Tropical Intersections and Shadow Points

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Abstract

In this paper we study tropical curves and the intersections created by the graph of two tropical polynomials in $\mathbb{R}^2$ via the valuation of the tropical polynomials. We seek to find a way to ease the process of finding shadow points by providing a detailed look at linear/conic intersections and conic/conic intersections. This includes looking at a technique that would ease finding shadow points and looking at examples.
1 Introduction

Tropical polynomial functions are functions with operations defined by tropical arithmetic. These polynomials can be created from traditional algebraic polynomials that have Puiseux series coefficients. These algebraic polynomials can be converted to tropical polynomials via the ORD map which maps a general algebraic polynomial with Puiseux series terms like:

\[ f = c(t) \cdot x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} + d(t) \cdot x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} + \ldots \]

to

\[ \text{trop}(f) = \text{ord}(c(t)) \odot x_1^{u_1} \odot x_2^{u_2} \odot \cdots \odot x_n^{u_n} \oplus \text{ord}(d(t)) \odot x_1^{v_1} \odot x_2^{v_2} \odot \cdots \odot x_n^{v_n} \oplus \ldots \]

via the valuation of each of the coefficients of \( f \). This tropicalized function can be graphed in \( \mathbb{R}^2 \) and this graph has bounded and unbounded edges, all edges have rational slope, and this graph satisfies a zero tension condition around each node. Since the ORD map is highly non-injective it is possible for an intersection in \( \mathbb{R}^2 \) to come from distinct points in \( K^2 \). These intersections are called shadow points. Tropical algebraic curves satisfy the Bezout Theorem. Since the ORD map is highly non-injective there exist the possibility that some intersections stack, in other words have a tropical multiplicity and thus shadow points form elsewhere in order for Bezout to hold. This paper seeks to illustrate methods that could lead to these conditions for two tropical linear functions intersecting, a tropical linear function intersecting a tropical conic function, and two tropical conic functions intersecting. These methods include both Maple programs combined with examples to illustrate how the process of looking for these shadow points begins and could be continued.

This paper is divided into five sections. The second section is background information in which we introduce tropical arithmetic and its link with Puiseux series as well as with tropical hypersurfaces. Additionally we discuss intersections and shadow points. The third section is a look at tropical intersections between two tropical linear functions. The fourth section looks at the tropical intersection of a linear and conic tropical function and how to search for shadow points including an in-depth case analysis. The fifth section looks at the tropical intersection of two conic functions. This final section details both the general strategy for finding shadow points but also a specific example.

2 Background

Tropical Arithmetic

This background is based on Bernd Sturmfels. [2] The algebraic structure in tropical geometry consists of the real numbers with the addition of infinity and consists of two operations called tropical sum and tropical product. The operations of addition and multiplication are redefined thus:
\[ x \oplus y := \max(x, y) \text{ and } x \odot y := x + y \]

The tropical sum is defined as the maximum of \(x\) and \(y\) and the tropical product is defined as the sum of \(x\) and \(y\). An example of how this works is:

\[ 1 \oplus 4 = 4 \text{ and } 1 \odot 4 = 5 \]

Many of the properties from arithmetic remain valid in tropical arithmetic. For example, both the Commutative law and Distributive law apply in tropical arithmetic.

\[ x \oplus y = y \oplus x \text{ and } x \odot y = y \odot x \]

\[ x \odot (y \oplus z) = x \odot y \oplus x \odot z \]

An identity element exists for tropical addition and is negative infinity. An identity element also exists for tropical multiplication and is zero.

\[ x \oplus -\infty = x \text{ and } x \odot 0 = x \]

Therefore tropical arithmetic can be summed up as the tropical semiring \( (\mathbb{R} \cup \{-\infty\}, \oplus, \odot) \).

**Puiseux Theorem**

The field of Puiseux series is defined as \( K = \mathbb{C}\{t\} \) where the elements of \( K \) are the power series

\[ c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + c_4 t^{a_4} + \ldots \]  

where \( c_n \) are non-zero complex numbers and \( a_n \) are increasing, rational fractions with a common denominator.

**Theorem 1**: The Puiseux Theorem

The reason this is compelling is that it provides the link between tropical geometry and classic geometry. First we introduce a few definitions. The *order* of the Puiseux series (1) is the additive inverse of the exponent of the lowest degree term, \( a_1 \). It should be noted that this map is highly non-injective. This map

\[ K \backslash \{0\} \rightarrow \mathbb{Q}, \quad c(t) \mapsto -a_1 = \text{order}(c(t)) \]

is called the *valuation* of \( c(t) \) which means that it satisfies

\[ \text{ord}(c + d) \geq \max(\text{ord}(c), \text{ord}(d)) \quad \text{and} \quad \text{ord}(c \cdot d) = \text{ord}(c) + \text{ord}(d) \]
Let $f \in \mathbb{K}$ be any polynomial with $n$ variables with coefficients in $\mathbb{K}$. The tropicalization of $f$ is defined to be the tropical polynomial gotten by replacing each coefficient by its order and replacing classic arithmetic with tropical arithmetic. For example:

$$f = (t^2 + t + 7) \cdot x + (t^9 + t^3 + t^5) \cdot y + (t^4 + t) \cdot x \cdot y$$

$$\text{trop}(f) = 0 \odot x \oplus 5 \odot y \oplus -1 \odot x \odot y$$

Or more generally:

$$f = c(t) \cdot x_1^{u_1} x_2^{u_2} \ldots x_n^{u_n} + d(t) \cdot x_1^{v_1} x_2^{v_2} \ldots x_n^{v_n} + \ldots$$

$$\text{trop}(f) = \text{ord}(c(t)) \odot x_1^{u_1} \ldots \odot x_n^{u_n} \oplus \text{ord}(d(t)) \odot x_1^{v_1} \ldots \odot x_n^{v_n} \oplus \ldots$$

This shows how to get a tropical polynomial from a classic algebra polygon with Puiseux series coefficients.

**Tropical Hypersurfaces and Graphs**

By interpreting addition and multiplication tropically the polynomial is a function from $\mathbb{R}^n \to \mathbb{R}$. This function is a piecewise linear function that is obtained as a max of the linear functions corresponding to monomials. A tropical polynomial function is given as the maximum of a finite number of linear functions $p: \mathbb{R}^n \to \mathbb{R}$. The hypersurface $T(p)$ is defined as the line segment where this maximum occurs twice. A tropical curve $T(p)$ is a finite graph in $\mathbb{R}^2$ which has both unbounded and bounded edges, all edges are rational, and this graph satisfies a zero tension condition around each node. The zero tension condition means that the sum of the nonzero primitive lattice vector on each ray emanating from $(x,y)$ has a sum of zero. For example a line in a plane:

$$p(x,y) = a \odot x \oplus b \odot y \oplus c$$

where $a,b,c \in \mathbb{R}$

The curve $T(p)$ consists of all points $(x,y)$ where the function

$$p: \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \max(a+x, b+y, c)$$

achieves a maximum twice. It consists of three rays emanating from the point $(c-a, c-b)$ in the southern, western and northeastern direction. The line exhibits the zero tension condition because each vector on each ray emanating from the central point has a sum of zero.

Here is a general method for drawing a tropical curve $T(p)$ in the plane. Consider any term $\alpha \odot x^i \odot y^j$ in $p$. Now assign the valuation of $\alpha$ to the point $(i,j)$. Compute the convex hull of these points in $\mathbb{R}^3$ and project the upper envelope of the hull into $\mathbb{R}^2$. The tropical curve $T(p)$ is the dual graph of this subdivision. For a detailed look at how this process occurs see *A combinatorial introduction to tropical geometry* by Bernd Sturmfels.

