Changes in Betti numbers of Hessenberg varieties on restricted tableaux

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Abstract

It has been observed in all known cases that moving to a less restrictive Hessenberg function on a fixed tableau design will never cause a decrease in the Betti numbers $b_i$. In this paper we prove this observation on certain restricted tableaux.

1 Introduction

In this paper we seek to find a solution to a question posed by Tymoczko regarding the way in which Betti numbers of Hessenberg varieties, a subvariety of the full flag variety, change with a change in the Hessenberg function. The Hessenberg function is a nondecreasing function $h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that $h(i) \geq i$ for all $i$. We seek to prove the observation that on a fixed Young Diagram changing the function $h$ to $g$ where $g(i) \geq h(i)$ for each $i$ results in Betti numbers $b'_i$ such that $b'_i \geq b_i$.

We will achieve this goal on certain restricted tableau which lend themselves to rearrangement of their boxes. In this paper I will follow some notation conventions established by Iveson in [1].

Before we can discuss the Betti numbers we need to take care of some preliminaries. A cell is a particular filling of a Young tableau. Given a Hessenberg function $h$ cell is valid or $h$-allowed if whenever $a \ b$ occurs in a cell $a \leq h(b)$ [1].

![Inversion Diagram](image)

Figure 1: An Inversion Diagram

Definition 1  The number of inversions on a box occupied by $k$ is the number of $i$ such that:

- $i$ is anywhere below or to the left of $k$,
- $k > i$, and


• if there is a box immediately to the right of \( k \) filled by \( j \) then \( i < h(j) \).

Rephrasing Tymoczko’s result in [2] we find

**Theorem 2**  The sum over all boxes in a cell of the number of inversions on each box is the dimension of the cell.

The Betti numbers \( b_i \) are the number of valid fillings of a tableau of dimension \( i \). For the purposes of this paper \( h \) is a Hessenberg function on a Young Diagram, \( h' \) is another Hessenberg function such that \( h'(j) = h(j) + 1 \) for exactly one \( j \) and \( h'(i) = h(i) \) for all \( i \neq j \).

**Lemma 1.1**  Given a fixed tableau design \( \lambda \) and Hessenberg functions \( h \) and \( h' \). When we move from \( h \) to \( h' \) the dimension of each cell is either unchanged or increases by exactly 1.

**Proof.** First we show that the dimension of a cell can never decrease. Each inversion corresponds to a unique triple \( i, j, k \) such that \( i \) is below or to the left of \( k \), \( k < i \), and if there is a box immediately to the right of \( k \) filled by \( j \), then \( i \leq h(j) \). Since \( h'(j) \geq h(j) \) for all \( j \) we know \( i \leq h(j) \) implies \( i \leq h'(j) \). Therefore all triples which correspond to an inversion in the case of \( h \) correspond to an inversion in the case of \( h' \), and the dimension of a cell cannot decrease.

Now suppose the dimension of the cell increases. The dimension increase must occur as an increase in the number of inversions on some box filled by \( i < h(j) + 1 \) where \( j \) is immediately to the right of \( i \). Furthermore we know \( h(j) + 1 \) is below or somewhere to the left of \( i \). This defines a unique triple in a given cell, because both \( j \) and \( h(j) + 1 \) can occur only once in the filling, and \( i \) is then specified because it must be the number immediately to the left of \( j \). This triple then increases the dimension by exactly one. \( \square \)

**Definition 3**  A bumped up cell is a cell whose dimension increases by one with the change in Hessenberg functions from \( h \) to \( h' \).

2  **First restricted tableaux**

![Figure 2: An example of a very restrictive tableau design](image-url)
Lemma 2.1  Given Hessenberg functions $h$ and $h'$ such that $h'(\beta) = h(\beta) + 1$ for some $\beta$ and $h'(j) = h(j)$ for all $j \neq \beta$, a valid cell $\lambda$ of design similar to Figure 2 (having 2 boxes in the first row, and any number of additional rows with a single box) will have a dimension increase if and only if $\beta$ is in the second column of the first row and $a_1 < h'(\beta)$.

Proof. Suppose we have $\beta$ in the second column and $a_1 < h'(\beta)$. Note that for $h'$ to be a valid Hessenberg function on a cell with $N$ boxes, we have $\beta < h'(\beta) \leq N$. Therefore $h'(\beta)$ exists somewhere in the sequence $a_1, a_2, \ldots, a_{N-3}, a_{N-2}$. We know $a_1 \leq h(\beta)$ and hence $a_1 \neq h'(\beta)$. The triple $a_1, h'(\beta)$, $\beta$ gives us an inversion with the new function $h'$ but not with the old $h$ since the third condition for an inversion was not previously met. By Lemma 1.1 this type of function increase from $h$ to $h'$ leaves all previous inversions intact. Therefore the dimension of our cell has increased by 1.

