FINITE GROUP ACTIONS ON REDUCTIVE GROUPS AND BUILDINGS AND TAMELY-RAMIFIED DESCENT IN BRUHAT-TITS THEORY

By Gopal Prasad

Dedicated to Guy Rousseau

Abstract. Let K be a discretely valued field with Henselian valuation ring and separably closed (but not necessarily perfect) residue field of characteristic p, H a connected reductive K-group, and Θ a finite group of automorphisms of H. We assume that p does not divide the order of Θ and Bruhat-Tits theory is available for H over K with $\mathcal{B}(H/K)$ the Bruhat-Tits building of $H(K)$. We will show that then Bruhat-Tits theory is also available for $G := (H^{\Theta})^{\circ}$ and $\mathcal{B}(H/K)^{\Theta}$ is the Bruhat-Tits building of $G(K)$. (In case the residue field of K is perfect, this result was proved in [PY1] by a different method.) As a consequence of this result, we obtain that if Bruhat-Tits theory is available for a connected reductive K-group G over a finite tamely-ramified extension L of K, then it is also available for G over K and $\mathcal{B}(G/K) = \mathcal{B}(G/L)^{\mathrm{Gal}(L/K)}$. Using this, we prove that if G is quasi-split over L, then it is already quasi-split over K.

Introduction. This paper is a sequel to our recent paper [P2]. We will assume familiarity with that paper; we will freely use results, notions and notations introduced in it.

Let $\mathcal{O}$ be a discretely valued Henselian local ring with valuation ω. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and K the field of fractions of $\mathcal{O}$. We will assume throughout that the residue field κ of $\mathcal{O}$ is separably closed. Let $\widehat{\mathcal{O}}$ denote the completion of $\mathcal{O}$ with respect to the valuation ω and $\widehat{K}$ the completion of K. For any $\mathcal{O}$-scheme $\mathscr{X}$, $\mathscr{X}(\mathcal{O})$ and $\mathscr{X}(\widehat{\mathcal{O}})$ will always be assumed to carry the Hausdorff-topology induced from the metric-space topology on $\mathcal{O}$ and $\widehat{\mathcal{O}}$ respectively. It is known that if $\mathscr{X}$ is smooth, then $\mathscr{X}(\mathcal{O})$ is dense in $\mathscr{X}(\widehat{\mathcal{O}})$, [GGM, Prop. 3.5.2]. Similarly, for any K-variety $\mathcal{X}$, $\mathcal{X}(K)$ and $\mathcal{X}(\widehat{K})$ will be assumed to carry the Hausdorff-topology induced from the metric-space topology on K and $\widehat{K}$ respectively. In case $\mathcal{X}$ is a smooth K-variety, $\mathcal{X}(K)$ is dense in $\mathcal{X}(\widehat{K})$, [GGM, Prop. 3.5.2].

Throughout this paper H will denote a connected reductive K-group. In this introduction, and beginning with §2 everywhere, we will assume that Bruhat-Tits theory is available for H over K [P2, 1.9, 1.10]. Then Bruhat-Tits theory is also available for the derived subgroup $\mathscr{D}(H)$ of H over K [P2, 1.11]. Thus there is an affine building called the Bruhat-Tits building of $H(K)$, that is a polysimplicial complex given with a metric, and $H(K)$ acts on it by polysimplicial isometries.

This building is also the Bruhat-Tits building of $\mathscr{D}(H)(K)$ and we will denote it by $\mathcal{B}(\mathscr{D}(H)/K)$. It is known (cf. [P2, 3.11, 1.11]) that Bruhat-Tits theory is also available over K for the centralizer of any K-split torus in H and for the derived subgroup of such centralizers.

Let $\mathfrak{Z}$ be the maximal K-split torus in the center of H. Let $V(\mathfrak{Z}) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Hom}_K(\mathrm{GL}_1, \mathfrak{Z}_K)$. Then there is a natural action of $H(K)$ on this Euclidean space by translations, with $\mathscr{D}(H)(K)$ acting trivially. The *enlarged* Bruhat-Tits building $\mathcal{B}(H/K)$ of $H(K)$ is the direct product $V(\mathfrak{Z}) \times \mathcal{B}(\mathscr{D}(H)/K)$. The apartments of this building, as well as that of $\mathcal{B}(\mathscr{D}(H)/K)$, are in bijective correspondence with maximal K-split tori of H. Given a maximal K-split torus T of H, the corresponding apartment of $\mathcal{B}(H/K)$ is an affine space under $V(T) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Hom}_K(\mathrm{GL}_1, T)$.

Given a nonempty bounded subset Ω of an apartment of $\mathcal{B}(\mathscr{D}(H)/K)$, there is a smooth affine $\mathcal{O}$-group scheme $\mathscr{H}_\Omega$ with generic fiber H, associated with Ω, such that $\mathscr{H}_\Omega(\mathcal{O})$ is the subgroup $H(K)^\Omega$ of $H(K)$ consisting of elements that fix $V(\mathfrak{Z}) \times \Omega\,(\subset \mathcal{B}(H/K))$ pointwise [P2, 1.9 1.10]. The neutral component $\mathscr{H}_\Omega^\circ$ of $\mathscr{H}_\Omega$ is an open affine $\mathcal{O}$-subgroup scheme of the latter; it is by definition the union of the generic fiber H of $\mathscr{H}_\Omega$ and the identity component of its special fiber. The group scheme $\mathscr{H}_\Omega^\circ$ is called the Bruhat-Tits group scheme associated to Ω. The special fiber of $\mathscr{H}_\Omega^\circ$ will be denoted by $\overline{\mathscr{H}}_\Omega^\circ$.