Consider the general quadratic polygon as an example.
\[ p(x,y) = a \odot x^2 \oplus b \odot y^2 \oplus c \odot x \oplus d \odot y \oplus e \odot x \odot y \oplus f. \]

Then the subdivision triangle, assuming that \(a, b, c, d, e, f \in \mathbb{R}\) are general solutions of

\[
2c \geq a + f, \quad 2d \geq b + f, \quad 2e \geq a + b,
\]

is

(figure 1 from Bernd Sturmfels)

**Intersections and Shadow Points**

It should be noted that tropical curves intersect and interpolate like algebraic curves and that Bezout’s Theorem holds. Therefore two curves of degree \(d\) and \(e\) intersect \(d \cdot e\) times. For a complete proof of this see *First Steps in Tropical Geometry* by Richter-Gebert, Sturmfels and Theobald [1].

Another way to think of a tropical curve is the tropicalization of a curve in \(K^2\), the “plane” with coordinates in the field of Puiseux series via the ORD map. Since the ORD map is non-injective it is possible for an intersection in \(R^2\) to come from distinct points in \(K^2\). These intersections are called *shadow points*. Shadow points are points that exist when intersections from \(K^2\) map to the same point in \(R^2\) and therefore in order for Bezout’s Theorem to hold, additional intersections called shadow points form. An exploration of the conditions for finding these points will be what is discussed in the remainder of this paper. A quick note about notation used in this paper, the term *multiplicity of a tropical intersection*, rather than being defined in the sense of the Bezout Theorem is here defined as the number of Puiseux pre-images in \(K^2\). Three cases will be looked at in terms of their shadow points; linear/linear intersections; linear/conic intersections; and conic/conic intersections.

### 3 Linear/Linear Intersection

These intersections are the simplest of intersections so it is best to start with this type. The intersection of two linear functions in tropical geometry must occur at either one point or the intersection may be over a line segment called a *degenerate graph*. Here are two functions which do not intersect in a non-degenerate fashion. On the left is the tropical representation and the function on the right is the same function represented in conventional algebraic terms.

\[
f(x) = (t^2) \odot x \oplus (t^2) \odot y \oplus t \quad \text{or} \quad f(x) = \max(2+x, 2+y, 1)
\]

(Image 2)

\[
g(x) = (t^2) \odot x \oplus y \oplus t^2 \quad \text{or} \quad g(x) = \max(2+x, y, 2)
\]

(Image 3)
Here the two functions intersect at (1,-1). The degenerate graph can also occur with two linear functions. The two functions

\[ f(x) = (t^2) \odot x \oplus (t^2) \odot y \oplus t \quad \text{or} \quad f(x) = \max(2+x, 2+y, 1) \]

(Image 2)

\[ g(x) = (t^2) \odot x \oplus (t^2) \odot y \oplus t^2 \quad \text{or} \quad g(x) = \max(2+x, 2+y, 2) \]

(Image 4)

intersect along the x=y segment beginning at (1,1). A degenerate graph does not imply the existence of a shadow point since it is a series of infinite points that would be counted as intersections. Therefore a degenerate graph is treated as its own case.

The question to be answered is do shadow points exist in an intersection of two tropical lines? The answer is clearly no since in order for two linear functions to have a shadow point they must intersect at least twice. Since all linear function have the same structure, meaning the south, west and northeast orientation, they cannot intersect at more than one point except in the degenerate case. Therefore no shadow points exist in the system of two linear functions intersecting.

The reason for presenting the two linear function case is to show in greater length how linear functions behave in tropical geometry and how certain changes to the Puiseux series will affect the graph of the function. It should be noted that is fairly easy to thus construct a linear function in a specific location. Since all linear functions have the same basic structure, altering the Puiseux series in the equation simply moves the vertex of the line. The Vertex or vertices of a curve is the point(s) where three edges of a curve meet.

It should be noted that the degenerate graph occurs when at least one vertex of either function is located on an edge of the other function.

4 Linear/Conic Intersections

The second type of intersections to look at are the intersection between a linear function and a conic function. For the purpose of clarity the structures of the functions are:

\[ f(x) = a \odot x \oplus b \odot y \oplus c \]

\[ g(x,y) = a \odot x^2 \oplus b \odot y^2 \oplus c \odot x \oplus d \odot y \oplus e \odot x \odot y \oplus f \]

In traditional algebra a conic and line interest projectively in exactly two points. In tropical geometry the Bezout Theorem still holds [1]. The purpose of this section of this paper is to determine if shadow
points exist for all functions, some functions or no functions and what, if any, conditions are required for a shadow point to exist in a linear/conic system. In order to aid in finding shadow points it is useful to look at the Newton polytopes of the function. The Newton polytope of a function is the subdivided triangle that is dual to the graph of the function. With linear functions it is clear that the Newton polytope will always be the triangle with points at \((0, 0), (1, 0)\) and \((0, 1)\). Since the only subdivision of this triangle is the triangle itself the structure of all linear functions is the same, an edge in the southern, western and northeastern direction.

Unlike the linear function, a conic function’s Newton polytope has more than one possible structure. Recall from before that the graph of a function depends on the convex hull of the evaluations of the various Puiseux series. An easy way to picture this is as below.

As you can see the X axis here represents the x variable, the Y axis represents the y variable, and the Z axis represents the valuation of the Puiseux series associated with that point in the XY plane. Thus the point \((1,1)\) in the XY plane corresponds to the xy term in the function and has height equal to the valuation of the Puiseux series in the xy term. Similarly the point \((2,0)\) in the XY plane corresponds to the \(x^2\) term in the function and has height equal to the valuation of the Puiseux series in the \(x^2\) term. Since the Newton polytope is the convex hull of this figure in \(\mathbb{R}^3\) there are many different possible structures for the graph of a conic function. These different possibilities and the conditions, listed as inequalities of the heights, needed to produce them are listed in the chart below.
Empty Triangle

Conditions:
\[ 2H_c < H_a + H_f \]
\[ 2H_d < H_b + H_f \]
\[ 2H_e < H_a + H_b \]

Unbounded Edges:
1 South
1 West
1 Northeast

X-XY connection

Conditions:
\[ 2H_c > H_a + H_f \]
\[ 2H_d < H_b + H_f \]
\[ 2H_e > H_a + H_b \]

Line E-F = Line C-D

Unbounded Edges:
2 South
1 West
1 Northeast
Y-XY connection

Conditions:
\[2H_c < H_a + H_f\]
\[2H_d > H_b + H_f\]
\[2H_e > H_a + H_b\]
Line E-F = Line A-D
Unbounded Edges:
1 South
2 West
1 Northeast

X-XY, Y-XY Connection

Conditions:
\[2H_c > H_a + H_f\]
\[2H_d > H_b + H_f\]
\[2H_e > H_a + H_b\]
C-D-E-F form a plane
Unbounded Edges:
2 South
2 West
2 Northeast

X-Y² Connection

Conditions:
\[2H_c > H_a + H_f\]
\[2H_d < H_b + H_f\]
\[2H_e < H_a + H_b\]
Unbounded Edges:
2 South
1 West
1 Northeast
**X²-Y Connection**

Conditions:
- \(2H_c < H_a + H_f\)
- \(2H_d > H_b + H_f\)
- \(2H_e < H_a + H_b\)

Unbounded Edges:
- 1 South
- 2 West
- 1 Northeast

**X-XY, K-XY Connection**

Conditions:
- \(2H_c > H_a + H_f\)
- \(2H_d < H_b + H_f\)
- \(2H_e > H_a + H_b\)
- Line E-F > Line B-C

Unbounded Edges:
- 1 South
- 1 West
- 1 Northeast

**Y-XY, K-XY Connection**

Conditions:
- \(2H_c > H_a + H_f\)
- \(2H_d < H_b + H_f\)
- \(2H_e > H_a + H_b\)
- Line E-F > Line A-D