Now suppose we have a cell whose dimension has changed. By Lemma 1.1 we know that $\beta$ must be somewhere to the right of $h'(\beta)$, which means $\beta$ must be in the only box in the second column. We also know $a_1$ is less than $h'(\beta)$ since $a_1 \leq h(\beta)$ is a condition for a valid cell.

Proposition 4  Uniquely mapping each bumped up cell to another cell of dimension reduced by one, which is either also bumped up, or was not previously valid, is enough to show that if $h$ gives us $k$ cells of some dimension $d$ and $h'$ gives us $j$ cells of dimension $d$, $j \geq k$.

Proof. This should be very clear, we will be forming chains of cells which take each other’s place in their former dimension, these chains will then be terminated by cells which were not previously valid. Thus if a cell $\lambda$ of dimension $d$ moves to dimension $d+1$ and we can always find a cell of dimension $d$ which takes it’s place we will always be left with at least as many cells of dimension $d$ as we had previously.

This proposition motivates the remainder of our work. We seek to find methods to establish these chains of cells which move up in dimension, terminating with a new cell. This will give us the results that we are searching for on the Betti numbers.

Theorem 5  Suppose we have a bumped up cell with 2 boxes in the first row and any number of additional rows with a single box as before. Exchanging $a_1$ with $a_l$ the smallest $a_j$ greater than $a_1$ will give us a unique cell with dimension reduced by one.

Proof. It should be clear that $a_l$ exists since $h'(\beta)$ is somewhere in the $a_j$, and $h'(beta)$ is necessarily greater than $a_1$ since a condition for validity of our cell is $a_1 \leq h(beta) < h'(beta)$.

Suppose $\beta \neq a_1+1$. Then $a_l + 1$ is in the $a_j$ and $a_l = a_1+1$. The condition for our cell to be valid with the function $h$ is $a_1 \leq h(\beta)$. This tells us $a_1 + 1 \leq h(\beta) + 1 = h'(\beta)$, so our new cell is valid. Now we look at the inversions in this cell. The number of inversions on the box occupied by $a_l$ before we exchange is the number of $a_j$ such that $a_j > a_l$ and $j > l$. After we exchange we see that the number of inversions on this box is constant since the sequence $a_{l+1}, \ldots, a_{N-2}$ of (nonrepeating) integers cannot contain any $a_j$ such that $a_1 < a_j < a_l$. The inversions on boxes occupied by $a_j$ such that $j > l$ must be constant since no changes were made to boxes below these numbers. Since we have no repetition if any $a_j$ of $a_1, \ldots, a_{l-1} < a_l$ then $a_j < a_1$ and thus the number of inversions on the boxes occupied by $a_1, \ldots, a_{l-1}$ is unchanged. The number of inversions on the box occupied by $\beta$ is unchanged because every number in the
sequence \(a_1, \ldots, a_{N-2}\) remains to the left of \(\beta\) after the permutation. Finally we come to the box previously occupied by \(a_1\). There is exactly one integer which is greater than \(a_1\) but not greater than \(a_l\), it is \(a_l\). Since we have switched these two numbers we get a reduction of one in the number of inversions on this box. This is exactly what we were looking for.

Now suppose \(\beta = a_1 + 1\) so now \(a_l = a_1 + 2\). We know \(a_l \leq h'(\beta)\) since \(h(\beta) \geq \beta\) and \(h'(\beta) = h(\beta) + 1\) so exchanging \(a_1\) and \(a_l\) gives us a valid cell. The previous arguments for the changes in the number of inversions on a cell hold if we add the fact that \(a_j \neq a_1 + 1\) for all \(a_j\).

To show uniqueness we can simply show that this process is reversible. We only need to reverse a cell if it has been bumped up, and it could come from some other cell. Thus we first eliminate all cells where \(\beta\) is not in the upper right, and all cells where \(a_1\) is the smallest of the \(a_j\), as these cells could not be generated by our previous algorithm. The rule for recovering our original cell is then exchange \(a_1\) and \(a_m\) where \(a_m\) is the maximum \(a_j\) such that \(a_j < a_1\).

