

The only boundary divisor which gives a nonzero intersection is shown in Figure 23. In this case, the boundary divisor actually contains $C_1$, and so, as described earlier, the intersection will be the sum of the $\psi$ classes at the marked node (labeled $\star$). The first twig gives a $\psi$ class on $\overline{M}_{0,4}$, which is -1 point (shown in Figure 23), and the second twig gives a $\psi$ class on $\overline{M}_{0,3}$, which is zero, giving $A_{12} = -1$.

![Figure 23: The intersection $C_1 \cdot \Delta_{3|n-3}$](image)

• $A_{1j} = (C_1) \cdot (\Delta_{j+1|n-(j+1)}) = 0$ for $j \neq 1, 2$

This case is similar to $A_{12}$ above in that the boundary divisor still contains $C_1$, but now, our marked node becomes the node between the $j^{th}$ and $(j + 1)^{th}$ marked points (see Figure 24). Since every twig after the first twig has exactly three marked points, the $\psi$ classes will all be zero, making $A_{1j} = 0$.

Row 2  In the second row, $A_{21} = 0$, $A_{22} = 2$, $A_{23} = -1$, and all other elements $A_{2j} = 0$.  

23
\[ A_{21} = (C_2) \cdot (\Delta_{2|n-2}) = 0 \]

The \( A_{21} \) term is zero because of one transversal point of intersection (+1), and one \( \psi \) class intersection (-1), which sum to give 0. The node at which the \( \psi \) classes are calculated is again denoted by \( \star \).

\[ \begin{pmatrix} \bullet & p_3 & p_4 & \cdots & p_{n-1} & p_n \\ p_1 & p_2 & p_5 & \cdots & p_{n-2} & p_{n-1} \end{pmatrix} \]

\[ \begin{pmatrix} \bullet & p_{j+1} & \cdots & p_{n-1} & p_n \\ p_1 & p_2 & p_3 & \cdots & p_{n-2} & p_{n-1} \end{pmatrix} \]

Figure 24: The intersection \( C_1.\Delta_{j+1|n-(j+1)} \)

\[ A_{22} = (C_2) \cdot (\Delta_{3|n-3}) = 2 \]

The intersection of \( C_2 \) with the boundary divisor \( \Delta_{(p_1,p_2,p_3)}|(p_4..p_n) \) is shown in Figure 26. But, in looking at the curve \( C_2 \), it is clear that \( p_3 \) and \( p_4 \) are on an equal footing, and so choosing the boundary divisor \( \Delta_{(p_1,p_3,p_4)}|(p_5..p_n) \) will give another point of intersection exactly the same as below, only with \( p_3 \) and \( p_4 \) swapped, making \( A_{22} = 2 \).

\[ \begin{pmatrix} p_3 & \cdots & p_{n-1} & p_n \\ p_1 & p_2 & p_5 & \cdots & p_{n-2} & p_{n-1} \end{pmatrix} \]

\[ \begin{pmatrix} p_4 & \cdots & p_{n-1} & p_n \\ p_1 & p_2 & p_3 & \cdots & p_{n-2} & p_{n-1} \end{pmatrix} \]

Figure 25: The intersection \( C_2.\Delta_{2|n-2} \) sums to 0.

\[ \begin{pmatrix} p_3 \ast \bullet & \cdots & p_{n-1} & p_n \\ p_1 & p_2 & p_5 & \cdots & p_{n-2} & p_{n-1} \end{pmatrix} \]

\[ \begin{pmatrix} \psi \text{-class} \end{pmatrix} \]

Figure 26: The intersection of \( C_2 \) with the boundary divisor \( \Delta_{(p_1,p_2,p_3)}|(p_4..p_n) \).

\( C_2 \) will intersect with the boundary divisor \( \Delta_{(p_1,p_2,p_4)}|(p_3,p_5..p_n) \) in a similar fashion.
\[ A_{23} = (C_2) \cdot (\Delta_{3|n-3}) = -1 \]

Again, \( C_2 \) is contained in the boundary divisor shown in Figure 27, and so our intersection will be a \( \psi \) class at the node between \( p_4 \) and \( p_5 \) (marked \( \star \)). Just as before, one twig is a copy of \( \bar{M}_{0,4} \), giving \( \psi = -1 \), and the other is a copy of \( \bar{M}_{0,3} \), giving \( \psi = 0 \). Therefore, \( A_{23} = -1 \).

\[
\begin{pmatrix}
\bullet
\vdots
p_3 & p_4 & \cdots & p_{n-1} & \bullet
\end{pmatrix}
- \begin{pmatrix}
p_2 & p_3 & p_4 & \cdots & p_{n-2} & p_n
\end{pmatrix} =
\begin{pmatrix}
p_1 & p_2 & p_3 & p_4 & \cdots & p_{n-1} & p_n
\end{pmatrix}
\]

Figure 27: The intersection \( C_2 \cdot \Delta_{3|n-3} \)

\[ A_{2j} = (C_2) \cdot (\Delta_{j+1|n-(j+1)}) = 0 \text{ for } j \neq 1, 2, 3 \]

This is similar to the case \( A_{1j} \) discussed in the row above. The curve \( C_2 \) will be contained in the boundary divisor \( \Delta_{(p_1, p_2)|(p_{j+1}, p_n)} \), and so we will once again need to calculate the \( \psi \) classes at the node between the \( j^{th} \) and \( (j+1)^{th} \) points. However, for all \( j > 3 \), this node will attach two copies of \( \bar{M}_{0,3} \), which means all \( \psi \) classes will be 0. (The picture is exactly the same as Figure 24, except on \( C_2 \), \( p_3 \) is on the second twig instead of the first).