Unbounded Edges:
- 1 South
- 1 West
- 1 Northeast
X-Y², X-XY Connection

Conditions:
- $2H_c < H_a + H_f$
- $2H_d < H_b + H_f$
- $2H_e < H_a + H_b$

Line E-F < Line B-C

Unbounded Edges:
- 2 South
- 1 West
- 2 Northeast

X²-Y, Y-XY Connection

Conditions:
- $2H_c < H_a + H_f$
- $2H_d < H_b + H_f$
- $2H_e < H_a + H_b$

Line E-F < Line A-D

Unbounded Edges:
- 1 South
- 2 West
- 2 Northeast

X-XY, Y-XY, X-Y Connection

Conditions:
- $2H_c > H_a + H_f$
- $2H_d > H_b + H_f$
- $2H_e > H_a + H_b$

Line C-D > Line E-F

Unbounded Edges:
- 2 South
- 2 West
- 2 Northeast
**X-XY, Y-XY, K-XY Connection**

Conditions:
\[ 2H_c > H_a + H_f \]
\[ 2H_d > H_b + H_f \]
\[ 2H_e > H_a + H_b \]

Line C-D < Line E-F

Unbounded Edges:
2 South
2 West
2 Northeast

**K-XY Connection**

Conditions:
\[ 2H_c < H_a + H_f \]
\[ 2H_d < H_b + H_f \]
\[ 2H_e > H_a + H_b \]

Unbounded Edges:
1 South
1 West
2 Northeast

**X-Y Connection**

Conditions:
\[ 2H_c > H_a + H_f \]
\[ 2H_d > H_b + H_f \]
\[ 2H_e < H_a + H_b \]

Line A-D = Line B-C

Unbounded Edges:
2 South
2 West
1 Northeast
Technique

Given the constraints on the different types of polytopes the next idea was to find combinations of polytopes that yield a shadow point. The technique here was to define two Puiseux series points that mapped to the same point when the ORD map is applied to it and construct a linear function and a conic function through these two points. The two resulting graphs, if no shadow point exists, should only intersect at one point with multiplicity two. However, the ORD map is not injective but it is a function and therefore two distinct points in $K^2$ can map to the same point in $R^2$. In order to find the linear function and the conic function it is first required to choose the two Puiseux points that under the ORD map are placed on the same point. Once the two points are chosen the linear and the conic functions need to be found. The linear function is very straightforward to find as it is in traditional algebra given two points. Since the goal of finding a shadow points depends on the specific graphs and therefore the
specific polytopes of the function, it is important to notice that the coefficients on the variables in the conic function determine the polytope type that is achieved. Thus given two Puiseux points, the system contains four free variables. For this the four variables were a, b, c, and d from this equation.

\[ g(x,y) = a \odot x^2 \oplus b \odot y^2 \oplus c \odot x \oplus d \odot y \oplus e \odot x \odot y \oplus f \]

Therefore e and f could both be expressed in terms of the four free variables. Since each of the letters a-f correspond to the heights of each of the points in the Newton polytope the heights of e and f could be written as the minus minimum of the valuation of 4 expressions each containing one of the free variables.

Since the polytope depends on the relationship between only 4 variables a, b, c, d, e and f are dependent on those 4 variables, 4 cases are tested each case allowing a different variable expression to be the minimum of the e and f expressions. For example, the first case would state that the a expression in the height of e equation and height of f equation was the minimum. From this the relationship between a, e and f is fixed and the relationship between a and the other three variables is an inequality since a must be the minimum. For figuring out the possible polytopes the minimum free variable expression was set equal to 0 and the heights of e and f are immediately known. Since the polytope is the convex hull and the other three heights are inequalities it is very easy to see what polytopes are possible by simply starting them at their highest possible value (remember height is inversely related to the valuation) and lowering them into all possible convex hull formations. Once all the possible polytopes are found for a given case the general expressions for a, b, c, and d can be defined in order to find that particular polytope. This expression is general and not unique since the valuation map only cares about the highest power of the expression and all lower powers can be general. This process is then repeated for b, c and d all being the minimum expression for e and f. Once completed, the result is a series of equations each of which represents a general polytope type that can be generated. Given these polytope types and the linear function, the two are mapped on the same graph and the hunt for shadow points begins. Each of the polytope types must either intersect the linear function at one and only one point, in which case no shadow point exists, or at more than one point, in which case a shadow does exist. Given the series of restrictions from the previous steps, many equations can be tried that give the same polytopes. A key concern is that some conic functions that are generated are degenerate graph. The goal, therefore, is to find a non-degenerate, conic graph that intersects the linear function in at least two points, one of which being the intersection with multiplicity two and at least one shadow point, or prove that this cannot happen.

**Example**

To begin the example, two sets of Puiseux points are chosen and a line and a conic are constructed through those two points. For this example the initial points are:

\[(t, -1-t), (t+t^2, -1-t-t^2) \mapsto (-1,0)\]
Linear case is already set, only care about conic) From the initial Puiseux points a general conic can be constructed and is one so using the following Maple commands:

\[
\begin{align*}
\textup{> } & C:=a x^2 + b y^2 + c x + d y + e x y + f; \\
& C := a x^2 + b y^2 + c x + d y + e x y + f \\
\textup{> } & C[1] := \text{subs}(x=t, y=-1-t, C); \\
& C_1 := a t^2 + b (-1-t)^2 + c t + d (-1-t) + e t (-1-t) + f \\
\textup{> } & F := \text{solve}(C[1], f); \\
& F := -a t^2 - b - 2 b t - b \ t^2 - c t + d + d t + e t + e \ t^2 \\
\textup{> } & C[2] := a x^2 + b y^2 + c x + d y + e x y + F; \\
& C_2 := a x^2 + b y^2 + c x + d y + e x y - a t^2 - b - 2 b t - b \ t^2 \\
& \quad - c t + d + d t + e t + e \ t^2 \\
\textup{> } & C[3] := \text{subs}(x=t+t^2, y=-1-t-t^2, C[2]); \\
& C_3 := a (t + t^2)^2 + b (-1-t-t^2)^2 + c (t + t^2) + d (-1-t-t^2) \\
& \quad - a t^2 - b - 2 b t - b \ t^2 \\
& \quad - c t + d + d t + e t + e \ t^2 \\
\textup{> } & E := \text{solve}(C[3], e); \\
& E := 2 a t + a t^2 + 2 b + 2 b t + b \ t^2 + c - d \ \frac{1}{1 + 2 t + t^2} \\
\textup{> } & \text{Final} := a x^2 + b y^2 + c x + d y + E x y + F; \\
& \text{Final} := a x^2 + b y^2 + c x + d y \\
& \quad + \frac{(2 a t + a t^2 + 2 b + 2 b t + b \ t^2 + c - d) x y}{1 + 2 t + t^2} \\
& \quad - a t^2 - b - 2 b t - b \ t^2 - c t + d + d t + e t + e \ t^2 \\
\textup{> } & e := E; \\
& e := 2 a t + a t^2 + 2 b + 2 b t + b \ t^2 + c - d \ \frac{1}{1 + 2 t + t^2} \\
\textup{> } & \text{Final}; \\
& a x^2 + b y^2 + c x + d y \\
& \quad + \frac{(2 a t + a t^2 + 2 b + 2 b t + b \ t^2 + c - d) x y}{1 + 2 t + t^2} \\
& \quad - 2 b t - b \ t^2 - c t + d + d t \\
& \quad + \frac{(2 a t + a t^2 + 2 b + 2 b t + b \ t^2 + c - d) t}{1 + 2 t + t^2} \\
& \quad + \frac{(2 a t + a t^2 + 2 b + 2 b t + b \ t^2 + c - d) \ t^2}{1 + 2 t + t^2}
This portion of Maple command takes the general conic equation and substitutes in the two Puiseux points. This equation can be rewritten:

\[
ax^2 + by^2 + cx + dy + \left(2at + a t^2 + 2b + 2bt + b t^2 + c - d\right)xy \quad \frac{1}{1 + 2t + t^2}
\]

\[
+ \frac{1}{1 + 2t + t^2} \left(-b + d + a t^2 - 2bt - 2bt^2 + 2dt + dt^3 + 2dt^2 - ct^3 - ct^2 - bt^3 + at^3\right)
\]

From this equation, the ORD map can be applied to the e and f terms. When the ORD map is applied the valuation of the e and f terms are determined by the minimum of the a through d terms. Therefore the height of e and f are determined by the following equations:

\[
H_e = -\min(a + 1, b, c, d)
\]

\[
H_f = -\min(a + 2, b, c + 2, d)
\]

Since the height of e and f is dependent on a minimum function, the next step is to fix a, b, c, and d as the minimums in a series of different cases and explore what polytopes are possible.