It is obvious that we get a \(h\)-allowed filling here since \(a_m < a_1 \leq h'(\beta)\) gives us \(a_m \leq h(\beta)\). We know that \(a_m\) exists in the first column since we have already eliminated all cells were \(a_1\) is the smallest \(a_j\). Since our algorithm is reversible we have one-to-one correspondence between a cell whose dimension increases by one, and another cell with the dimension of the original cell.

Note that if the cell formed by this rule was valid with the function \(h\) it is of the form given by Lemma 2.1 and this formula can be applied again to find a cell which has its dimension reduced by one. These chains are terminated when we place \(h'(\beta)\) in front of \(\beta\) which gives us a cell which was not valid with the function \(h\).

This case motivates the three lemmas that follow. First we look at the possibility of reducing big cells to smaller ones by examining which parts of a cell are important when we exchange the contents of two boxes. Then we will establish methods for changing the number of inversions on a single box by 1, either up or down, through exchanging the contents of that box.

**Lemma 2.2** Assume we exchange two elements in any cell diagram such that the result is a valid cell. The number of inversions on boxes either directly above or anywhere to the right of each exchanged box is unchanged.

*Proof.* Let us exchange \(i_1\) and \(i_2\) and let \(k\) represent a box which is either directly above or to the right of \(i_1\) and \(i_2\). Since the position criteria are met on both locations an inversion on \(k\) by either \(i_1\) or \(i_2\) remains in the final cell. \(\square\)

**Lemma 2.3** Let \(a_{l(i)}\) be the \(\max(L(a_i))\) where \(L(a_i) = \{a_j | j > i \text{ and } a_j > a_i\}\). Thus \(a_{l(i)}\) is the least \(a_j\) simultaneously greater than and below \(a_i\). Assuming a valid cell is created swapping \(a_i\) with \(a_{l(i)}\) will result in a reduction by one of the number of inversions on the \(i\)-th box in the first column.

This swap will also increase by one the number of inversions on any other box occupied by \(a_n\) if and only if the following conditions are met

- \(i < n < l\)
- \(n \leq k\) (the \(n\)-th row contains 2 boxes)
• \( a_n < a_i < a_{l(i)}, \) and

• \( a_i \leq h(b_n) < a_{l(i)}. \)

The number of inversions on all other boxes is left alone.

Proof. Without loss of generality we assume \( a_j = a_1 \). Since boxes anywhere above \( a_i \) do not affect our result by Lemma 2.2. The permutations are occurring on boxes which are to the left of \( b_1, ..., b_k \). Therefore the number of inversions on these boxes is also unchanged by Lemma 2.2.

The number of inversions on the first box in the first column is the number of \( a_j \) such that \( a_1 < a_j \leq h(b_1) \). There is no repetition in the \( a_j \). Therefore, for all \( j \neq l \) if \( a_j > a_1 \) then \( a_j > a_{l(i)} \), and similarly \( a_j < a_1 \) implies \( a_j < a_{l(i)} \). However the inversion caused by \( a_{l(i)} \) on the first box has been removed, so the number of inversions on this box has been reduced by one.

The number of inversions on the \( l \)-th box in the first column has not decreased since \( a_j > a_{l(i)} \) implies \( a_j > a_1 \). It has also not increased because \( a_{l(i)} > a_j > a_1 \) contradicts our conditions on \( a_{l(i)} \). The number of inversions on boxes occupied by \( a_{l+1}, ..., a_{N-k} \) remains the same because no permutations occur below these boxes.

Now we will look \( a_n \) where \( 1 < n < l \). If \( n > k \) (that is to say if the \( n \)-th row has only one box) the number of inversions cannot change on \( a_n \). This is because we know either \( a_1 < a_{l(i)} < a_n \) or \( a_n < a_1 < a_{l(i)} \), and without a \( b_n \) these conditions are sufficient to maintain the previous inversion or lack of inversion by \( a_{l(i)} \) on \( a_n \). Therefore we know \( n > k \). If \( a_n > a_1 \) then \( a_n > a_{l(i)} \) by our conditions on \( a_{l(i)} \). In this case there was no inversion on \( a_n \) by \( a_{l(i)} \) and there cannot be an inversion on \( a_n \) by \( a_1 \). Now suppose \( a_n < a_1 \). If \( a_{l(i)} \leq h(b_n) \) then \( a_1 < h(b_n) \) and the inversion on \( a_n \) by \( a_{l(i)} \) is maintained by \( a_1 \). Similarly if \( a_1 \geq h(b_n) \) then \( a_{l(i)} > h(b_n) \) and no inversion by \( a_{l(i)} \) on \( a_n \) remains no inversion after the swap. However if \( a_1 \leq h(b_n) < a_{l(i)} \) no inversion by \( a_{l(i)} \) on \( a_n \) becomes an inversion by \( a_1 \) on \( a_n \). In this case the number of inversions on the box occupied by \( a_n \) increases by one.