Let Θ be a finite group of automorphisms of H. We assume that the order of Θ is not divisible by the characteristic of the residue field κ. Let $G = (H^\Theta)^\circ$. This group is also reductive, see [Ri, Prop. 10.1.5] or [PY1, Thm. 2.1]. The goal of this paper is to show that Bruhat-Tits theory is available for G over K, and the *enlarged* Bruhat-Tits building of $G(K)$ can be identified with the subspace $\mathcal{B}(H/K)^\Theta$ of $\mathcal{B}(H/K)$ consisting of points fixed under Θ (see §3). These results have been inspired by the main theorem of [PY1], which implies that if the residue field κ is algebraically closed (then every reductive K-group is quasi-split [P2, 1.7], so Bruhat-Tits theory is available for any such group over K), the enlarged Bruhat-Tits building of $G(K)$ is indeed $\mathcal{B}(H/K)^\Theta$.

In §4, we will use the above results to obtain "tamely-ramified descent": (1) We will show that if a connected reductive K-group G is quasi-split over a finite tamely-ramified extension L of K, then it is quasi-split over K (Theorem 4.4); this result has been proved by Philippe Gille in [Gi] by an entirely different method. (2) The enlarged Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$ can be identified with the subspace of points of the enlarged Bruhat-Tits building of $G(L)$ that are fixed under the action of the Galois group $\mathrm{Gal}(L/K)$. This latter result was proved by Guy Rousseau in his unpublished thesis [Rou, Prop. 5.1.1]. It is a pleasure to dedicate this paper to him for his important contributions to Bruhat-Tits theory.

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For a K-split torus S, let $\mathrm{X}_*(S) = \mathrm{Hom}(\mathrm{GL}_1, S)$ and $V(S) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$. Then for a maximal K-split torus T of H, the apartment $A(T)$ of $\mathcal{B}(H/K)$ corresponding to T is an affine space under $V(T)$.

1. Passage to completion

We begin by proving the following well-known result.

Proposition 1.1. K-rank $H = \widehat{K}$-rank H.

Proof. Let T be a maximal K-split torus of H and Z be its centralizer in H. Let Z_{a} be the maximal K-anisotropic connected normal subgroup of Z. Then

$$\widehat{K}\text{-rank}\, H = \widehat{K}\text{-rank}\, Z = \dim(T) + \widehat{K}\text{-rank}\, Z_{\mathrm{a}} = K\text{-rank}\, H + \widehat{K}\text{-rank}\, Z_{\mathrm{a}} .$$

So to prove the proposition, it suffices to show that Z_{a} is anisotropic over $\widehat{K}$. But according to Theorem 1.1 of [P2], Z_{a} is anisotropic over $\widehat{K}$ if and only if $Z_{\mathrm{a}}(\widehat{K})$ is bounded. The same theorem implies that $Z_{\mathrm{a}}(K)$ is bounded. As $Z_{\mathrm{a}}(K)$ is dense in $Z_{\mathrm{a}}(\widehat{K})$, we see that $Z_{\mathrm{a}}(\widehat{K})$ is bounded. □

Proposition 1.2. *Bruhat-Tits theory for H is available over K if and only if it is available over $\widehat{K}$. Moreover, if Bruhat-Tits theory for H is available over K, then the enlarged Bruhat-Tits buildings of $H(K)$ and $H(\widehat{K})$ are equal.*

It was shown by Guy Rousseau in his thesis that the enlarged Bruhat-Tits buildings of $H(K)$ and $H(\widehat{K})$ coincide [Rou, Prop. 2.3.5]. Moreover, every apartment in the building of $H(K)$ is also an apartment in the building of $H(\widehat{K})$; however, the latter may have many more apartments.

Proof. We assume first that Bruhat-Tits theory is available for H over K and let $\mathcal{B}(H/K)$ denote the enlarged Bruhat-Tits building of $H(K)$. We begin by showing that the action of $H(K)$ on $\mathcal{B}(H/K)$ extends to an action of $H(\widehat{K})$ by isometries. For this purpose, we recall that $H(K)$ is dense in $H(\widehat{K})$ and the isotropy at any point $x \in \mathcal{B}(H/K)$ is a bounded open subgroup of $H(K)$. Now let $\{h_i\}$ be a sequence in $H(K)$ which converges to a point $\widehat{h} \in H(\widehat{K})$, then given any open subgroup of $H(K)$, for all large i and j, $h_i^{-1}h_j$ lies in this open subgroup. Thus for any point x of $\mathcal{B}(H/K)$, the sequence $h_i \cdot x$ is eventually constant, i.e., there exists a positive integer n such that $h_i \cdot x = h_n \cdot x$ for all $i \geqslant n$. We define $\widehat{h} \cdot x = h_n \cdot x$. This gives a well-defined action of $H(\widehat{K})$ on $\mathcal{B}(H/K)$ by isometries.

For a nonempty bounded subset Ω of an apartment of the Bruhat-Tits building $\mathcal{B}(\mathscr{D}(H)/K)$, let $\mathscr{H}_\Omega$ and $\mathscr{H}_\Omega^\circ$ be the smooth affine $\mathcal{O}$-group schemes as in the Introduction. Then as $\mathscr{H}_\Omega(\widehat{\mathcal{O}})$ is a closed and open subgroup of $H(\widehat{K})$ containing $\mathscr{H}_\Omega(\mathcal{O})$ as a dense subgroup, we see that $\mathscr{H}_\Omega(\widehat{\mathcal{O}})$ equals the subgroup $H(\widehat{K})^\Omega$ of $H(\widehat{K})$ consisting of elements that fix $V(\mathfrak{Z}) \times \Omega$ pointwise.