The case where a is the minimum (for the sake of example we will set it to zero):

\[
H_a = 0 \quad H_e = -1 \quad H_f = -2
\]

\[
H_b < H_a - 2 \quad H_c < H_a - 1 \quad H_d < H_a - 2
\]

Given these conditions it is easy to determine which polytopes are possible in this case. Those polytopes are:
Now we simply repeat this procedure for the other cases starting with b minimum.

\[ H_b = 0 \quad | \quad H_e = 0 \quad | \quad H_f = 0 \]

\[ H_a < H_b + 2 \quad | \quad H_c < H_b + 2 \quad | \quad H_d < H_b \]

Now we apply the same to c minimum case.

\[ H_c = 0 \quad | \quad H_e = 0 \quad | \quad H_f = -2 \]

\[ H_a < H_c \quad | \quad H_b < H_c - 2 \quad | \quad H_d < H_c - 2 \]
Now we apply the same to the minimum case.

\[
H_d = 0 \quad | \quad H_e = 0 \quad | \quad H_f = 0 \\
H_a < H_d + 2 \quad | \quad H_b < H_d \quad | \quad H_c < H_d + 2
\]

From each of the different polytopes the next step is to find a non-degenerate tropical conic graph. In order to do that recall that the vertex of a tropical graph is determined by the three vertices of the polytope section around the vertex being equal and minimal.

For the c minimum case we give the initial conditions for a through d such:

\[
a = t, \ b = t^3, \ c = 1, \ d = t^3
\]

\[
eval(a) = -1 \quad | \quad eval(b) = -3 \quad | \quad eval(c) = 0 \quad | \quad eval(d) = -3 \quad | \quad eval(e) = 0 \quad | \quad eval(f) = 2
\]

From this information it is easy to determine both the graph and the vertices of the graph. The Newton polytope is
and the vertices can be calculated as such:

\[
\text{eval}(e) + x + y = \text{eval}(b) + 2y = \text{eval}(f) \quad | \quad x + y = -3 + 2y = -2
\]

\((-5/2, 1/2)\)

\[
\text{eval}(e) + x + y = \text{eval}(c) + x = \text{eval}(f) \quad | \quad x + y = x = -2
\]

\((-2, 0)\)

\[
\text{eval}(e) + x + y = \text{eval}(c) + x = \text{eval}(a) + 2x \quad | \quad x + y = x = -1 + 2x
\]

\((-1, 0)\)

The last equation gives a vertex that lands exactly on the point given by the initial conditions thus this is a degenerate example. Since the vertex is dictated by the final equation and that when c is the minimum those three values exist in a fixed ratio, \(y=0\) for all possible c minimum cases. Therefore every tropical graph with the c minimum case is degenerate. Thus no shadow points exist in for the c minimum case.

For the d minimum case

\[a = t^2, \ b = t, \ c = t^{-1}, \ d = 1\]

\[
\text{eval}(a) = -2 \quad | \quad \text{eval}(b) = -1 \quad | \quad \text{eval}(c) = 1 \quad | \quad \text{eval}(d) = 0 \quad | \quad \text{eval}(e) = 0 \quad | \quad \text{eval}(f) = 0
\]

With Newton polytope:
The second equation leads to a degenerate graph since by fixing that vertex at (-1, 0) there is a line segment which lies on the linear graph. Given that c, d and f always exist in the same ratio then any polytope which contains the C-D connection will yield a degenerate graph. This does not disqualify all D minimum polytopes and it is necessary to look at an additional D connection case to determine what condition may enforce that D minimum is always a degenerate case.

\[
\begin{align*}
\text{eval}(a) + 2x &= \text{eval}(c) + x = \text{eval}(e) + x + y \quad | \quad -2 + 2x = 1 + x = x + y \\
(3, 1) \\
\text{eval}(c) + x &= \text{eval}(d) + y = \text{eval}(f) \quad | \quad 1 + x = y = 0 \\
(-1, 0) \\
\text{eval}(c) + x &= \text{eval}(d) + y = \text{eval}(e) + x + y \quad | \quad 1 + x = y = x + y \\
(0, 1)
\end{align*}
\]

Given that \( a = t^{-1}, \ b = t^{2}, \ c = 1, \ d = 1 \)

\[
\begin{align*}
\text{eval}(a) &= 1 \quad | \quad \text{eval}(b) = -2 \quad | \quad \text{eval}(c) = 0 \quad | \quad \text{eval}(d) = 0 \quad | \quad \text{eval}(e) = 0 \quad | \quad \text{eval}(f) = 0
\end{align*}
\]
eval(a) + 2x = eval(f) = eval(d) + y | 1 + 2x = 0 = y

(-1/2, 0)

eval(a) + 2x = eval(e) + x + y = eval(d) + y | 1 + 2x = x + y = y

(0, 1)

eval(d) + y = eval(e) + x + y = eval(b) 2y | y = x + y = -2 + 2y

(0, 2)

Notice that this case does not have a vertex of the tropical graph on the initial point. However, this example is still degenerate since there is a line segment from (-1/2, 0) that overlays the linear graph. Therefore this leads to a degenerate graph. Additionally it should be noted that since this is D minimum graph there will always be an equation containing the valuation of f and the valuation of d. We notice, therefore, that all D minimum graphs contain a vertex with a y=0 component and thus each D minimum graph is a degenerate graph due to the line segment overlapping with the linear graph. Therefore the D minimum case cannot have any shadow points.

Now we can look at the B minimum case.

\[ a = t^3, \ b = 1, \ c = t, \ d = t \]

\[ \text{eval}(a) = -3 \ | \ \text{eval}(b) = 0 \ | \ \text{eval}(c) = -1 \ | \ \text{eval}(d) = -1 \ | \ \text{eval}(e) = 0 \ | \ \text{eval}(f) = 0 \]

With Newton polytope:
\[ \text{eval}(a) + 2x = \text{eval}(c) + x = \text{eval}(e) + x + y \ | \ -3 + 2x = -1 + x = x + y \]
\[ (2, -1) \]

\[ \text{eval}(c) + x = \text{eval}(e) + x + y = \text{eval}(f) \ | \ -1 + x = x + y = 0 \]
\[ (1, -1) \]

\[ \text{eval}(f) = \text{eval}(b) + 2y = \text{eval}(e) + x + y \ | \ 0 = 2y = x + y \]
\[ (0, 0) \]

For this specific example we find again that the vertex, while being the initial point, still causes a degenerate case for the same reason as before, the y=0 vertex. Since f and b have the same height, the height of d must be less than the line between them. Therefore the last equality must occur and if it does then there is always a y=0 vertex. Therefore the b minimum case is always degenerate.