**Rows \( C_3 \ldots C_{n/2-2} \)** These rows will all take exactly the same form: \( A_{k1} = 1 \) (the first element), \( A_{k(k-1)} = \psi_{(k)} = 0 \) (the elements on either side of the diagonal), \( A_{kk} = 0 \) (the diagonal). All other elements \( A_{kj} = 0 \).

\[ A_{k1} = (C_k) \cdot (\Delta_{2|n-2}) = 1 \]

The one transversal point of intersection is between \( C_k \) and the boundary divisor \( \Delta_{(p_1, p_2)|(p_3, p_n)} \) as shown in Figure 28. It arises from the points \( p_k \) and \( p_{k+1} \) coming together and sprouting a new twig. This situation only arises when intersecting \( C_k \) with boundary divisors of the form \( \Delta_{2|n-2} \). It cannot happen if there are more than 2 marks on a Twig of the boundary divisor because we have at most two marks on any twig of \( C_k \).

\[ A_{kj} = (C_k) \cdot (\Delta_{j+1|n-(j+1)}) = 0 \text{ for } j = 2, 3, \ldots (k - 2) \]

We will first fix \( k \) from \( k = 3, 4, \ldots (\frac{n}{2} - 2) \) and study what happens as \( j \) varies from \( j = 2 \) to \( j = k - 2 \). Similarly to the case \( A_{1j} \) discussed above, the curve \( C_k \) will be contained in \( \Delta_{(p_1, \ldots p_{j+1}):(p_{j+2}, \ldots p_n)} \), and so the intersection will be a \( \psi \) class at the node between the \( (j + 1)^{th} \) and \( (j + 2)^{th} \) points. But, recall that the \( (j + 1)^{th} \) and \( (j + 2)^{th} \) points are on the \( j^{th} \) and \( (j + 1)^{th} \) twigs, respectively. This means
that $2 \leq j \leq k - 2$ and $3 \leq j + 1 \leq k - 1$, and so each twig we are considering will be a copy of $M_{0,3}$ (because the only twig on $C_k$ that is NOT a copy of $M_{0,3}$ is the $k^{th}$ twig - see Figure 29). Since the $\psi$ classes on $M_{0,3}$ are 0, $A_{kj} = 0$.

$A_{(k)(k-1)} = (C_k).\Delta_{k\mid n-k-1} = -1$

The argument follows exactly as in the case above, except now $j = k - 1$, so we calculate the $\psi$ class at the node between the $(k-1)^{th}$ and $k^{th}$ twigs. Recalling that the $k^{th}$ twig contains an extra point, it is a copy of $M_{0,4}$, and so will give a contribution of -1 to the intersection. The $(k-1)^{th}$ twig is still $M_{0,3}$ and so the corresponding $\psi$ class will still be 0, giving $A_{(k)(k-1)} = -1$.

$A_{kk} = (C_k).\Delta_{k+1\mid n-(k+1)} = 2$

Here, the curve $C_k$ is actually transversal to the relevant boundary divisors, and so we will not need to calculate $\psi$ classes. The intersection of $C_k$ with the boundary divisor $\Delta_{(p_1\ldots p_{k+1})(p_{k+2}\ldots p_n)}$ is shown in Figure 31. Notice again that $p_{k+1}$ and $p_{k+2}$ are on equal footing, and so we will have a second, similar transversal intersection with the boundary divisor that has $p_{k+1}$ and $p_{k+2}$ switched ($\Delta_{(p_1\ldots p_k p_{k+2})(p_{k+1} p_{k+3}\ldots p_n)}$), giving $A_{kk} = 2$. 

Figure 28: The intersection $C_k \Delta_{2\mid n-2}$

Figure 29: The curve $C_k$ with the $j^{th}$ and $k^{th}$ twigs pointed out. The $\star$ represents where to calculate the $\psi$ classes for the intersection. Note that all twigs are copies of $M_{0,3}$ except for the $k^{th}$ twig, and so for any $2 \leq j \leq k - 2$, the intersection will be 0.
Figures 30 and 31: The * represents the node where the ψ class is calculated when \( j = k - 1 \).

- \( A_{(k)(k+1)} = -1 \)
  This is exactly the same as the case \( j = k - 1 \) (\( A_{(k)(k-1)} \)) above, but instead of calculating the ψ class at the node between the \((k-1)\)th and \(k\)th twigs, we now must use the node between the \(k\)th and \((k+1)\)th twigs. Again, the \(k\)th twig is a copy of \( \bar{M}_{0,4} \) whose ψ class gives a contribution of -1, and the \((k+1)\)th twig is \( \bar{M}_{0,3} \), for which all ψ classes are 0. Therefore, \( A_{(k)(k+1)} = -1 \).

- \( A_{kj} = 0 \) for \( k + 2 \leq j \leq n/2 - 1 \)
  This is similar to the case \( 2 \leq j \leq k - 2 \) above, except now we have passed the \(k\)th twig. But, since all twigs beyond the \(k\)th twig also have only 3 special points, all of the ψ classes will be 0, just as described above.