Let T be a maximal K-split torus of H, then by Proposition 1.1, $T_{\widehat{K}}$ is a maximal $\widehat{K}$-split torus of $H_{\widehat{K}}$. Let A be the apartment of $\mathcal{B}(H/K)$, or of $\mathcal{B}(\mathscr{D}(H)/K)$, corresponding to T. Then every maximal $\widehat{K}$-split torus of $H_{\widehat{K}}$ is of the form $\widehat{h}T_{\widehat{K}}\widehat{h}^{-1}$ for an $\widehat{h} \in H(\widehat{K})$, and we define the corresponding apartment to be $\widehat{h} \cdot A$. We now declare $\mathcal{B}(H/K)$ (resp. $\mathcal{B}(\mathscr{D}(H)/K)$) to be the enlarged Bruhat-Tits building (resp. the Bruhat-Tits building) of $H(\widehat{K})$ with these apartments.

Let A be an apartment of the Bruhat-Tits building of $H(K)$ corresponding to a maximal K-split torus T of H and $\widehat{h} \in H(\widehat{K})$. Given a nonempty bounded subset $\widehat{\Omega}$ of $\widehat{A} := \widehat{h} \cdot A$, the subset $\Omega := \widehat{h}^{-1} \cdot \widehat{\Omega}$ is contained in A. The closed and open subgroup $\widehat{h}H(\widehat{K})^{\Omega}\widehat{h}^{-1} = \widehat{h}\mathscr{H}_{\Omega}(\widehat{\mathcal{O}})\widehat{h}^{-1}$ of $H(\widehat{K})$ is the subgroup $H(\widehat{K})^{\widehat{\Omega}}$ consisting of elements that fix $V(\mathfrak{Z}) \times \widehat{\Omega}$ pointwise. Now as $H(K)$ is dense in $H(\widehat{K})$ and $H(\widehat{K})^{\widehat{\Omega}}$ is an open subgroup, $H(\widehat{K}) = H(\widehat{K})^{\widehat{\Omega}} \cdot H(K)$, so $\widehat{h} = h' \cdot h$, with $h' \in H(\widehat{K})^{\widehat{\Omega}}$ and $h \in H(K)$. Thus the apartment $\widehat{A} = \widehat{h} \cdot A = h' \cdot hA$, and hA is an apartment of the Bruhat-Tits building of $H(K)$. As $h' \in H(\widehat{K})^{\widehat{\Omega}}$, the apartment hA contains $\widehat{\Omega}$. This shows that any bounded subset $\widehat{\Omega}$ of an apartment of the Bruhat-Tits building of $H(\widehat{K})$ is contained in an apartment of the Bruhat-Tits building of $H(K)$. We define the $\widehat{\mathcal{O}}$-group schemes $\mathscr{H}_{\widehat{\Omega}}$ and $\mathscr{H}^{\circ}_{\widehat{\Omega}}$ associated to $\widehat{\Omega}$ to be the group schemes obtained from the corresponding $\mathcal{O}$-group schemes (given by considering $\widehat{\Omega}$ to be a nonempty bounded subset of an apartment of the building of $H(K)$) by extension of scalars $\mathcal{O} \hookrightarrow \widehat{\mathcal{O}}$.

Let us assume now that Bruhat-Tits theory is available for H over $\widehat{K}$. Then Bruhat-Tits theory is also available for $\mathscr{D}(H)$ over $\widehat{K}$ [P2, 1.11]. The action of $H(\widehat{K})$ on its building $\mathcal{B}(\mathscr{D}(H)/\widehat{K})$ restricts to an action of $H(K)$ by isometries. Let T be a maximal K-split torus of G and A be the apartment of $\mathcal{B}(\mathscr{D}(H)/\widehat{K})$ corresponding to $T_{\widehat{K}}$. We consider the polysimplicial complex $\mathcal{B}(\mathscr{D}(H)/\widehat{K})$, with apartments $h \cdot A$, $h \in H(K)$, as the building of $H(K)$ and denote it by $\mathcal{B}(\mathscr{D}(H)/K)$.

Let $\widehat{\Omega}$ be a nonempty bounded subset of the apartment $\widehat{A} = \widehat{h} \cdot A$, $\widehat{h} \in H(\widehat{K})$, in the building $\mathcal{B}(\mathscr{D}(H)/\widehat{K})$. As $H(K)$ is dense in $H(\widehat{K})$, the intersection $\mathscr{H}_{\widehat{\Omega}}(\widehat{\mathcal{O}})\widehat{h} \cap H(K)$ is nonempty. For any h in this intersection, $\widehat{\Omega}$ is contained in the apartment $h \cdot A$ of $\mathcal{B}(\mathscr{D}(H)/K)$. This implies, in particular, that any two facets lie on an apartment of $\mathcal{B}(\mathscr{D}(H)/K)$. We now note that the $\widehat{\mathcal{O}}$-group schemes $\mathscr{H}_{\widehat{\Omega}}$ and $\mathscr{H}^{\circ}_{\widehat{\Omega}}$ admit unique descents to smooth affine $\mathcal{O}$-group schemes with generic fiber H, [BLR, Prop. D.4(b) in §6.1]; the affine rings of these descents are $K[H] \cap \widehat{\mathcal{O}}[\mathscr{H}_{\widehat{\Omega}}]$ and $K[H] \cap \widehat{\mathcal{O}}[\mathscr{H}^{\circ}_{\widehat{\Omega}}]$ respectively. $\square$

In view of the preceding proposition, we may (and do) replace $\mathcal{O}$ *and* K *with* $\widehat{\mathcal{O}}$ *and* $\widehat{K}$ *respectively to assume in the rest of this paper that* $\mathcal{O}$ *and* K *are complete.*

2. Fixed points in $\mathcal{B}(H/K)$ under a finite automorphism group Θ of H

We will henceforth assume that Bruhat-Tits theory is available for H over K.