Finally we look at the a minimum case.

\[ a = 1, \ b = t^6, \ c = t^2, \ d = t^3 \]

\[ \text{eval}(a) = 0 \ | \ \text{eval}(b) = -6 \ | \ \text{eval}(c) = -2 \ | \ \text{eval}(d) = -3 \ | \ \text{eval}(e) = -1 \ | \ \text{eval}(f) = -2 \]

With Newton polytope:
\[ \text{eval}(e) + x + y = \text{eval}(d) + y = \text{eval}(b) + 2y \quad | \quad -1 + x + y = -3 + y = -6 + 2y \]
\[ (-2, 3) \]

\[ \text{eval}(e) + x + y = \text{eval}(d) + y = \text{eval}(f) \quad | \quad -1 + x + y = -3 + y = -2 \]
\[ (-2, 1) \]

\[ \text{eval}(f) = \text{eval}(a) + 2x = \text{eval}(e) + x + y \quad | \quad -2 = 2x = -1 + x + y \]
\[ (-1, 0) \]

As in the other cases we find that a vertex lands on our initial point making this a degenerate graph.

Since all of the minimum cases are degenerate it is not possible for a shadow point to exist for this specific example. Further study could attempt to generalize the initial conditions in such a way that the intersection point is preserved but the initial conditions are made as general as possible.

### 5 Conic/Conic Intersection

Similar to the linear/conic intersections to goal is to find a non-degenerate conic graph that intersects the other conic function in one more point then initially expected. For example, if the two conics should intersect in one point with multiplicity four than the goal is to find that the two functions intersect in two points, one with multiplicity four and the other point being a shadow point. Since the two conics intersect in four points, or less with multiplicity greater than one, four initial points are chosen rather than two. Additionally choosing where these points map to under the ORD map is different. In the two conic intersection you can choose all four points to map to the same point, three points to one point and then one point mapping elsewhere, two points mapping to one point and two points mapping to a different point, or two points to one point and the other two points to two different points.
In order to find a conic through the four chosen points the following method is used. First the four points are chosen. Two lines are then taken through the four points and they are multiplied together to form a conic. For example let \((x_n, y_n)\) be the \(n^{th}\) point, then the equations for two conics would look like:

\[
C_1 = [(y_2 - y_1) \cdot (x - x_1)/(x_2 - x_1) + y_1 - y] \cdot [(y_4 - y_3) \cdot (x - x_3)/(x_4 - x_3) + y_3 - y]
\]

\[
C_2 = [(y_3 - y_1) \cdot (x - x_1)/(x_3 - x_1) + y_1 - y] \cdot [(y_4 - y_2) \cdot (x - x_2)/(x_4 - x_2) + y_2 - y]
\]

From these two conics the general conic through these four points can be found by taking all possible linear combinations of these two conics. This equation looks like:

\[
C_{\text{gen}} = \alpha \cdot C_1 + \beta \cdot C_2
\]

Using the general equation and the method from the linear/conic intersection it is possible to work out the heights for the Newton polytope. A noticeable difference between the two is the former represented \(e\) and \(f\) in terms of \(a, b, c,\) and \(d\) the conic/conic heights are all represented in terms of alpha and beta. In order to do this somewhat long and tedious calculation I wrote a Maple program the does the calculations for me and outputs the four lines through the four points used to create the conics, the general conic, and the height of each in terms of alpha and beta as well as gives the portion of the equation that applies to that term. This is a relatively simple calculation despite the fact that some terms are quite lengthy. This is because the heights are dependent on the valuation of the term and not on the entire term and thus making the heights of the polytope a min system with the first minimum being in terms of alpha and the second minimum being in terms of beta.

Here I will break down the program and explain what is going on.
This section of the procedure shows the inputs as being the x and y coordinates (each being a Puiseux series) for the four points and defines the various variables being used as well as shows how the functions through the four points and the general conic are found as stated above. This also prepares the section on Equations and Heights.

This section of the code removes all the components of the equation that are not $x^2$. It does this by taking two derivatives of the equation with respect to x and multiplies by one half. This remaining equation is then fed into the X2run procedure which I will talk about now and then return to the Conic procedure.

The purpose of the X2run procedure was to have a program which found the valuation of an expression that is a minimum of an alpha term and a beta term. This program could then be called by the Conic procedure to do this calculation for all the variable combinations in the general conic function.
This portion of code has inputs of an equation in terms of alpha and beta as well as a number which helps keep track of which height is being worked on, more on this later, as well as defines the variables being used in the equation. The second portion focuses on the part of the input equation that has an alpha component by taking the derivative with respect to alpha. If this part is not equal to zero, the equation has an alpha component, then the variable Alphalead is defined as the lead term of the alpha portion of the equation written in series notation. Finally the op command looks at the exponent on the lead term. This process chooses the highest exponent in the alpha portion of the equation which is the valuation of the alpha portion of the equation.

```plaintext
X2run := proc(X2, f)
  local AlphaX2, BetaX2, K, L, a, b, X2alphacon, X2betacon, g,
          Alphalead, Betalead;
  AlphaX2 := diff(X2, alpha);
  if AlphaX2 <> 0 then
    K := 1;
    Alphalead := series('leadterm'(AlphaX2), t);
    X2alphacon := op(2, Alphalead)
  else
    K := 0; X2alphacon := 0
  end if;

BetaX2 := diff(X2, beta); if BetaX2 <> 0 then
  L := 1;
  Betalead := series('leadterm'(BetaX2), t);
  X2betacon := op(2, Betalead)
else
  L := 0; X2betacon := 0 end if;
```

This does the exact same thing as the alpha section but does it for beta instead.
if \( f = 1 \) then \\
\( g := Ha \) \\
elif \( f = 2 \) then \\
\( g := Hb \) \\
elif \( f = 3 \) then \\
\( g := Hc \) \\
elif \( f = 4 \) then \\
\( g := Hd \) \\
elif \( f = 5 \) then \\
\( g := He \) \\
elif \( f = 6 \) then \\
\( g := Hf \) \\
else \\
\( g := X \) \\
end if; \\
\textit{print}(- g = (alpha + X2alphacon) * K); \\
\textit{print}(- g = (beta + X2betacon) * L) \\
end proc

This section, using the second input, determines which height is being looked and prints that height and the valuation of the equation in terms of alpha and beta. All of the heights are listed as the minimum of these two components. Once this section is complete the Conic procedure continues.

\[ \text{if } Y2 <> 0 \text{ then } X2run(Y2, 2) \text{ end if;} \]
\[ XY := expand\left(\text{diff}\left(\text{diff}\left(C3, x, y\right)\right)\right); \]
\[ X1 := expand\left(\text{diff}\left(C3 - X2*x^2 - XY*x*y, x\right)\right); \]
\textit{print}(X1); \\
\textit{if } X1 <> 0 \text{ then } X2run(X1, 3) \text{ end if;} \\
\[ Y1 := expand\left(\text{diff}\left(C3 - Y2*y^2 - XY*x*y, y\right)\right); \]
\textit{print}(Y1); \\
\textit{if } Y1 <> 0 \text{ then } X2run(Y1, 4) \text{ end if;} \\
\textit{print}(XY); \\
\textit{if } XY <> 0 \text{ then } X2run(XY, 5) \text{ end if;} \\
\[ Con := expand\left(C3 - X2*x^2 - Y2*y^2 - XY*x*y - X1 \right. \]
\[ \left.*x - Y1*y\right); \]
\textit{print}(Con); \\
\textit{if } Con <> 0 \text{ then } X2run(Con, 6) \text{ end if} \\
end proc
This portion of code in the Conic procedure continues to call the X2run procedure for all the different combinations of variables. Notice that in order for the derivative method to work, the variables with higher powers must be removed before taking the derivative. The X2run procedure is run for the $y^2$, $x$, $y$, $xy$ and constant component of the equation giving the heights of all of these as the minimum of an alpha term and a beta term.

From these heights the polytope can be constructed and the dual of the convex hull of the polytope shows the tropical graph. The general equation can also be plugged into Cingular which would give immediately the graph and vertices of the equation. From these height equations the next step is to determine what graphs are possible and from that information determine if a non-degenerate intersection can occur. For example, in the case where all four points have valuations at one point all that would be required to find would be to find two tropical graphs that intersect in a non degenerate way in two points. This process is repeated for all four types of mappings for the initial four Puiseux series points.