Row \( C_{n/2-1} \) (the bottom row) The bottom row takes exactly the same form as the general row described above with the one exception that \( A_{(n/2-1)(n/2-2)} = -2 \). The arguments for every other entry follow exactly as above, and so we will only discuss the \( j = n/2 - 2 \) case.

- \( A_{(n/2-1)(n/2-2)} = (C_{n/2-1}):(\Delta_{n/2}|n/2) = -2 \)
The picture for this situation is the same as in Figure 30, but now $k = \frac{n}{2} - 1$, and so the special twig with 2 marked points lies exactly in the middle of the chain (there are $\frac{n}{2} - 2$ twigs to the left of it and $(n - 3) - (\frac{n}{2} - 1) = \frac{n}{2} - 2$ twigs to the right). By the symmetry of the situation, we can intersect with (1) the boundary divisor that has the first $\frac{n}{2} - 1$ points ($p_1, p_2, ..., p_{\frac{n}{2}-1}$) on one twig and (2) the boundary divisor that has the last $\frac{n}{2} - 1$ points ($p_{\frac{n}{2}+1}, ..., p_n$) on one twig. Each will give a $\psi$ class on $M_{0,4}$, giving a total of -2.

Putting all of the above together will yield the matrix $A$ in equation (3.2) above.

Now that we have proven the form of the matrix $A$, we must prove that intersecting $\pi^*(\lambda_1)$ with each of the curves $C_j$ gives us the solution vector $[1/4 \ 0 \ 1/4 \ 0 \cdots]^T$.

To do so, we first use the projection formula, which allows us to to write the intersection we are looking for (which is on $M_{0,n}$) in terms of an intersection on $\overline{Adm}_g$, which we know how to handle:

$$\int_{C_j} \pi_*(\lambda_1) = \int_{\pi^*(C_j)} \lambda_1 \quad (3.3)$$

Now, the RHS of equation (3.3) boils down to finding the first Chern class of the Hodge bundle, restricted to covers lying over points parametrized by $C_j$.

Now, recall that a boundary component in $M_{0,n}$ is a product of copies of $\overline{M}_{0,m}$, where $m < n$. The same thing holds true in the space $\overline{Adm}_g$, where a genus $g$ cover on the boundary is formed by gluing together covers of smaller genus at the nodes. This is exactly what happens in Eq. 3.3 above, where $C_j \in M_{0,n}$ can be written as the product

$$C_j = \overline{M}_{m_1} \times \overline{M}_{m_2} \times \cdots \times \overline{M}_{m_{n-3}}$$

and $\pi^*(C_j) \in \overline{Adm}_g$ can be written as the product

$$\pi^*(C_j) = \overline{Adm}_{g_1} \times \overline{Adm}_{g_2} \times \cdots \times \overline{Adm}_{g_{n-3}}$$

Now, recall section 2.5, where we discussed how to glue together the individual covers sitting over each of the twigs. There, we noted that there are two possibilities for the gluing: (1) the node is a branch point, in which case, the two covers are glued at exactly one point, or (2) the node is not a branch point, in which case the adjacent covers are glued at exactly 2 points.

Now, all of the twigs in $C_j$ have 3 special points (marks or nodes) except for the $j^{th}$ twig. Hence, $C_j$ is a product of irreducible components, all of which are zero dimensional (3 special points) except for the $j^{th}$ component. This means that the cover of every twig except twig $j$ will be of dimension 0 and have genus 0, since at least 4 special points are required for $g > 0$. The $j^{th}$ twig, however, has dimension 1, while the genus of the cover of the $j^{th}$ twig can be 0 or 1, depending on what $j$ is. This happens because the nodes on the $j^{th}$ twig can
either be both branch points or both not branch points, a fact which depends on the number of marks on all previous twigs.

If \( j \) is odd, then, both nodes are branch points and the cover of the \( j \)th twig has 4 ramification points (\( g = 1 \)), meaning \( \pi^*(C_j) \) from above can be written as

\[
\prod_{1}^{n-4} \overline{Adm}_{0,*} \times \overline{Adm}_1
\]  

(3.4)

If \( j \) is even, on the other hand, then neither of the nodes on the \( j \)th twig are branch points, and so the \( j \)th twig has only 2 ramification points (\( g = 0 \)), meaning that \( \pi^*(C_j) \) becomes

\[
\prod_{1}^{n-3} \overline{Adm}_{0,*}
\]  

(3.5)

Now, we can use an analogue of Fubini’s Theorem from calculus, which states that

\[
\int_{X \times Y} f(x)g(y)dxdy = \int_X f(x)dx \int_Y g(y)dy
\]

In our specific case, we can rewrite the RHS of 3.3 as

\[
\int_{\pi^*(C_j)} \lambda_1 = \int_{\overline{Adm}_{g_1} \times \overline{Adm}_{g_2} \times \cdots \times \overline{Adm}_{g_{n-3}}} \lambda_1 1^{n-4} \overline{Adm}_{g_j}
\]

where the superscript \( \overline{Adm}_{g_j} \) denotes that that particular \( \lambda_1 \) is defined on the space \( \overline{Adm}_{g_j} \). Then, using Fubini’s Theorem, the above reduces to

\[
\int_{\overline{Adm}_{g_1}} 1 \cdots \int_{\overline{Adm}_{g_{j-1}}} 1 1 \int_{\overline{Adm}_{g_j}} \lambda_1 \int_{\overline{Adm}_{g_{j+1}}} 1 \cdots \int_{\overline{Adm}_{g_{n-3}}} 1
\]

(3.6)

where the \( \overline{Adm}_{g_1}, \overline{Adm}_{g_2}, \cdots, \overline{Adm}_{g_{n-3}} \) are those from (3.4) if \( j \) is odd or (3.5) if \( j \) is even.