2.1. Let G be a smooth affine K-group and $\mathscr{G}$ be a smooth affine $\mathcal{O}$-group scheme with generic fiber G. According to [BrT2, 1.7.1-1.7.2] $\mathscr{G}$ is "étoffé" and hence by (ET) of [BrT2, 1.7.1] its affine ring has the following description:

$$\mathcal{O}[\mathscr{G}] = \{f \in K[G] \mid f(\mathscr{G}(\mathcal{O})) \subset \mathcal{O}\}.$$

Let Ω be a nonempty bounded subset of an apartment of $\mathcal{B}(\mathscr{D}(H)/K)$. As the $\mathcal{O}$-group scheme $\mathscr{H}_\Omega$ is smooth and affine and its generic fiber is H, the affine ring of $\mathscr{H}_\Omega$ has thus the following description:

$$\mathcal{O}[\mathscr{H}_\Omega] = \{f \in K[H] \mid f(H(K)^\Omega) \subset \mathcal{O}\}.$$

Proposition 2.2. *Let Ω be a nonempty bounded subset of an apartment of $\mathcal{B}(\mathscr{D}(H)/K)$. Let $\mathscr{H}_\Omega$ and $\mathscr{H}_\Omega^\circ$ be as above. Let G be a smooth connected K-subgroup of H and $\mathscr{G}$ be a smooth affine $\mathcal{O}$-group scheme with generic fiber G and connected special fiber. Assume that a subgroup $\mathcal{G}$ of $\mathscr{G}(\mathcal{O})$ of finite index fixes Ω pointwise (i.e., $\mathcal{G} \subset H(K)^\Omega$). Then there is a $\mathcal{O}$-group scheme homomorphism $\varphi : \mathscr{G} \to \mathscr{H}_\Omega^\circ$ that is the natural inclusion $G \hookrightarrow H$ on the generic fibers. So the subgroup $\mathscr{G}(\mathcal{O})$ of $G(K)$ is contained in $\mathscr{H}_\Omega^\circ(\mathcal{O})$ and hence it fixes Ω pointwise. If F is a facet of $\mathcal{B}(\mathscr{D}(H)/K)$ that meets Ω, then $\mathscr{G}(\mathcal{O})$ fixes F pointwise.*

Let S be a K-split torus of H and $\mathscr{S}$ the $\mathcal{O}$-torus with generic fiber S. If a subgroup of the maximal bounded subgroup $\mathscr{S}(\mathcal{O})$ of $S(K)$ of finite index fixes Ω pointwise, then there is a maximal K-split torus T of H containing S such that Ω is contained in the apartment of $\mathcal{B}(\mathscr{D}(H)/K)$ corresponding to T.

Proof. Since the fibers of the smooth affine group scheme $\mathscr{G}$ are connected and the residue field κ is separably closed, the subgroup $\mathcal{G}$ is Zariski-dense in G, and its image in $\mathscr{G}(\kappa)$ is Zariski-dense in the spacial fiber of $\mathscr{G}$. Using this observation, we easily see that the affine ring $\mathcal{O}[\mathscr{G}]\,(\subset K[G])$ of $\mathscr{G}$ has the following description (cf. [BrT2, 1.7.2]):

$$\mathcal{O}[\mathscr{G}] = \{f \in K[G] \mid f(\mathcal{G}) \subset \mathcal{O}\}.$$

This description of $\mathcal{O}[\mathscr{G}]$ implies at once that the inclusion $\mathcal{G} \hookrightarrow H(K)^\Omega$ induces a $\mathcal{O}$-group scheme homomorphism $\varphi : \mathscr{G} \to \mathscr{H}_\Omega$ that is the natural inclusion $G \hookrightarrow H$ on the generic fibers. Since $\mathscr{G}$ has connected fibers, the homomorphism φ factors through $\mathscr{H}_\Omega^\circ$.

Any facet F of $\mathcal{B}(\mathscr{D}(H)/K)$ that meets Ω is stable under $\mathscr{G}(\mathcal{O})\,(\subset H(K))$, so a subgroup of $\mathscr{G}(\mathcal{O})$ of finite index fixes it pointwise. Now applying the result of the preceding paragraph, for F in place of Ω, we see that there is a $\mathcal{O}$-group scheme homomorphism $\mathscr{G} \to \mathscr{H}_F^\circ$ that is the natural inclusion $G \hookrightarrow H$ on the generic fibers and hence $\mathscr{G}(\mathcal{O})$ fixes F pointwise.

Now we will prove the last assertion of the proposition. It follows from what we have shown above that there is a $\mathcal{O}$-group scheme homomorphism $\iota : \mathscr{S} \to \mathscr{H}_\Omega^\circ$ that is the natural inclusion $S \hookrightarrow H$ on the generic fibers (ι is actually a closed immersion, see [PY2, Lemma 4.1]). Applying [P2, Prop. 2.1(i)] to the centralizer of $\iota(\mathscr{S})$ (in $\mathscr{H}_\Omega^\circ$) in place of $\mathscr{G}$, and $\mathcal{O}$ in place of $\mathfrak{o}$, we see that there is a closed $\mathcal{O}$-torus

$\mathscr{T}$ of $\mathscr{H}_\Omega^\circ$ that commutes with $\iota(\mathscr{S})$ and whose generic fiber T is a maximal K-split torus of H. The torus T clearly contains S, and [P2, Prop. 2.2(ii)] implies that Ω is contained in the apartment corresponding to T. □

The following is a simple consequence of the preceding proposition.