**Long Method Example:**

**3 Puiseux points to 1 tropical point and 1 Puiseux point to a different tropical point:**

The purpose of this section is to show the procedure for looking for the different polytope types for specific initial points. Initial points:

\[(t, 1), (t+t^2, 1), (t+t^2, 1+t) \rightarrow (-1,0)\]

\[(1/t^2, t) \rightarrow (2, -1)\]

Plugging these values into procedure the result is:

Here is the height information rewritten:

\[H_a = -(eval(\beta) + 1)\]
\[H_b = -\min(eval(\alpha), eval(\beta))\]
\[H_c = -\min(eval(\alpha) + 2, eval(\beta) - 1)\]
\[H_d = -\min(eval(\alpha), eval(\beta))\]
\[H_e = -\min(eval(\alpha) + 2, eval(\beta) - 1)\]
\[H_f = -\text{eval}(\alpha)\]

Recall the three inequalities that determine the different polytopes. Let’s revisit those here:
Now each possible combination of inequalities results in different polytope so looking at this equation allows us to determine whether a certain polytope type, and its dual graph, occurs for the initial conditions. Three connection polytope:

\[
2H_c > H_a + H_f \quad | \quad 2H_d > H_b + H_f \quad | \quad 2H_e > H_a + H_b
\]

Substituting the values from the height equations gives the following system:

\[
-2\min(\text{eval}(\alpha)+2, \text{eval}(\beta)-1) > -(\text{eval}(\beta)+1) - \text{eval}(\alpha)
\]

\[
-2\min(\text{eval}(\alpha), \text{eval}(\beta)) > -\min(\text{eval}(\alpha), \text{eval}(\beta)) - \text{eval}(\alpha)
\]

\[
-2\min(\text{eval}(\alpha)+2, \text{eval}(\beta)-1) > -(\text{eval}(\beta)+1) - \min(\text{eval}(\alpha), \text{eval}(\beta))
\]

Or rewriting:

\[
2\min(\text{eval}(\alpha)+2, \text{eval}(\beta)-1) < (\text{eval}(\beta)+1) + \text{eval}(\alpha)
\]

\[
2\min(\text{eval}(\alpha), \text{eval}(\beta)) < \min(\text{eval}(\alpha), \text{eval}(\beta)) + \text{eval}(\alpha)
\]

\[
2\min(\text{eval}(\alpha)+2, \text{eval}(\beta)-1) < (\text{eval}(\beta)+1) + \min(\text{eval}(\alpha), \text{eval}(\beta))
\]

Now we look at the look at all the possible minimums in each of the above equations and from that develop the different possible cases that could occur:

Case 1: \( \text{eval}(\alpha) < \text{eval}(\alpha) + 3 < \text{eval}(\beta) \)

Case 2: \( \text{eval}(\alpha) < \text{eval}(\beta) < \text{eval}(\alpha) + 3 \)
Case 3: \( \text{eval}(\beta) < \text{eval}(\alpha) < \text{eval}(\alpha) + 3 \)

Since these cases do not depend on the polytope but rather on the initial conditions, these cases apply to all polytopes with these initial conditions. Next we apply each of these 3 cases to the system of equations. If a contradiction occurs then that case cannot happen. If all three cases are contradictions then the polytopes in that are associated with that set of inequalities cannot occur ever.

Case 1

\[
\begin{align*}
2\text{eval}(\alpha) + 4 &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 \\
\text{eval}(\alpha) + 3 &< \text{eval}(\beta) \\
2\text{eval}(\alpha) &< \text{eval}(\alpha) + \text{eval}(\alpha) \\
0 &< 0 \\
2\text{eval}(\alpha) + 4 &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 \\
\text{eval}(\alpha) + 3 &< \text{eval}(\beta)
\end{align*}
\]

Since the second equation can never occur, this case can never give this polytope. Now we will try the second case.

Case 2:

\[
\begin{align*}
2\text{eval}(\beta) - 2 &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 \\
\text{eval}(\beta) - 3 &< \text{eval}(\alpha) \\
2\text{eval}(\alpha) &< \text{eval}(\alpha) + \text{eval}(\alpha) \\
0 &< 0 \\
2\text{eval}(\beta) - 2 &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 \\
\text{eval}(\beta) - 3 &< \text{eval}(\alpha)
\end{align*}
\]

Again we see that the second equation can never occur and thus this case will never give this polytope.
Case 3:

\[ 2\text{eval}(\beta) - 2 < \text{eval}(\beta) + \text{eval}(\alpha) + 1 \]

\[ \text{eval}(\beta) - 3 < \text{eval}(\alpha) \]

\[ 2\text{eval}(\beta) < \text{eval}(\beta) + \text{eval}(\alpha) \]

\[ \text{eval}(\beta) < \text{eval}(\alpha) \]

\[ 2\text{eval}(\beta) - 2 < \text{eval}(\beta) + \text{eval}(\beta)) + 1 \]

\[-2 < 1\]

Here we have a case where all 3 conditions hold true and therefore when the conditions hold true on the inequalities AND on the specific case, the polytope occurs.

Now it is time to look at another polytope for the same initial conditions, the same technique is used as above. Here we will look at this polytope:

\[ 2H_c \leq H_a + H_f \]
\[ 2H_d \leq H_b + H_f \]
\[ 2H_e \leq H_a + H_b \]

\[ 2\min(\text{eval}(\alpha)+2, \text{eval}(\beta)-1) \geq (\text{eval}(\beta)+1) + \text{eval}(\alpha) \]

\[ 2\min(\text{eval}(\alpha), \text{eval}(\beta)) \geq \min(\text{eval}(\alpha), \text{eval}(\beta)) + \text{eval}(\alpha) \]

\[ 2\min(\text{eval}(\alpha)+2, \text{eval}(\beta)-1) \geq (\text{eval}(\beta)+1) + \min(\text{eval}(\alpha), \text{eval}(\beta)) \]
It is important to point out that if the points on the midpoints of the outside of the polytope lie in line with the two adjacent corners they, since we are looking at the convex hull, act as though they are below the line connecting the two. Now we can look at the various cases, notice that these cases are identical to the cases before but the signs have been changed.

Case 1

\[
2\text{eval}(\alpha) + 4 > \text{eval}(\beta) + \text{eval}(\alpha) + 1
\]

\[
\text{eval}(\alpha) + 3 > \text{eval}(\beta)
\]

\[
2\text{eval}(\alpha) > \text{eval}(\alpha) + \text{eval}(\alpha)
\]

\[
0 > 0
\]

\[
2\text{eval}(\alpha) + 4 > \text{eval}(\beta) + \text{eval}(\alpha) + 1
\]

\[
\text{eval}(\alpha) + 3 > \text{eval}(\beta)
\]

As before, the second equation causes this situation to never occur. Since Case 1 does not work in any direction of the inequality, it is safe to not consider any future case 1 situations contributing to the polytope.

Case 2:

\[
2\text{eval}(\beta) - 2 > \text{eval}(\beta) + \text{eval}(\alpha) + 1
\]

\[
\text{eval}(\beta) - 3 > \text{eval}(\alpha)
\]

\[
2\text{eval}(\alpha) > \text{eval}(\alpha) + \text{eval}(\alpha)
\]

\[
0 > 0
\]

\[
2\text{eval}(\beta) - 2 > \text{eval}(\beta) + \text{eval}(\alpha) + 1
\]

\[
\text{eval}(\beta) - 3 > \text{eval}(\alpha)
\]

As with case 1, case 2 fails in both directions of the inequality and therefore can be disregarded when looking at future polytopes for these initial conditions.