(For our specific case, almost all of the integrals are of the class 1 because every space except for the \( j \)th is 0 dimensional, since they have only 3 special points. The \( j \)th space has 4 special points, and so it is 1-dimensional, meaning we must integrate \( \lambda_1 \) on \( \overline{Adm}_{g_j} \).)

Now, all that is left is to calculate each of the individual Hodge integrals in (3.6). Actually calculating the integrals on the 0-dimensional spaces gives a result of 1/2 from the automorphisms of the cover, which is exactly canceled by the factor of 2 which arises from the degree of the map \( \pi \) being 1/2, making almost every integral equal to 1.

The only integral that is not on a 0-dimensional space is the \( j \)th integral, which is on a 1-dimensional space. This space can have either genus 0 (\( j \) even)
or genus 1 \((j \text{ odd})\) as explained above, meaning that there are 2 integrals we must know to complete our calculation:

\[
\int_{Adm_0} \lambda_1 \quad \text{and} \quad \int_{Adm_1} \lambda_1
\]

Clearly, if \(g = 0\), then \(\lambda_1 = 0\) on this space, and the left integral above is 0. Also, it is shown in \([2]\) that

\[
\int_{Adm_1} \lambda_1 = 1/4 \quad (3.7)
\]

Therefore, every integral produces a factor of 1, except for the \(j^{th}\) integral, which depends on whether \(j\) is even or odd:

\[
\int_{Adm_j} \lambda_1 = \begin{cases} \int_{Adm_1} \lambda_1 = 1/4, & j \text{ odd} \\ \int_{Adm_0} \lambda_1 = 0, & j \text{ even} \end{cases} \quad (3.8)
\]

which gives exactly the solution vector \([1/4 \ 0 \ 1/4 \ 0 \ \cdots]^T\).

This establishes the form of the linear system given in (3.2). Now, in order to prove Theorem (2.1), we need only insert the proposed values of \(\alpha_i\) into our matrix equation, and check that we get the expected solution vector. Again, the first, second, and last row must be treated separately.

- **The first row**
  Plugging in the numbers, we get
  \[
  3\alpha_2 - 1\alpha_3 = 3 \frac{2(n-2)}{16(n-1)} - \frac{2(n-4)}{16(n-1)} = \frac{4(n-1)}{16(n-1)} = \frac{1}{4}
  \]

- **The second row**
  \[
  2\alpha_3 - 1\alpha_4 = 2 \frac{2(n-4)}{16(n-1)} - \frac{4(n-4)}{16(n-1)} = 0
  \]

- **The last row \((n/2 - 1 \text{ odd})\)**
  \[
  \alpha_2 - 2\alpha_{n/2-1} + 2\alpha_{n/2} = \frac{2(n-2)}{16(n-1)} - 2 \frac{(\frac{n}{2} - 1)(n - (\frac{n}{2} - 1) - 1)}{16(n-1)} + 2 \frac{(\frac{n}{2})(n - \frac{n}{2})}{16(n-1)} = 1/4
  \]

- **The last row \((n/2 - 1 \text{ even})\)**
  \[
  \alpha_2 - 2\alpha_{n/2-1} + 2\alpha_{n/2} = \frac{2(n-2)}{16(n-1)} - 2 \frac{(\frac{n}{2} - 1)(n - (\frac{n}{2} - 1))}{16(n-1)} + 2 \frac{(\frac{n}{2})(n - \frac{n}{2} - 1)}{16(n-1)} = 0
  \]
• Any other row, \( j \) even

\[
\alpha_2 - \alpha_j + 2\alpha_{j+1} - \alpha_{j+2} = \frac{2(n-2)}{16(n-1)} - \frac{j(n-j)}{16(n-1)} + 2\frac{j(n-j-2)}{16(n-1)} - \frac{(j+2)(n-j-2)}{16(n-1)}
\]

\[
= 0
\]

• Any other row, \( j \) odd

\[
\alpha_2 - \alpha_j + 2\alpha_{j+1} - \alpha_{j+2} = \frac{2(n-2)}{16(n-1)} - \frac{(j-1)(n-j-1)}{16(n-1)} + 2\frac{(j+1)(n-j-1)}{16(n-1)} - \frac{(j+1)(n-j-3)}{16(n-1)}
\]

\[
= \frac{1}{4}
\]

This completes the proof of Theorem (3.1).