Corollary 2.3. *Let G, S, $\mathscr{G}$, and $\mathscr{S}$ be as in the preceding proposition. Then the set of points of $\mathcal{B}(\mathscr{D}(H)/K)$ that are fixed under $\mathscr{G}(\mathcal{O})$ is the union of facets pointwise fixed under $\mathscr{G}(\mathcal{O})$. The set of points of the enlarged building $\mathcal{B}(H/K)$ that are fixed under a finite-index subgroup $\mathcal{S}$ of the maximal bounded subgroup $S(K)_b\,(=\mathscr{S}(\mathcal{O}))$ of $S(K)$ is the enlarged Bruhat-Tits building $\mathcal{B}(Z_H(S)/K)$ of the centralizer $Z_H(S)(K)$ of S in $H(K)$.*

2.4. Let Θ be a finite group of automorphisms of the reductive K-group H. There is a natural action of Θ on the Bruhat-Tits building $\mathcal{B}(\mathscr{D}(H)/K)$ of $H(K)$ by polysimplicial isometries such that for all $h \in H(K)$, $x \in \mathcal{B}(\mathscr{D}(H)/K)$ and $\theta \in \Theta$, we have $\theta(h\cdot x) = \theta(h)\cdot\theta(x)$.

Let Ω be a nonempty bounded subset of an apartment of $\mathcal{B}(\mathscr{D}(H)/K)$. Assume that Ω is stable under the action of Θ on $\mathcal{B}(\mathscr{D}(H)/K)$. Then $\mathscr{H}_\Omega(\mathcal{O})$ is stable under the action of Θ on $H(K)$, so the affine ring $\mathcal{O}[\mathscr{H}_\Omega]$ is stable under the action of Θ on $K[H]$. This implies that Θ acts on the group scheme $\mathscr{H}_\Omega$ by $\mathcal{O}$-group scheme automorphisms. The neutral component $\mathscr{H}_\Omega^\circ$ of $\mathscr{H}_\Omega$ is of course stable under this action.

In the following we assume that the characteristic p of the residue field κ does not divide the order of Θ. Then $G := (H^\Theta)^\circ$ is a reductive group, see [Ri, Prop. 10.1.5] or [PY1, Thm. 2.1]. We will prove that Bruhat-Tits theory is available for G over K and the enlarged Bruhat-Tits building of $G(K)$, as a metric space, can be identified with the subspace $\mathcal{B}(H/K)^\Theta$ of points of $\mathcal{B}(H/K)$ fixed under Θ.

Let C be the maximal K-split central torus of G and H' be the derived subgroup of the centralizer of C in H. Then H' is a connected semi-simple subgroup of H stable under the group Θ of automorphisms of H; $({H'}^\Theta)^\circ\,(\subset G)$ contains the derived subgroup of G and its central torus is K-anisotropic. Replacing H with H' *we assume in the sequel that H is semi-simple and the central torus of G is K-anisotropic* (cf. [P2, 3.11, 1.11]).

For a subset X of a set given with an action of Θ, we denote by X^Θ the subset of points of X that are fixed under Θ. We will denote $\mathcal{B}(H/K)^\Theta$ by $\mathcal{B}$ in the sequel.

If a facet of $\mathcal{B}(H/K)$ is stable under the action of Θ, then its barycenter is fixed under Θ. Conversely, if a facet F contains a point x fixed under Θ, then being the unique facet containing x, F is stable under the action of Θ.

2.5. We introduce the following partial order "$\prec$" on the set of nonempty subsets of $\mathcal{B}(H/K)$: Given two nonempty subsets Ω and Ω', $\Omega' \prec \Omega$ if the closure $\overline{\Omega}$ of Ω contains Ω'. If F and F' are facets of $\mathcal{B}(H/K)$, with $F' \prec F$, or equivalently, $\mathscr{H}_F^\circ(\mathcal{O}) \subset \mathscr{H}_{F'}^\circ(\mathcal{O})$, we say that F' is a *face* of F. In a collection $\mathcal{C}$ of facets, thus a facet is *maximal* if it is not a proper face of any facet belonging to $\mathcal{C}$, and a facet is *minimal* if no proper face of it belongs to $\mathcal{C}$.

Now let X be a convex subset of $\mathcal{B}(H/K)$ and $\mathcal{C}$ be the set of facets of $\mathcal{B}(H/K)$, or facets lying in a given apartment A, that meet X. Then the following assertions are easy to prove (see Proposition 9.2.5 of [BrT1]): (1) All maximal facets in $\mathcal{C}$ are of equal dimension and a facet $F \in \mathcal{C}$ is maximal if and only if $\dim(F \cap X)$ is maximal. (2) Let F be a facet lying in an apartment A. Assume that F is maximal among the facets of A that meet X, and let A_F be the affine subspace of A spanned by F. Then every facet of A that meets X is contained in A_F and $A \cap X$ is contained in the affine subspace of A spanned by $F \cap X$.

The subset $\mathcal{B} = \mathcal{B}(H/K)^{\Theta}$ of $\mathcal{B}(H/K)$ is closed and convex. Hence the assertions of the preceding paragraph hold for $\mathcal{B}$ in place of X. We will show in this section that $\mathcal{B}$ is an affine building with apartments described below. We begin with the following proposition which has been suggested by Proposition 1.1 of [PY1], and the proof given here is an adaptation of the proof of that proposition.