Case 3:

\[
2\text{eval}(\beta) - 2 > \text{eval}(\beta) + \text{eval}(\alpha) + 1
\]
\[
\begin{align*}
\text{eval(\(\beta\)) - 3} & > \text{eval(\(\alpha\))} \\
2\text{eval(\(\beta\))} & > \text{eval(\(\beta\)) + eval(\(\alpha\))} \\
\text{eval(\(\beta\))} & > \text{eval(\(\alpha\))} \\
2\text{eval(\(\beta\))} - 2 & > \text{eval(\(\beta\)) + eval(\(\beta\))) + 1} \\
-2 & > 1
\end{align*}
\]

Unlike the previous example, case 3 does not hold when the inequality is in the given direction. Therefore this polytope cannot occur. Additionally, since case 1 and case 2 never work then the only case to consider is case 3 and that is only when the 3\textsuperscript{rd} inequality looks like:

\[
2H_e > H_a + H_b
\]

The signs on the other two inequalities can go either way since all they do is help define the relationship between alpha and beta and do not affect whether or not the inequality is numerically correct. The next step is to explore other combinations of the inequalities:

1) \(2H_c < H_a + H_f \mid 2H_d < H_b + H_f \mid 2H_e > H_a + H_b\)
2) \(2H_c > H_a + H_f \mid 2H_d < H_b + H_f \mid 2H_e > H_a + H_b\)
3) \(2H_c < H_a + H_f \mid 2H_d > H_b + H_f \mid 2H_e > H_a + H_b\)

Let's look at case I first:

\[
\begin{align*}
2\text{eval(\(\beta\))} - 2 & > \text{eval(\(\beta\)) + eval(\(\alpha\)) + 1} \\
\text{eval(\(\beta\)) - 3} & > \text{eval(\(\alpha\))} \\
2\text{eval(\(\beta\))} & > \text{eval(\(\beta\)) + eval(\(\alpha\))} \\
\text{eval(\(\beta\))} & > \text{eval(\(\alpha\))} \\
2\text{eval(\(\beta\))} - 2 & < \text{eval(\(\beta\)) + eval(\(\beta\))) + 1} \\
-2 & < 1
\end{align*}
\]

At first glance this appears to be a perfectly fine solution since the numerical inequality holds but there is one other problem that needs to be considered. The second equation is directly opposite of the initial condition for case 3 which has the inequality going in the other direction. Since the end result is a direct
contradiction to the initial condition then polytopes of this structure cannot be made by these initial condition either.

Now on to case II:

\[
\begin{align*}
2 \text{eval}(\beta) - 2 &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 \\
\text{eval}(\beta) - 3 &< \text{eval}(\alpha) \\
2 \text{eval}(\beta) &> \text{eval}(\beta) + \text{eval}(\alpha) \\
\text{eval}(\beta) &> \text{eval}(\alpha) \\
2 \text{eval}(\beta) - 2 &< \text{eval}(\beta) + \text{eval}(\beta) + 1 \\
-2 &< 1
\end{align*}
\]

Case II fails for the same reasons as case I. Now we look at case III:

\[
\begin{align*}
2 \text{eval}(\beta) - 2 &> \text{eval}(\beta) + \text{eval}(\alpha) + 1 \\
\text{eval}(\beta) - 3 &> \text{eval}(\alpha) \\
2 \text{eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\alpha) \\
\text{eval}(\beta) &< \text{eval}(\alpha) \\
2 \text{eval}(\beta) - 2 &< \text{eval}(\beta) + \text{eval}(\beta) + 1 \\
-2 &< 1
\end{align*}
\]

Here the second equation is a direct contradiction to the first equation and therefore this cannot be a valid polytope either. Since all possible options have been exhausted the conclusion is that the only polytope that passes through the four initial conditions is of the form:
This can be shown graphically as well. Consider the plane with the valuation of alpha on the horizontal axis and the valuation of beta on the vertical axis. The graph can be divided into areas that show, given the initial alpha and beta, what the polytope map would look like but not give the vertices. It is unclear what polytope would be represented in the upper portion of the graph.

(insert α β graph)

2 Puiseux points to 1 tropical point and 2 Puiseux points to a different tropical point:

This is very similar to the previous section except there are two points with multiplicity two. The initial points are:

\[(t,1), (t+t^2, 1) \rightarrow (-1, 0)\]
\[(1, t), (2, t+t^2) \rightarrow (0, -1).\]

Plugging these points into the Maple procedure yields the following heights.

\[
H_a = -\text{eval}(\beta)
\]
\[
H_b = -\text{min}(\text{eval}(\alpha), \text{eval}(\beta))
\]
\[
H_c = -\text{min}(\text{eval}(\alpha)+2, \text{eval}(\beta))
\]
\[
H_d = -\text{min}(\text{eval}(\alpha), \text{eval}(\beta))
\]
\[
H_e = -\text{min}(\text{eval}(\alpha)+2, \text{eval}(\beta))
\]
\[
H_f = -\text{min}(\text{eval}(\alpha)+1, \text{eval}(\beta))
\]

Continuing in the same fashion as before, first we will look at the 3 polytope type and then the empty polytope.

\[
2\text{min}(\text{eval}(\alpha)+2, \text{eval}(\beta)) < \text{eval}(\beta) + \text{min}(\text{eval}(\alpha)+1, \text{eval}(\beta))
\]
\[
2\text{min}(\text{eval}(\alpha), \text{eval}(\beta)) < \text{min}(\text{eval}(\alpha), \text{eval}(\beta)) + \text{min}(\text{eval}(\alpha)+1, \text{eval}(\beta))
\]
\[
2\text{min}(\text{eval}(\alpha)+2, \text{eval}(\beta)) < \text{eval}(\beta) + \text{min}(\text{eval}(\alpha), \text{eval}(\beta))
\]

Now we can look at the various cases, notice that these cases are not the same as the previous sections cases since the initial conditions have changed. Note that since Puiseux series are not required to have integer powers that case 2 is a valid case.

Case 1: \(\text{eval}(\alpha) + 2 < \text{eval}(\beta)\)
Case 2: \[ \text{eval}(\alpha) + 1 < \text{eval}(\beta) < \text{eval}(\alpha) + 2 \]

Case 3: \[ \text{eval}(\alpha) < \text{eval}(\beta) < \text{eval}(\alpha) + 1 \]

Case 4: \[ \text{eval}(\beta) < \text{eval}(\alpha) \]

As before these cases do not depend on the polytope but rather on the initial conditions, these cases apply to all polytopes with these initial conditions. Next we apply each of these 3 cases to the system of equations.

\[
\begin{align*}
2\text{eval}(\alpha) + 4 &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 & \text{eval}(\alpha) &< \text{eval}(\beta) - 3 \\
2\text{eval}(\alpha) &< \text{eval}(\alpha) + \text{eval}(\alpha) + 1 & 0 &< 1 \\
2\text{eval}(\alpha) + 4 &< \text{eval}(\beta) + \text{eval}(\alpha) & \text{eval}(\alpha) &< \text{eval}(\beta) - 4
\end{align*}
\]

This case holds since both the numeric and the conditions on alpha and beta do not contradict the case conditions. We take the in order to insure that all three equations hold we say that this polytope can occur, for this case, when \( \text{eval}(\alpha) < \text{eval}(\beta) - 4 \).

Now we will look at the second case.

\[
\begin{align*}
2\text{eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\alpha) + 1 & \text{eval}(\beta) &< \text{eval}(\alpha) + 1 \\
2\text{eval}(\alpha) &< \text{eval}(\alpha) + \text{eval}(\alpha) + 1 & 0 &< 1 \\
2\text{eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\alpha) & \text{eval}(\beta) &< \text{eval}(\alpha)
\end{align*}
\]

This case does not hold true since both the first and third equation contradict the case two conditions. Now we will look at the third case.

\[
\begin{align*}
2 \text{ eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\beta) & 0 &< 0 \\
2 \text{ eval}(\alpha) &< \text{eval}(\alpha) + \text{eval}(\beta) & \text{eval}(\alpha) &< \text{eval}(\beta) \\
2 \text{ eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\alpha) & \text{eval}(\beta) &< \text{eval}(\alpha)
\end{align*}
\]

Since the second and third equations are opposite inequalities it is impossible for case three hold here. Finally we can look at case four for this polytope type.

\[
\begin{align*}
2 \text{ eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\beta) & 0 &< 0 \\
2 \text{ eval}(\beta) &< \text{eval}(\beta) + \text{eval}(\beta) & 0 &< 0
\end{align*}
\]
2 eval(β) < eval(β) + eval(β)  |  0 < 0

Here we have another case that cannot occur since these inequalities are not true.