3.1 A more geometrical expression for \( \alpha_i \)

We have proved that we can express \( \pi_*(\lambda_1) \) in terms of a sum of boundary divisors \( \Delta_{i|n-i} \) together with coefficients \( \alpha_i \). The expression that we have for \( \alpha_i \) is

\[
\alpha_i = \frac{i(n-i)}{16(n-1)} \quad i = 2, 4, 6, \ldots 
\]

(3.9)

\[
\alpha_i = \frac{(i-1)(n-i-1)}{16(n-1)} \quad i = 3, 5, 7, \ldots
\]

(3.10)

However, these coefficients can be given a much more elegant geometrical description. Recall section 1.5, where we described how to find the genera \( g_1 \) and \( g_2 \) of the two components of the cover of any boundary divisor. There, we noted that if \( i \) is even, the cover of \( \Delta_{i|n-i} \) has a loop at the node. If \( i \) is odd, then the node is twisted (no loop). If we define \( g_1 \) and \( g_2 \) to be the genera that we get for each half of the cover by attaching the two preimages of the node which form the loop, then, we will have that

\[
g_j = \frac{i}{2} \quad i \text{ even} \\
g_j = \frac{(i-1)}{2} \quad i \text{ odd} \quad (j = 1, 2)
\]

But, again, if \( i \) is even, then \( n-i \) is even, and so the genera of the two halves of the cover will be \( g_1 = i/2, g_2 = (n-i)/2 \). Conversely, if \( i \) is odd, then, the two genera will be \( g_1 = (i-1)/2, g_2 = (n-i-1)/2 \). Now, if we multiply \( g_1g_2 \), we get

\[
g_1g_2 = \frac{i(n-i)}{4} \quad (i \text{ even}) \\
g_1g_2 = \frac{(i-1)(n-i-1)}{4} \quad (i \text{ odd})
\]

(3.11)
We can substitute these expressions in the above formulas for \( \alpha_i \), which allows us to write one completely general expression for \( \pi_*(\lambda_1) \):

\[
\pi_*(\lambda_1) = \frac{1}{2} \frac{1}{4(2g + 1)} \sum_{i=2}^{n-1} g_1 g_2 \Delta_{i|n-i} \tag{3.12}
\]

where \( g_1, g_2 \) refer to the genera of the two components of the cover of \( \Delta_{i|n-i} \). Also, we have replaced \( n-1 \) with \( 2g + 1 \) and let the sum run to \( n-1 \) instead of \( n/2 \), which we can do since the \( \Delta \)'s are symmetric (\( \Delta_{i|n-i} = \Delta_{n-i|i} \)) and we have placed a factor of \( 1/2 \) in front of the entire expression. The formula (3.12) will be the most useful form of the theorem in the following sections.

4 Computing Hodge Integrals

The Hodge integrals we are studying (i.e., those that are restricted to the space \( \overline{\text{Adm}}_g \)) take the form

\[
\int_{\overline{\text{Adm}}_g} \prod_{i=1}^k \lambda_{i_k}
\]

where all \( i_k \leq g \) and \( \sum i_k = 2g - 1 \). With the help of Theorem 3.1, we can prove the following:

**Theorem 4.1** All Hodge integrals in which at least one of the \( i_k \)'s is equal to 1 can be reduced to a product of Hodge integrals on spaces of lower genus.

**Proof.** We once again start with the initial condition \( \int \lambda_1 = 1/4 \). Then, for any other integral with at least one \( \lambda_1 \), we can write

\[
\pi_*(\lambda_1) = \sum_{i=2}^{n/2} \alpha_i \Delta_{i|n-i} \tag{4.1}
\]

where the \( \alpha_i \)'s are known from Theorem 3.1. Because \( \pi \) is a degree 1/2 map, we also know that if we push \( \lambda_1 \) forward and then pull it back, we get

\[
\lambda_1 = 2\pi^* \pi_*(\lambda_1) \tag{4.2}
\]

Combining these two equations, we can replace one of the \( \lambda_1 \)'s in our Hodge integral with the expression

\[
\lambda_1 = 2\pi^* \left( \sum \alpha_i \Delta_{i|n-i} \right) \tag{4.3}
\]

Now, all that remains is to restrict the Hodge bundle to the pullback of each of the boundary divisors represented by \( \Delta_{i|n-i} \), which will be made up of two components of lower genus, leaving Hodge integrals on components of lower genera. ■
While we can reduce any Hodge integral that contains one \( \lambda \) to a product of Hodge integrals on spaces of lower genus, this does not necessarily mean we will know how to compute these new integrals. However, for a few specific forms of Hodge integrals, we do know how to do this, and we can actually find a recursion which can be used to calculate any Hodge integral of those forms. Before proving these recursions, however, we first give a concrete example to illustrate the strategy used in the proof above.

### 4.1 An example: \( \int \lambda_2 \lambda_1 \)

To illustrate this strategy, we will first do a concrete example and compute the integral \( \int \lambda_2 \lambda_1 \). It is well known that

\[
\sum \left( \int \lambda_g \lambda_{g-1} \right) \frac{x^{2g-1}}{(2g-1)!} = \frac{1}{2} \tan \left( \frac{x}{2} \right)
\]

is a generating function for Hodge integrals of the form \( \int \lambda_g \lambda_{g-1} \). In particular, from this formula, we know that \( \lambda_2 \lambda_1 = 1/8 \), which will serve as a check for our new method of computation.