Proposition 2.6. *Let A be an apartment of $\mathcal{B}(H/K)$ and F a facet of A that meets $\mathcal{B}$. Let Ω be a nonempty bounded subset of the affine subspace A_F of A spanned by F. We assume that Ω contains F and is stable under the action of Θ on $\mathcal{B}(H/K)$. Let $\mathscr{H} := \mathscr{H}_{\Omega}^{\circ}$ be the Bruhat-Tits smooth affine $\mathcal{O}$-group scheme with generic fiber H, and connected special fiber $\overline{\mathscr{H}}$, associated with Ω. Let $\overline{\mathscr{H}}^{\rm pred} := \overline{\mathscr{H}}/\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}})$ be the maximal pseudo-reductive quotient of $\overline{\mathscr{H}}$. Then there exist K-split tori $S \subset T$ in H such that*

(i) *T is a maximal K-split torus of H and Ω is contained in the apartment $A(T)$ corresponding to T;*

(ii) *S is stable under Θ and the special fiber of the schematic closure $\mathscr{S}$ of S in $\mathscr{H}$ maps onto the central torus of $\overline{\mathscr{H}}^{\rm pred}$.*

Proof. Let $\mathcal{T}$ be the set of maximal K-split tori T of H such that $\Omega \subset A(T)$. Then the automorphism group Θ clearly permutes $\mathcal{T}$, and the subgroup $\mathcal{P} := \mathscr{H}(\mathcal{O})$ acts transitively on $\mathcal{T}$ [P2, Prop. 2.2(i)]. Hence, for every $T \in \mathcal{T}$, Ω is contained in the affine subspace of $A(T)$ spanned by the facet F.

For $T \in \mathcal{T}$, let S_T be the lift of the central torus of $\overline{\mathscr{H}}^{\rm pred}$ in T. It is clear that the pair (S, T) satisfy (i) and (ii) if S is Θ-stable. We consider $\mathcal{S} := \{S_T \mid T \in \mathcal{T}\}$; Θ acts by permutation on $\mathcal{S}$ and $\mathcal{P}$ acts transitively on it. We will find an element of $\mathcal{S}$ that is Θ-stable. We first prove the following lemma.

Lemma 2.7. *Let $T \in \mathcal{T}$ and $S := S_T$ be as above. Then*

(i) *The normalizer of S in $\mathcal{P}$ centralizes S.*

(ii) *$\mathcal{P} = \mathcal{P}_S \cdot \mathcal{U}$, where $\mathcal{P}_S$ is the centralizer of S in $\mathcal{P}$ and $\mathcal{U}$ is the kernel of the natural homomorphism $\mathscr{H}(\mathcal{O}) \to \overline{\mathscr{H}}^{\rm pred}(\kappa)$.*

Proof. (i) The affine subspace $A(T)_F$ of $A(T)$ spanned by F is an affine space under the $\mathbb{R}$-vector space $V(S)$. So for any $x \in F$, $V(S) + x = A(T)_F$. Now let h be an element of $\mathcal{P}$ that normalizes S. Then h takes $A(T)_F = V(S) + x\,(\subset A(T))$ to $V(S) + h \cdot x = V(S) + x\,(\subset A(hTh^{-1}))$ by an affine transformation whose derivative gives the action of h on $V(S)$. As h fixes the open subset F of $A(T)_F$ pointwise, its derivative acts trivially on $V(S)$ and hence h centralizes S.

(ii) Let $\mathscr{S}$ and $\mathscr{T}$ be the closed $\mathcal{O}$-tori in $\mathscr{H}$ with generic fibers S and T respectively. Then the centralizer $\mathscr{H}^{\mathscr{S}}$ of $\mathscr{S}$ in $\mathscr{H}$ is a smooth affine $\mathcal{O}$-subgroup scheme [CGP, Prop. A.8.10(2)]. Let $\overline{\mathscr{S}}$ be the special fiber of $\mathscr{S}$ and $\overline{\mathscr{H}}^{\overline{\mathscr{S}}}$ be the centralizer of $\overline{\mathscr{S}}$ in the special fiber $\overline{\mathscr{H}}$ of $\mathscr{H}$. Since $\mathcal{O}$ is Henselian, the natural map $(\mathcal{P}_S =)\,\mathscr{H}^{\mathscr{S}}(\mathcal{O}) \to \overline{\mathscr{H}}^{\overline{\mathscr{S}}}(\kappa)$ is surjective [EGA IV$_4$ 18.5.17]. As the image of $\overline{\mathscr{S}}$ in $\overline{\mathscr{H}}^{\,\mathrm{pred}}$ is central, the natural homomorphism $\overline{\mathscr{H}}^{\overline{\mathscr{S}}} \to \overline{\mathscr{H}}^{\,\mathrm{pred}}$ is surjective (see [Bo, Prop. 9.6]). On the other hand, $\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}}) \cap \overline{\mathscr{H}}^{\overline{\mathscr{S}}} = \mathscr{R}_{u,\kappa}(\overline{\mathscr{H}}^{\overline{\mathscr{S}}})$ ([CGP, Prop. A.8.14]; note that as $\overline{\mathscr{S}}$ is a torus, both $\overline{\mathscr{H}}^{\overline{\mathscr{S}}}$ and $(\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}}))^{\overline{\mathscr{S}}} = \mathscr{R}_{u,\kappa}(\overline{\mathscr{H}}) \cap \overline{\mathscr{H}}^{\overline{\mathscr{S}}}$ are smooth and connected). So the natural map $\overline{\mathscr{H}}^{\overline{\mathscr{S}}} / \mathscr{R}_{u,\kappa}(\overline{\mathscr{H}}^{\overline{\mathscr{S}}}) \to \overline{\mathscr{H}}^{\mathrm{pred}}$ is an isomorphism. Since κ is separably closed, this implies that $\overline{\mathscr{H}}^{\overline{\mathscr{S}}}(\kappa) \to \overline{\mathscr{H}}^{\,\mathrm{pred}}(\kappa)$ is surjective. Hence, the map $\mathcal{P}_S \to \overline{\mathscr{H}}^{\,\mathrm{pred}}(\kappa)$ is surjective too. From this we conclude that $\mathcal{P} = \mathcal{P}_S \cdot \mathcal{U}$. $\square$