After looking at all the different cases this polytope occurs when eval(α) < eval(β) - 4. Now we can look at the opposite initial inequalities which give the empty polytope.

\[
2 \min(\text{eval}(\alpha)+2, \text{eval}(\beta)) \geq \text{eval}(\beta) + \min(\text{eval}(\alpha)+1, \text{eval}(\beta))
\]

\[
2 \min(\text{eval}(\alpha), \text{eval}(\beta)) \geq \min(\text{eval}(\alpha), \text{eval}(\beta)) + \min(\text{eval}(\alpha)+1, \text{eval}(\beta))
\]

\[
2 \min(\text{eval}(\alpha)+2, \text{eval}(\beta)) \geq \text{eval}(\beta) + \min(\text{eval}(\alpha), \text{eval}(\beta))
\]

Now the four cases can be looked at as before, starting with case 1:

\[
2 \text{eval}(\alpha)+ 4 \geq \text{eval}(\beta) + \text{eval}(\alpha)+ 1 \quad | \quad \text{eval}(\alpha) \geq \text{eval}(\beta) - 3
\]

\[
2 \text{eval}(\alpha) \geq \text{eval}(\alpha) + \text{eval}(\alpha)+1 \quad | \quad 0 \geq 1
\]

\[
2 \text{eval}(\alpha)+ 4 \geq \text{eval}(\beta) + \text{eval}(\alpha) \quad | \quad \text{eval}(\alpha) \geq \text{eval}(\beta) - 4
\]

This case cannot occur due to the second equation not holding true but the other two equations are fine. Therefore the only time that case one can hold is when the height of d is greater than the height of the line adjoining b and f at its midpoint. Now we can look at case two:

\[
2 \text{eval}(\beta)) \geq \text{eval}(\beta) + \text{eval}(\alpha)+ 1 \quad | \quad \text{eval}(\beta) \geq \text{eval}(\alpha)+ 1
\]

\[
2 \text{eval}(\alpha) \geq \text{eval}(\alpha) + \text{eval}(\alpha)+1 \quad | \quad 0 \geq 1
\]

\[
2 \text{eval}(\beta)) \geq \text{eval}(\beta) + \text{eval}(\alpha) \quad | \quad \text{eval}(\beta) \geq \text{eval}(\alpha)
\]

Here as seen before, the second inequality prevents this case from occurring. The other two equations do hold without problem. Now the third case:

\[
2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\beta) \quad | \quad 0 \geq 0
\]

\[
2 \text{eval}(\alpha) \geq \text{eval}(\alpha) + \text{eval}(\beta) \quad | \quad \text{eval}(\alpha) \geq \text{eval}(\beta)
\]

\[
2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\alpha) \quad | \quad \text{eval}(\beta) \geq \text{eval}(\alpha)
\]

Observe that the valuation of alpha must be strictly less than the valuation of beta and therefore this case cannot occur. Finally case four can be observed:

\[
2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\beta) \quad | \quad 0 \geq 0
\]
\[ 2 \text{eval}(β) \geq \text{eval}(β) + \text{eval}(β) \quad | \quad 0 \geq 0 \]

\[ 2 \text{eval}(β) \geq \text{eval}(β) + \text{eval}(β) \quad | \quad 0 \geq 0 \]

This case does hold true. Therefore whenever the valuation of beta is less than the valuation of alpha a structure of this form is observed.

Now that the two most general cases have been found it is time to look at the more particular cases. Something to notice is that each of the four cases yields three equations that must be taken together except the direction of the inequality can change freely.

**Case 1:** \( \text{eval}(α) + 2 < \text{eval}(β) \)

\[ 2\text{eval}(α) + 4 \geq \text{eval}(β) + \text{eval}(α) + 1 \quad | \quad \text{eval}(α) \geq \text{eval}(β) - 3 \]

\[ 2\text{eval}(α) < \text{eval}(α) + \text{eval}(α) + 1 \quad | \quad 0 < 1 \]

\[ 2\text{eval}(α) + 4 \geq \text{eval}(β) + \text{eval}(α) \quad | \quad \text{eval}(α) \geq \text{eval}(β) - 4 \]

As we can see, the second equation is a fixed inequality since otherwise it is a numerical contradiction. It is also clear from earlier analysis that the other two inequalities can go either way and still hold true under given conditions. From this we get the following polytopes:

\( (\text{eval}(α) < \text{eval}(β) - 3, 0 < 1, \text{eval}(α) \geq \text{eval}(β) - 4) \)

\( (\text{eval}(α) \geq \text{eval}(β) - 3, 0 < 1, \text{eval}(α) \geq \text{eval}(β) - 4) \)
Now for case 2:

Case 2: $\text{eval}(\alpha) + 1 < \text{eval}(\beta) < \text{eval}(\alpha) + 2$

$2\text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\alpha) + 1$ \quad | \quad \text{eval}(\beta) \geq \text{eval}(\alpha) + 1$

$2\text{eval}(\alpha) < \text{eval}(\alpha) + \text{eval}(\alpha) + 1$ \quad | \quad 0 < 1$

$2\text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\alpha)$ \quad | \quad \text{eval}(\beta) \geq \text{eval}(\alpha)$

Same as case 1, the second equation is fixed or it is a numerical contradiction. From the two previous case 2 analysis we see there is only one set of conditions that works for this case:

$\left( \text{eval}(\beta) \geq \text{eval}(\alpha) + 1, 0 < 1, \text{eval}(\beta) \geq \text{eval}(\alpha) \right)$

Now case 3:
Case 3: \[ \text{eval}(\alpha) < \text{eval}(\beta) < \text{eval}(\alpha) + 1 \]

\[ 2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\beta) \quad | \quad 0 \geq 0 \]

\[ 2 \text{eval}(\alpha) < \text{eval}(\alpha) + \text{eval}(\beta) \quad | \quad \text{eval}(\alpha) < \text{eval}(\beta) \]

\[ 2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\alpha) \quad | \quad \text{eval}(\beta) \geq \text{eval}(\alpha) \]

Only this set of inequalities holds with the case three conditions. This is a polygon of the form:

![Polygon Diagram]

Now case 4:

Case 4: \[ \text{eval}(\beta) < \text{eval}(\alpha) \]

\[ 2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\beta) \quad | \quad 0 \geq 0 \]

\[ 2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\beta) \quad | \quad 0 \geq 0 \]

\[ 2 \text{eval}(\beta) \geq \text{eval}(\beta) + \text{eval}(\beta) \quad | \quad 0 \geq 0 \]

This is the only set of inequalities that works for this case and it was covered previously. Therefore no new information is gained from this.

Now that all possible polytopes and the conditions that produce them have been explored we can lay all the cases out on one graph to see what values of alpha and beta give certain polytopes.

(insert polytope end chart)

Here we can see for given alpha and beta valuations what the associated polytope would be.

**Conic/conic conclusion**
Whether it be from the Maple program or from the long method the next step would be to try to determine if there were intersections that did not occur on the initial points. If there were intersections then those would show the existence of shadow points. If not, that would only show that there were no shadow points for that specific case. This paper does not explore whether or not these shadow points occur generally but rather sought to lay the framework for a method to ease the process. If initial points are chosen somewhat randomly or certain initial conditions were desired to be tested then either the Maple program could be used or the values could be placed within the general cases listed above.

References