To proceed, note that, in \( g = 2 \),

\[
\pi_*(\lambda_1) = \frac{1}{10} \Delta_{2|4} + \frac{1}{20} \Delta_{3|3}
\]

Then, Eq. (4.3) above becomes

\[
\lambda_1 = 2 \pi^* \pi_*(\lambda_1) = 2 \pi^* \left( \frac{1}{10} \Delta_{2|4} + \frac{1}{20} \Delta_{3|3} \right)
\]

(4.4)

Now, replacing \( \lambda_1 \) in \( \int \lambda_2 \lambda_1 \) with Eq. (4.4):

\[
\int \lambda_2 \lambda_1 = 2 \int \lambda_2 \pi^* \left( \frac{1}{10} \Delta_{2|4} + \frac{1}{20} \Delta_{3|3} \right)
\]

\[
= 2 \int \lambda_2 \pi^* \left( \frac{1}{10} \Delta_{2|4} \right) + 2 \int \lambda_2 \pi^* \left( \frac{1}{20} \Delta_{3|3} \right)
\]

\[
= \frac{2}{10} \int \lambda_2 \pi^* (\Delta_{2|4}) + \frac{2}{20} \int \lambda_2 \pi^* (\Delta_{3|3})
\]

(4.5)

But now, \( \pi^*(\Delta_{j|k}) \) is just the pullback of those strata represented by \( \Delta_{j|k} \), and so, we calculate this intersection just as before: by restricting the Hodge bundle to the boundary over \( \Delta_{j|k} \) and finding \( \lambda_2 \) on it.

In the first term, \( \mathbb{E}^2 \) restricted to \( \Delta_{2|4} \) will be \( \mathbb{E}^2_{\Delta_{2|4}} = \mathcal{O} \oplus \mathbb{E}^1 \). Using the Whitney formula, we get

\[
(1 + \lambda_1 + \lambda_2) = (1)(1 + \lambda_0^R)
\]

where the \( R \) denotes the cover of the right-hand twig (here, there is no \( L \) because the left-hand twig becomes an \( \mathcal{O} \), and so, the only non-zero Chern class is \( \lambda_0 = 1 \)).
It is clear from the formula above that $\lambda_2 = 0$, and so, in (3.3) $\int \lambda_2 \pi^*(\Delta_{2|4}) = 0$.

Now, the second term in (3.3) is found by restricting $E^2$ to $\Delta_{3|3}$, which is $E^2_{3|3} = E^1 \oplus E^1$. This time, the Whitney formula gives

$$(1 + \lambda_1 + \lambda_2) = (1 + \lambda_1^L)(1 + \lambda_1^R)$$

from which, we get that

$$\lambda_2 = \lambda_1^L \lambda_1^R$$  \hspace{1cm} (4.6)

This means that (by Fubini’s Theorem)

$$\int \lambda_2 \pi^*(\Delta_{2|4}) = \int_{\mathcal{M}_{0,n}} \lambda_1^L \int_{\mathcal{M}_{0,n}} \lambda_1^R$$

Both factors on the RHS of this equation are exactly the same, and we know from before that both equal 1/4.

Now, this is not entirely correct, because $\Delta_{3|3}$ represents the sum of all $\binom{6}{3}/2$ boundary divisors with 3 points on each twig. Each of these boundary divisors will contribute in exactly the same way, and so, we must multiply the above example by $\binom{6}{3}/2 = 10$. Also, because of the nuances of the degree $1/2$ map between $\mathcal{M}_{0,n}$ and our space of admissible covers, we must multiply by another factor of 2 for the 2 ways that we can glue the node of the cover. Adding all of these factors in, and recalling the factor of 2/20 from (3.3), we get that

$$\int \lambda_2 \lambda_1 = 0 + \frac{2}{20} \times \frac{1}{4} \times \frac{1}{4} \times 10 \times 2 = \frac{1}{8}$$

which is the answer we expected from the generating function above.

### 4.2 Integrals of the form $\int \lambda_g \lambda_{g-i}(\lambda_1)^{i-1}$

We know from Theorem 4.1 that we can reduce any Hodge integral that contains at least one $\lambda_1$. However, for certain forms of Hodge integrals, the reduced form will always be something we know how to compute. In particular, we can use our method to prove the following recursive relationship:

**Theorem 4.2** Let $L^g_i = \int \lambda_g \lambda_{g-i}(\lambda_1)^{i-1}$. Then,

$$L^g_i = \frac{1}{2g + 1} \sum_{g_1=1}^{g-1} \sum_{i_1=1}^{g_1-1} g_1 g_2 \left(\frac{2g + 2}{2g_1 + 1}\right) \binom{i-2}{i_1-1} L^{g_1}_{i_1} L^{g_2}_{i_2}$$

where $g_2 = g - g_1, i_2 = i - i_1$

The initial condition $L^g_1$ is known from the generating function $(1/2)\tan(x/2)$, and so, we can compute $L^g_i$ for all $g, i$.

**Proof.** First, note that when computing Hodge integrals with a $\lambda_g$ term, only boundary divisors with no loops (i.e., $\Delta_{g|n-i}$ where $i$ is odd) will contribute, since, if there is a loop, $\lambda_g = 0$ (see section 1.5).
Next, use the same technique as in Theorem 4.1, pushing forward and pulling back one of the \( \lambda_1 \) terms, to get

\[
L^g_{i} = 2\int \lambda_g \lambda_{g-i} (\lambda_1)^{i-2} \pi^* \left( \frac{1}{2} \frac{1}{4(2g+1)} \sum g_1 g_2 \Delta_{i|n-i} \right)
\]

But, as we said above, only the terms with no loops will contribute, and so, we only need to consider terms where \( i \) is odd, which means that \( g_2 = g - g_1 \).