Now if there is a box filled by \( a_n \), \( (n \neq 1) \) such that the number of inversions on \( a_n \) increases we know \( a_n \) is above \( a_{l(i)} \). This is because if \( a_n \) is below \( a_{l(i)} \) the permutations occur in boxes which cannot cause inversions on \( a_n \). Furthermore we know \( a_1 > a_n \), and \( a_1 \leq h(b_n) \) because \( a_1 \) will cause the new inversion on \( a_n \). If \( a_{l(i)} \leq h(b_n) \) there was a preexisting inversion on \( a_n \) by \( a_{l(i)} \) which is continued by \( a_1 \). Therefore in order to increase the number of inversions on \( a_n \) by one \( a_{l(i)} > h(b_n) \). \( \square \)

Example 2.1 As an example let us take the first cell below with the function \( h(6) = 7 \) and \( h(i) = i \) for \( i \neq 6 \), and preform a swap on the first box in the first column.

\[
\begin{align*}
\begin{array}{|c|c|}
\hline
4 & 6 \\
\hline
3 & 5 \\
\hline
2 \\
\hline
1 \\
\hline
7 \\
\hline
\end{array}
& \quad \begin{array}{|c|c|}
\hline
7 & 6 \\
\hline
3 & 5 \\
\hline
2 \\
\hline
1 \\
\hline
4 \\
\hline
\end{array}
& \quad \begin{array}{|c|c|}
\hline
7 & 6 \\
\hline
4 & 5 \\
\hline
2 \\
\hline
1 \\
\hline
3 \\
\hline
\end{array}
\end{align*}
\]

(a) Initial cell \quad (b) After first swap \quad (c) Final cell

Notice that we first move \( a_1 = 4 \), and this creates a change in the number of inversions on
3. We can then iterate the process on 3 to end with a cell whose total dimension is decreased by one.

**Lemma 2.4** Assume we have a valid cell and swap $a_i$ with $a_m$ such that $a_m$ is the greatest $a_j$ such that $j > i$ and $a_j < a_i$. Assuming this also gives us a valid cell (that is $a_1 \leq h(b_m)$ if $b_m$ exists) will result in an increase by one of the number of inversions on the $i$-th box in the first column.

This swap will also reduce by one the number of inversions on any box occupied by $a_n$ if and only if

- $i < n < m$
- $n \leq k$ (the $n$-th row contains 2 boxes)
- $a_n < a_m$ and
- $a_m \leq h(b_n) < a_i$.

The number of inversions on all other boxes is left alone.

**Proof.** Similar to above all boxes in columns above the $i$-th are unchanged and we remove them without loss of generality. The number of inversions on the boxes in the second column remains the same by Lemma 2.2. The reasoning for the number of inversions being unchanged on boxes in the $m$-th column and below is equivalent to that of the $l$-th column and below in the proof of Lemma 2.3.

The number of inversions on the $a_1$ box is the number of $a_j$ such that $a_1 < a_j \leq h(b_1)$. For all $j \neq m$ if $a_j > a_1$ then $a_j > a_m$. Also $a_j < a_1$ implies $a_j < a_m$ by our conditions on $a_m$. There is a new inversion caused by $a_1$ on the box now occupied by $a_m$ since $a_1 > a_m$ and $a_1 \leq h(b_1)$ by our cell’s previous validity. Thus the number of inversions on this box increases by one.

Again we examine $a_n$ with $1 < n < m$. We know $a_n < a_m$ if $a_n < a_1$ by our conditions on $a_m$. This gives us no new inversions if $a_n > a_m$. If no $b_n$ exists ($n > k$) the number of inversions on $a_n$ remains the same by the reasons in the previous Lemma. Now suppose $a_1 < a_m$. As before if $a_m < a_1 \leq h(b_n)$ or $h(b_n) \leq a_m < a_1$ we get no change on the number of inversions on $a_n$. However if $a_m \leq h(b_n) < a_1$ an inversion by $a_m$ on $a_n$ has been removed, and the number of inversions on this box is reduced by one.