We will now complete the proof of Proposition 2.6. As in the preceding lemma, let $\mathcal{U}$ be the kernel of the natural homomorphism $\mathscr{H}(\mathcal{O}) \to \overline{\mathscr{H}}^{\,\mathrm{pred}}(\kappa)$. Since Ω has been assumed to be stable under the action of Θ on $\mathcal{B}(H/K)$, the group Θ acts on $\mathscr{H}$ by $\mathcal{O}$-group scheme automorphisms. So $\mathcal{U}$ is stable under the induced action of Θ on $\mathcal{P} = \mathscr{H}(\mathcal{O})$. We will now describe a descending Θ-stable filtration of the subgroup $\mathcal{U}$. For a non-negative integer i, let $\mathcal{U}_i$ be the kernel of the homomorphism $\mathcal{P} = \mathscr{H}(\mathcal{O}) \to \mathscr{H}(\mathcal{O}/\mathfrak{m}^{i+1})$. Then each $\mathcal{U}_i$ is a normal subgroup of $\mathcal{P}$ and is stable under the action of Θ on the latter, $\mathcal{U}_i \supset \mathcal{U}_{i+1}$, and $\mathcal{U}_i/\mathcal{U}_{i+1}$ is a κ-vector space for all $i \geqslant 0$ [CGP, Prop. A.5.12]. The quotient $\mathcal{U}/\mathcal{U}_0$ is isomorphic to $\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}})(\kappa)$. If $p = 0$, we consider the ascending filtration of the nilpotent group $\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}})(\kappa)$ given by its ascending central series, and if $p \neq 0$ we consider the ascending filtration of the unipotent group $\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}})$ given by Corollary B.3.3 of [CGP] to obtain an ascending filtration of $\mathcal{U}/\mathcal{U}_0$. The inverse image in $\mathcal{U}$ of this filtration of $\mathcal{U}/\mathcal{U}_0$ gives us a descending filtration $\mathcal{U} = \mathcal{U}_{-n} \supset \mathcal{U}_{-n+1} \supset \mathcal{U}_{-n+2} \cdots \supset \mathcal{U}_0$, where n is a non-negative integer. For all $j \geqslant -n$, $\mathcal{U}_j$ is a normal subgroup of $\mathcal{P}$ that is stable under the action of Θ on the latter, $\mathcal{U}_j/\mathcal{U}_{j+1}$ is a commutative group of exponent p if $p \neq 0$, and is a vector space over $\mathbb{Q}$ if $p = 0$. For convenience, we will denote $\mathcal{U}_j$ by $\mathcal{U}^{(j+n+1)}$ for all j. Thus we have a decreasing filtration $\mathcal{U} = \mathcal{U}^{(1)} \supset \mathcal{U}^{(2)} \supset \mathcal{U}^{(3)} \cdots$.

For $S \in \mathcal{S}$, let $\mathcal{Z}_S^{(j)}$ be the centralizer of S in $\mathcal{U}^{(j)}$. If for $\theta \in \Theta$, there exists $u(\theta) \in \mathcal{U}^{(j)}$ such that $\theta(S) = u(\theta)^{-1} S u(\theta)$, then $\mathcal{Z}_S^{(j)} \mathcal{U}^{(j+1)}$ is Θ-stable. To see this, let $\theta \in \Theta$, and pick $u(\theta) \in \mathcal{U}^{(j)}$ such that $\theta(S) = u(\theta)^{-1}Su(\theta)$. Then $\theta(\mathcal{Z}_S^{(j)}) = u(\theta)^{-1}\mathcal{Z}_S^{(j)}u(\theta)$. So $\theta(\mathcal{Z}_S^{(j)}\mathcal{U}^{(j+1)}) = u(\theta)^{-1}\mathcal{Z}_S^{(j)}u(\theta)\mathcal{U}^{(j+1)} = \mathcal{Z}_S^{(j)}\mathcal{U}^{(j+1)}$ since $\mathcal{U}^{(j)}/\mathcal{U}^{(j+1)}$ is commutative. This shows that $\mathcal{Z}_S^{(j)}\mathcal{U}^{(j+1)}$ is Θ-stable. Now as Θ is a finite group of order prime to p if $p \neq 0$, and $\mathcal{U}^{(j)}/\mathcal{Z}_S^{(j)}\mathcal{U}^{(j+1)}$ is a commutative divisible group if $p = 0$, we conclude that $\mathrm{H}^1(\Theta, \mathcal{U}^{(j)}/\mathcal{Z}_S^{(j)}\mathcal{U}^{(j+1)}) = 0$ for all p.