So, when we restrict the Hodge bundle to each term that contributes, it will break down as \( E^g = E^{g_1} \oplus E^{g_2} \), where \( g_1 = 1, 2, ... (g-1) \).

Let \( L \) denote the cover of the left twig, with genus \( g_1 \), and let \( R \) denote the cover of the right twig, with genus \( g_2 \). The Whitney formula then says

\[
(1 + \lambda_1 + \cdots + \lambda_g) = (1 + \lambda_1^L + \cdots + \lambda_{g_1}^L)(1 + \lambda_1^R + \cdots + \lambda_{g_2}^R)
\]

(4.7)

Clearly, since \( g = g_1 + g_2 \),

\[
\lambda_g = \lambda_{g_1}^L \lambda_{g_2}^R
\]

Next, we must find \( \lambda_{g-i} \). From (4.7), \( \lambda_{g-i} \) will be the sum of all binomials \( \lambda_j^L \lambda_k^R \) where \( j + k = g - i \):

\[
\lambda_{g-i} = \sum_{j=1}^{g_1-1} \lambda_j^L \lambda_{g-i-j}^R
\]

(Note that \( j = g_1 \) will not work because then, our codimensions on the \( \lambda_j^L \) terms will sum to \( 2g_1 \), which is greater than \( 2g_1 - 1 \). A similar thing happens with the \( \lambda^R \) terms when \( j = 0 \)).

To find \( (\lambda_1)^{i-2} \), note that \( \lambda_1 = \lambda_1^L + \lambda_1^R \) (from (4.7)). Then, we must raise both sides to the power \( i - 2 \), which, from the Binomial Theorem, gives

\[
(\lambda_1)^{i-2} = \sum_{k=0}^{i-2} \binom{i-2}{k} (\lambda_1^L)^k (\lambda_1^R)^{i-2-k}
\]

Now, we must multiply together all of our expressions, to get

\[
\lambda_g \lambda_{g-i} (\lambda_1)^{i-2} = \lambda_{g_1}^L \lambda_{g_2}^R \sum_{j=1}^{g_1-1} \lambda_j^L \lambda_{g-i-j}^R \sum_{k=0}^{i-2} \binom{i-2}{k} (\lambda_1^L)^k (\lambda_1^R)^{i-2-k}
\]

When we multiply out the above, every term will contain \( \lambda_{g_1}^L \lambda_{g_2}^R \). However, the only nonzero terms in the expansion will be the terms in which the codimensions of each part of the cover (i.e., the subscripts) add correctly. That is, the subscripts for the \( \lambda^L \) terms must add to \( 2g_1 - 1 \), and similarly for the \( \lambda^R \) terms. However, we know that each term contains \( \lambda_{g_1}^L \), and so, this implies that the only nonzero terms will be those for which \( j + k = g_1 - 1 \), subject to the constraints that \( 0 \leq k \leq i-2 \) and \( 1 \leq j \leq g_1 - 1 \) (the corresponding constraint for the \( \lambda^R \) terms will be satisfied automatically).
After doing all of this, what we are left computing is

\[
\sum \int_{Adm_{g_1}} \lambda^L_{g_1} \lambda^R_j (\lambda^L_j)^{g_1-j-1} \int_{Adm_{g_2}} \lambda^R_{g_2} \lambda^R_{g_2-i-j} (\lambda^1)_{g_2-g+i+j-1} = \sum_{j=1}^{g_1-1} L_{g_1-j}^{g_1} L_{g_2-g+i+j}^{g_2}
\]

Now, just as in the \( \lambda_2 \lambda_1 \) example above, we must multiply by \( (2g+2)_{g_1+1} \) to account for all possible ways to distribute the marks on the boundary divisor. After doing this, and summing over all possible values of \( g_1 \), we get as a final answer

\[
L_i^g = \frac{1}{2g+1} \sum_{g_1} \sum_{g_2} g_1 g_2 \left( \frac{2g+2}{2g_1+1} \right) \left( i - 2 \right) \left( i_1 - 1 \right) L_{i_1}^{g_1} L_{i_2}^{g_2}
\]

where we have relabeled the indices \( (i_1 = g_1 - j, i_2 = g_2 - g + i + j) \) to make the equation more symmetric in \( g_1 \) and \( g_2 \) (also, note that, from this relabeling, \( i = i_1 + i_2 \)).

As was noted before, \((1/2) \tan(x/2)\) is a generating function for Hodge integrals of the form \( \lambda_3 \lambda_{g-1} \). We can take this slightly farther and define the following function \( u \) to be a generating function for all numbers \( L_i^g \):

\[
u(x, q) = \sum L_i^g \frac{x^{2g} q^{i-1}}{2g! (i-1)!}
\]

Starting with the recursion to find any \( L_i^g \) from Theorem 4.2:

\[
L_i^g = \frac{1}{2g+1} \sum_{g_1+g_2=g} \sum_{i_1+i_2=i} g_1 g_2 \left( \frac{2g+2}{2g_1+1} \right) \left( i - 2 \right) \left( i_1 - 1 \right) L_{i_1}^{g_1} L_{i_2}^{g_2}
\]

Then, expanding the binomial coefficients, we can rearrange (4.8) as

\[
\frac{8L_i^g}{(2g+2)(2g)!(i-2)!} = \sum_{g_1+g_2=g} \sum_{i_1+i_2=i} \frac{L_{i_1}^{g_1}}{(2g_1+1)(2g_1-1)!(i_1-1)!} \frac{L_{i_2}^{g_2}}{(2g_2+1)(2g_2-1)!(i_2-1)!}
\]

Then, take the function \( u(x, q) \) as defined above, differentiate w.r.t. \( q \), multiply by \( x \), and then integrate w.r.t. \( x \) to get:

\[
\int xu(x, q) = \sum L_i^g \frac{x^{2g+2} q^{i-2}}{(2g+2)(2g)!(i-2)!}
\]

The coefficient of \( x^{2g+2} q^{i-2} \) in the summation is the LHS of (4.9), up to a factor of 8.