Suppose the number of inversions on a box filled by $a_n$, ($n \neq 1$) decreases. We know $1 < n < m$ since boxes below $a_m$ are not affected by our swap. We also have $a_m > a_n$, and $a_m \leq h(b_n)$ since we need an inversion to take away in our swap. If $a_1 \leq h(b_n)$ we would still have this last inversion. Therefore in order to decrease the number of inversions on $a_n$ by one $a_1 > h(b_n)$.

**Definition 6** An Unexpected Inversion Change is the increase or decrease of the number of inversions on some box $a_n$ as the result of an exchange of boxes $a_i$ and $a_j$ with $n \neq i, j$. 


Theorem 7  Suppose we have a cell generated by a partition of the form $2, 2, 1^{N-2}$, as in figure 2. Given $h$ and $h'(\beta) = h(\beta)+1$ and a $h$ allowed cell which is bumped up $\lambda$ we can generate a cell with total dimension reduced by one with two rules (supposing $\beta$ is $b_1$):

1. exchange $a_i$ and $a_{l(i)}$ rename $a_{l(i)}$ to $a'_i$

2. if $i = 1$ and the dimension of $a_2$ changes from our previous move, exchange $a_2$ and $a_{l(2)}$, rename $a_2$ to $a'_2$

Proof. First note that if $i = 2$ we will not make any permutations on $a_1$ and $b_1$. We can then strike the first row from our tableau and reduce to the case proved in Theorem 5. Therefore let $i = 1$. By Lemma 2.3 our first swap produces a reduction by 1 in the number of inversions on the box occupied by $a'_1$. If $a_2$ satisfies the conditions outlined in Lemma 2.3 for an unexpected inversion change it will receive an increase by one in the number inversions on that box. We then note that applying the second rule is similar to removing the first row from our diagram and reducing to the case of Theorem 5 since we know $a_2 < a_1 \leq h'(b_2)$ there exists a valid $a_1$ for our swap, and taking the smallest one will reduce our total cell dimension to exactly one less than what we started with.

We can ensure uniqueness by reversing this process. Given some cell were if either $\beta$ is in the left column or $h'(\beta)$ is in the right the cell was not bumped up and no changes are necessary. Else apply the rules

1. if $i = 1$, $a'_2 < a'_1$ and $a'_{m(1)} \leq h(b_2) < a'_1$ exchange $a'_2$ and $a'_{m(2)}$ remove primes

2. exchange $a'_i$ and $a'_{m(i)}$ remove primes

The first step reverses the final step of our previous algorithm. The conditions are a check to ensure that the final step was run. In this step we know that $a'_{m(2)}$ exists since the conditions are sufficient to ensure that $a_2$ was exchanged with $a_{l(2)}$ before, and then by definition $a'_{m(2)} = a_2$. A valid cell is created here since $a'_{m(2)} < a'_2 \leq h(b_2)$.
The second step reverses the first step of the initial algorithm, since $a_m' < a_i'$ and $h(i) = h'(i) - 1$ we have a valid cell from this swap. Also note that if $i = 1$ we will not illegally move $a_1'$ into the $a_2'$ space, since an illegal move there would require that $a_2' < a_1'$ and $a_m'(1) \leq h(b_2) < a_1'$, in which case we will have already moved $a_2'$ down, into a cell valid for this swap. These conditions cannot be met twice, that is if a first rule exchange is made we know $a_m(1) \neq a_2$, since $a_m(1) < a_2' < a_1'$.

As was our goal we now have either another cell which meets the conditions for being bumped up, or a cell which was not previously valid. Again by Proposition 4 we are assured that in this cell design the Betti numbers $b_i$ never decrease under an $h \rightarrow h'$ function change.

Refer to Example 2.1 to see what this looks like in practice. Later we will develop this idea into a method for all tableaux with rows of length no greater than 2 but first lets look at another small addition to our tableaux design from Theorem 5 adding another box to the first row. For this proof we require a slight modification to the idea of $a_{l(a)}$. Let $a_{l(a)}$ be exactly $a_l(1)$ if $a_1 = \alpha$, and if $\alpha = b_1$ take $L(\alpha)$ as the set of all $a_i > b_1$ and let $a_{l(a)} = \min(L(\alpha))$.

**Theorem 8** Suppose we have a cell generated by a partition of the form $3, 1^N - 2$, as in figure 2. Given $h$ and $h'$ $(h'(\beta) = h(\beta) + 1)$ and a $h$ allowed cell which is bumped up $\lambda$ we can generate a cell with total dimension reduced by one with two rules (let the box to the left of $\beta$ be occupied by $\alpha$);

1. exchange $\alpha$ and $a_{l(a)}$

2. if we just moved $b_1$ and the dimension of $a_1$ changes from our previous move, exchange $a_1$ and $a_{l(1)}$, repeat until the number of inversions on the $a_1$ box is back to its original value

**Proof.** Again if $\beta = b_1$ we may remove the last box in the first row and this reduces to Theorem 5. We need to show that we have corrected for the case of $\beta = c_1$ moving $a_{l(1)}$ into the $b_1$ box. For simplicity call the new $b_1 b'_1$. First lets ensure that we have created a valid cell. Since $a_{l(a)} > b_1$ we have $h'(a_{l(a)}) > h'(b_1)$, furthermore either $a_1$ decreases if $l(\alpha) = 1$ or it stays the same if $l(\alpha) \neq 1$. Therefore we have an $h$-allowed filling.

There will now be 1 unexpected inversion change on $a_1$ for every $a_j$ such that $h'(b_1) < a_j \leq h'(b'_1)$. Note that each of these $a_j$ is in the set $L(a_1)$. And so we can preform Lemma 2.3 swaps on $a_1$ until the number of inversions on that box are back to its previous number.

To show this is unique we will show that no two distinct cells yield the same final cell. Suppose we have two cells $\lambda$ and $\lambda'$ which yield $\gamma$ such that $\lambda \neq \lambda'$. We see immediately that $\beta$ cannot be in the $b_1$ box because we already know this process is reversible. Furthermore $b_1 = b'_1$ since we preform exactly one switch on this box and $a_{l(b_1)} = a_{l(b'_1)}$ in order to receive the same final cell $\gamma$. Therefore $a_1 \neq a_1'$ else $\lambda = \lambda'$. Without loss of generality assume $a_1' < a_1$. The number of unexpected inversion changes on $a_1$ is equal to the number of unexpected inversion changes on $a_1'$ since they are both equal to the number of $a_j$ such that $h'(b_1) < a_j \leq h'(b'_1)$. Note there is no requirement on $a_1$ necessary since $a_j > h'(b_1)$ and $a_1 < h'(b'_1)$, our valid cell condition, implies $a_j > a_1$. At each exchange we know $a_{l(1)}' \leq a_1 < a_{l(1)}$ because $a_1$ must exist in the cell $\lambda'$ and since we have shown it is not in the first row it must be below $a_1'$ in the first column. Since both $a_1$ and $a_1'$ need to take the same number of steps we can be assured that they do not end on the same number. □
Example 2.2  Let's take an example with the function $h = 1, 2, 5, 5, 5, 6$ and $h' = 1, 2, 5, 5, 6, 6$

We can see that the initial swap produces a decrease by 1 in the number of inversions on the $b_1$ box, and also an increase by 2 on the $a_1$ box. Each swap after that reduces the number of inversions on the $a_1$ box by 1.

3  Generic two column tableaux

We will now look at general tableaux where all rows have no more than 2 boxes. These tableaux lend themselves to our methods because swaps on the $a_i$ need only be checked for validity against the values of the Hessenberg function on the second column.

![Figure 3: A cell $\lambda$ generated by the partition $2^k, 1^{N-k}$, where $k$ is the number of two box rows.](image)

Lemma 3.1  Any box in the left column of a cell receives a maximum of a plus one unexpected inversion change from $a_i$ and $a_{l(i)}$ (as in Lemma 2.3) swaps made in the left column.

Proof. Suppose a we have a box $a_n$ which has undergone an unexpected inversion change by exchange of $a_i$ and $a_{l(i)}$. We then know that $a_i \leq h(b_n) < a_{l(i)}$. A further unexpected inversion
change would require that there exists some \( a_k \) such that \( a_k \leq h(b_n) < a_{l(k)} \). We will show that no such \( a_k \) exists. We know that \( a_k \neq a_i \) because \( a_k \) must be above \( a_n \) and \( a_i \) below. If \( a_k > a_i \) we have a contradiction on the definition of \( a_{l(i)} \) because \( a_k < a_{l(i)} \) and then \( a_{l(i)} \) not the smallest \( a_j \) greater than and below \( a_i \). If \( a_k < a_i \) we have a contradiction on the definition of \( a_{l(k)} \) because \( a_i < a_{l(k)} \) and \( a_{l(k)} \) not the smallest \( a_j \) greater than and below \( a_k \).

\[ \square \]

**Theorem 9** Given a bumped up cell \( \lambda \) generated by a partition of the form \( 2^k, \ 1^{N-k} \), where \( h'(b_i) = h(b_i) + 1 \). The following rules will generate unique cell \( \gamma \) of total dimension reduced by one.