As for the RHS of (4.9), we first find \( xu_x \) and then integrate w.r.t. \( x \), and then square:

\[
\left( \int xu_x \right)^2 = \left( \sum L_i^g \frac{x^{2g+1} q^{i-1}}{(2g+1)(2g-1)!(i-1)!} \right) \left( \sum L_i^g \frac{x^{2g+1} q^{i-1}}{(2g+1)(2g-1)!(i-1)!} \right)
\]

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In this equation (after multiplying the two summations), the $x^{2g+2}q^{i-2}$ coefficient will be (once again letting $g_2 = g - g_1$ and $i_2 = i - i_1$)

$$\sum_{g_1 + g_2 = g_{1+} + i_{1+}} \sum_{i_{1+} = i} \frac{L_{g_1}^{g_1}}{(2g_1 + 1)(2g_1 - 1)!} \frac{L_{g_2}^{g_2}}{(2g_2 + 1)(2g_2 - 1)!}$$

But, this is exactly the RHS of (4.9)! So, the recursion we found for computing the numbers $L_g$ can be expressed in the following equivalent ways:

1. $8 \int_x x u_q = (\int_x x u_x)^2$
2. $4u_q = u_x \int_x x u_x$
3. $x u_x^3 + u_q u_{xx} - 4 u_x u_x = 0$

We already know from before that

$$\sum L_g^{x^{2g-1}} = \frac{1}{2} \tan \left( \frac{x}{2} \right)$$

which gives us the initial condition

$$u_x(x, 0) = \sum L_g^{x^{2g-1}} = 1/2 \tan(x/2)$$

However, these differential equations are not easily solvable.

### 4.3 Integrals of the form $\int (\lambda_1)^{2g-1}$

Our final result will be a recursion to calculate top intersections (integrals of the form $\int (\lambda_1)^{2g-1}$) for any $g$.

**Theorem 4.3** Let $K^g = \int (\lambda_1)^{2g-1}$. Then,

$$K^g = \frac{1}{2} \frac{1}{2g + 1} \sum_{g_1 = 1}^{g-1} g_1 g_2 \left( \frac{2g + 2}{2g_1 + 1} \right) \left( \frac{2g - 2}{2g_1 - 1} \right) K^{g_1} K^{g_2}$$

where, again, $g_2 = g - g_1$.

**Proof.** We once again replace one $\lambda_1$ with our known expression $\pi^* \left( \frac{1}{2} \frac{1}{4(2g+1)} \sum g_1 g_2 \Delta_{i|n-i} \right)$ to give

$$K^g = 2 \int (\lambda_1)^{2g-2} \pi^* \left( \frac{1}{2} \frac{1}{4(2g+1)} \sum g_1 g_2 \Delta_{i|n-i} \right)$$

(4.10)

When we restrict the Hodge bundle to the boundary divisors, only the divisors whose covers have no loops will contribute once again, meaning that we can say that $g_2 = g - g_1$ and sum from $g_1 = 1$ to $g_1 = g - 1$. The Hodge bundle will
once again split into two parts, $E^g = E^{g_1} \oplus E^{g_2}$, and, just as above, the Whitney formula tells us that

$$\lambda_1 = \lambda_1^L + \lambda_1^R$$

When we raise the above equation to the power $2g - 2$, the Binomial Theorem says

$$(\lambda_1)^{2g-2} = \sum_{k=0}^{2g-2} \binom{2g-2}{k} (\lambda_1^L)^k (\lambda_1^R)^{2g-2-k}$$

But then, remembering the $L$ part of the cover has genus $g_1$ and the $R$ part has genus $g_2$, we only want the $k = 2g_1 - 1$ term in the above summation (all other terms are zero because the codimensions do not add correctly). That is, the term

$$\binom{2g-2}{2g_1-1} (\lambda_1^L)^{2g_1-1} (\lambda_1^R)^{2g_2-1} = \binom{2g-2}{2g_1-1} K^{g_1} K^{g_2}$$

When we put this into Eq.(4.10) and tack on the usual factor of $\binom{2g+2}{2g_1+1}$ for every possible way to distribute the marked points across a cover of genus $g_1$ and a factor of 2 for the one node and the degree 1/2 map, we get as a final answer

$$K^g = \frac{1}{2} \frac{1}{2g+1} \sum_{g_1=1}^{g-1} g_1 g_2 \binom{2g+2}{2g_1+1} \binom{2g-2}{2g_1-1} K^{g_1} K^{g_2}$$

The initial condition for this recursion is again $K^1 = \int (\lambda_1)^1 = 1/4$. ■

References