1. Exchange \( a_i \) and \( a_{l(i)} \) rename all \( a_j \) to \( a'_j \)

2. Starting with the highest \( a'_n \) which has an unexpected inversion change (minimum \( n \) value). Swap \( a'_n \) and \( a'_{l(n)} \) repeat until all unexpected inversion changes have been eliminated.

Furthermore this is enough to show that if \( h \) gives us \( k \) cells of some dimension \( d \) and \( h' \) gives us \( j \) cells of dimension \( d \), \( j \geq k \).

**Proof.** **Part I: proof of reduction algorithm** First without loss of generality we remove all \( a_j \) and \( b_j \) where \( j < i \). These boxes will be unaffected by permutations made on boxes below them by Lemma 2.2. Reindex the cell so that \( a_i \) is now \( a_1 \). Our first swap \( a_1 \) and \( a_{l(i)} \) is valid because we know \( h'(b_1) \) is below \( a_1 \) so there exists at least one \( a_j \) such that \( a_j < a_j \leq h(b_1) \), then \( a_{l(i)} \) is the smallest such. By Lemma 2.3 we have reduced the number of inversions on \( a_1 \) by one.

Now we must account for all unexpected inversion changes. Our second rule starts with the highest, applying Lemma 2.3 again to remove each increase in dimension. Once again we know that each swap is valid because an unexpected inversion change on \( a_n \) requires moving some \( a_i \) below \( a_n \) such that

- \( i < n < l \)
- \( n \leq k \) (the \( n \)-th row contains 2 boxes)
- \( a_n < a_i \), and
- \( a_i \leq h(b_n) < a_{l(i)} \).

The last condition tells us that there exists at least one valid swap for every unexpected inversion change. Furthermore since we have to make at most one swap on each box in the left column by Lemma 3.1 so this process terminates by the finite number of boxes in our cell.

We have now generated a cell from our previous cell which has its overall dimension reduced by one. Note that if \( a_{l(i)} < h'(b_1) \) this is another bumped up cell, and if \( a_{l(i)} = h'(b_1) \) this cell is a new cell, not valid under the function \( h \).

**Part II: proof of uniqueness** Suppose two cells \( \lambda^1 \) and \( \lambda^2 \) with \( \lambda^1 \neq \lambda^2 \) give the same final cell \( \gamma \). For all \( i < k \) we know \( b^1_i = b^2_i \) because we do not make any changes to the \( b_j \). We also know \( a^1_{l^1(1)} = a^2_{l^2(1)} \) in order to receive the same final cell. This tells us that \( a^1_1 = a^2_1 \), because this is required for \( a^1_{l^1(1)} = a^2_{l^2(1)} \). We can see however that \( l^1(1) \) is not necessarily equal to \( l^2(1) \), because we have no restriction on the initial location of \( a^1_{l^1(1)} \) or \( a^2_{l^2(1)} \).
Take the set \( D = \{ d | a_d^1 \neq a_d^2 \} \). As shown previously unexpected inversion changes must occur between two swapped numbers, therefore there exists at least one \( d \) such that, \( 1 < d \leq \min(l^1(1), l^2(1)) \). This is because if \( l^1(1) \neq l^2(1) \) then \( \min(l^1(1), l^2(1)) \neq a_{\min(l^1(1), l^2(1))} \). Also if \( l^1(1) = l^2(1) \) then one of \( \lambda^1 \) and \( \lambda^2 \) has an unexpected inversion change from swapping \( a_1 \) and \( a_{l(1)} \). Otherwise, the algorithm terminates for both cells after the first swap. Since the same values in the same positions were exchanged, and the same cell \( \gamma \) was created, it follows that \( \lambda^1 = \lambda^2 \). This contradicts our hypothesis that \( \lambda^1 \neq \lambda^2 \). An unexpected inversion change can only occur between two swapped numbers, so some \( d \) exists with \( 1 < d \leq \min(l(1), l(1)) \).

Let us take the \( \min(D) \) and call it \( \delta \). Furthermore without loss of generality let \( a_\delta^1 > a_\delta^2 \).

It is possible that the \( a_\delta \) are involved in previous algorithm swaps. Swapping some \( a_j \) with \( j < \delta \) cannot give us \( a_\delta^1 = a_\delta^2 \) because then \( a_j^1 \neq a_j^2 \). Also \( a_\delta^2 \) remains less than \( a_\delta^1 \) since either \( a_j < a_\delta^2 < a_\delta^1 \) in which case \( a_j \) and \( a_\delta^2 \) are swapped, or \( a_\delta^2 < a_j < a_\delta^1 \) in which case \( a_j \) and \( a_\delta^1 \). By Lemma 3.1 each \( a_\delta \) can receive only one swap. As some \( a_j^2 = a_\delta \) and all \( a_j^1 = a_j^2 \) for \( j < \delta \) by our definition of \( \delta \) we know some \( a_k^2 = a_k^1 \) for \( k > \delta \). Therefore \( a_{\overline{\delta}2}^\lambda \leq a_\delta^1 \). Suppose both \( a_\delta^1 \) and \( a_\delta^2 \) receive unexpected inversion changes \( a_{\overline{\delta}2}^\lambda \neq a_{\overline{\delta}1}^\lambda \) as \( a_{\overline{\delta}2}^\lambda \leq a_\delta^1 \) and the final cells cannot be equal. Then an unexpected inversion change must occur on \( a_\delta^2 \) and not on \( a_\delta^1 \). We have \( b_{\overline{\delta}}^1 = b_{\overline{\delta}}^2 \) which implies \( h'(b_{\overline{\delta}}^1) = h'(b_{\overline{\delta}}^2) \), and since the unexpected inversion change must occur from some \( a_j \), \( a_{\overline{\delta}j} \) exchange we know \( a_\delta^2 < a_j < a_\delta^1 \) and \( a_{\overline{\delta}j} \) cannot equal \( a_\delta^1 \) (it must be no greater than \( a_j \)), a contradiction.

Again we see that we have retained the important element of our scheme, that is either \( h'(b_1) \) is in front of \( b_1 \) which means that we have a cell which was not previously valid, or \( h'(b_1) \) is below \( a_1 \) and we have another bumped up cell which we can then apply the algorithm to again find a cell of further reduced dimension. This fits with Proposition 4 and ensures that \( b_i \leq b_i' \) for all \( i \). \( \square \)

**Example 3.1** An example of a chain with function \( h = 2, 2, 5, 5, 6, 7, 10, 10, 10, 11, 12 \) and \( h'(11) = 12 \) is

```
  1   11  
  2    10  
  4    8   
  6    9   
  5    3   
  7    7   
 12   12   
 12   12   
  1    10  
  2    8   
  5    9   
  4    3   
  6    1   
 12   7   
```

Here each cell maps to the cell to its immediate right under our algorithm. The last mapping is (by design) where all the unexpected inversion changes occur and so we will do a
step by step on this cell in the next example.

**Example 3.2** We are going to do the final rearrangement in the last example in more detail. I will use +1 to the left of a box to indicate an increase in the number of inversions on that box by 1. We start with a single increase on the $a_1$ box because this is a bumped up cell.

\[
\begin{array}{ccc}
+1 & 7 & 11 \\
1 & 10 \\
2 & 8 \\
5 & 9 \\
4 & 3 \\
6 \\
12 \\
\end{array}
\]

\[
\begin{array}{ccc}
+1 & 12 & 11 \\
1 & 10 \\
2 & 8 \\
5 & 9 \\
4 & 3 \\
6 \\
7 \\
\end{array}
\]

\[
\begin{array}{ccc}
+1 & 12 & 11 \\
2 & 10 \\
4 & 8 \\
6 & 9 \\
5 & 3 \\
1 \\
7 \\
\end{array}
\]

\[
\begin{array}{ccc}
+1 & 12 & 11 \\
2 & 10 \\
4 & 8 \\
6 & 9 \\
5 & 3 \\
1 \\
7 \\
\end{array}
\]

In this example each new cell represents a single swap from our algorithm.

4 Conclusions

**Theorem 10** For all the cell designs we have studied so far whenever we have two general Hessenberg functions $h$ and $g$ with $g(j) \geq h(j)$ for all $j$ the number of cells of a particular dimension $d$ under $h$ is less than or equal to the number of cells of dimension $d$ under the function $g$.

Proof. We can step from $h$ increasing each $h(i)$ individually until we reach $g$. While multiple paths may be legal, making our increases starting at the greatest $i$ such that $h(i) < g(i)$ will always be legal. The result is then immediate since our previous theorems show that each increase results in no decrease in the number of cells of dimension $d$.

This then is the result we were looking for since each of the Betti numbers $b_i$ are the number of cells of dimension $i$ we can say that moving from $h$ to some other Hessenberg function $g$ with each $g(i) \geq h(i)$ yields nondecreasing Betti numbers.

References