Now we fix an $S_0 \in \mathcal{S}$. Then for $\theta \in \Theta$, clearly $\theta(S_0) \in \mathcal{S}$, and since $\mathcal{P}$ acts transitively on $\mathcal{S}$, we see using Lemma 2.7(ii) (for S_0 in place of S) that $\theta(S_0) = u_1(\theta)^{-1}S_0u_1(\theta)$ with $u_1(\theta) \in \mathcal{U}^{(1)}\,(= \mathcal{U})$. As $\mathcal{Z}_{S_0}^{(1)}$ is the normalizer of S_0 in $\mathcal{U}^{(1)}$ (Lemma 2.7(i)), we see that $\theta \mapsto u_1(\theta) \pmod{\mathcal{Z}_{S_0}^{(1)}\mathcal{U}^{(2)}}$ is a 1-cocycle on Θ with values in $\mathcal{U}^{(1)}/\mathcal{Z}_{S_0}^{(1)}\mathcal{U}^{(2)}$, and hence it is a 1-coboundary. This means that there is a $v_1 \in \mathcal{U}^{(1)}$ such that $u_1'(\theta) := v_1^{-1}u_1(\theta)\theta(v_1) \in \mathcal{Z}_{S_0}^{(1)}\mathcal{U}^{(2)}$ for all $\theta \in \Theta$.

Let $S_1 = v_1^{-1}S_0v_1$. Then for $\theta \in \Theta$, we have $\theta(S_1) = u_1'(\theta)^{-1}S_1u_1'(\theta)$. Observe that $u_1'(\theta) \in \mathcal{Z}_{S_0}^{(1)}\mathcal{U}^{(2)} = v_1\mathcal{Z}_{S_1}^{(1)}v_1^{-1}\mathcal{U}^{(2)} = \mathcal{Z}_{S_1}^{(1)}\mathcal{U}^{(2)}$ as $\mathcal{U}^{(1)}/\mathcal{U}^{(2)}$ is commutative. So for each $\theta \in \Theta$, there is an element $u_2(\theta)$ of $\mathcal{U}^{(2)}$ such that $\theta(S_1) = u_2(\theta)^{-1}S_1u_2(\theta)$. Now, as above, using the fact that the normalizer of S_1 in $\mathcal{U}^{(2)}$ is the centralizer $\mathcal{Z}_{S_1}^{(2)}$, we see that $\theta \mapsto u_2(\theta) \pmod{\mathcal{Z}_{S_1}^{(2)}\mathcal{U}^{(3)}}$ is a 1-cocycle on Θ with values in $\mathcal{U}^{(2)}/\mathcal{Z}_{S_1}^{(2)}\mathcal{U}^{(3)}$, and hence it is a 1-coboundary. Therefore, there is a $v_2 \in \mathcal{U}^{(2)}$ such that $u_2'(\theta) := v_2^{-1}u_2(\theta)\theta(v_2) \in \mathcal{Z}_{S_1}^{(2)}\mathcal{U}^{(3)}$ for all $\theta \in \Theta$.

Repeating the above argument, we construct a sequence $\{S_i\}$ of tori in $\mathcal{S}$, and a sequence of elements $v_i \in \mathcal{U}^{(i)}$, such that

- $S_i = v_i^{-1}S_{i-1}v_i$, and for each $\theta \in \Theta$, there is an element $u_{i+1}(\theta)$ of $\mathcal{U}^{(i+1)}$ such that $\theta(S_i) = u_{i+1}(\theta)^{-1}S_iu_{i+1}(\theta)$, and $\theta \mapsto u_{i+1}(\theta) \pmod{\mathcal{Z}_{S_i}^{(i+1)}\mathcal{U}^{(i+2)}}$ is a 1-cocycle on Θ with values in $\mathcal{U}^{(i+1)}/\mathcal{Z}_{S_i}^{(i+1)}\mathcal{U}^{(i+2)}$.

For $i \geqslant 1$, let $w_i = v_1v_2\cdots v_i$. Then $S_i = w_i^{-1}S_0w_i$. Since $v_j \in \mathcal{U}^{(j)}$, and $\mathcal{O}$ has been assumed to be complete, $w := \lim_{i\to\infty} w_i$ exists in $\mathcal{U}$. Let $S = w^{-1}S_0w$. For $\theta \in \Theta$, as $\theta(S_i) = u_{i+1}(\theta)^{-1}S_iu_{i+1}(\theta)$, we see that $u_1(\theta)\theta(w_i)u_{i+1}(\theta)^{-1}w_i^{-1}$ normalizes S_0. Since the normalizer of S_0 in $H(K)$ is closed, taking $i \to \infty$, we conclude that $u_1(\theta)\theta(w)w^{-1}$ normalizes S_0. This implies that $\theta(S) = S$ for all $\theta \in \Theta$. □

2.8. Let $x,\, y \in \mathcal{B} = \mathcal{B}(H/K)^{\Theta}$. Let F be a facet of $\mathcal{B}(H/K)$ which contains x in its closure and is maximal among the facets that meet $\mathcal{B}$, and let $\Omega = F \cup \{y\}$. Let $S \subset T$ be a pair of K-split tori with properties (i) and (ii) of Proposition 2.6, and S_G and T_G be the maximal subtori of S and T respectively contained in G. Let A be the apartment of $\mathcal{B}(H/K)$ corresponding to the maximal K-split torus T of H. Then A contains y and the closure of F, and so it also contains x. Moreover, A is an affine space under $V(T)$, the affine subspace $V(S) + x$ of A contains F and is spanned by it. The affine subspaces $V(S_G) + x \subset V(T_G) + x$ of A are clearly contained in $\mathcal{B} = \mathcal{B}(H/K)^{\Theta}$. As $V(S)^{\Theta} = V(S_G)$ and $F \subset V(S) + x$, we see that F^{Θ} is contained in $V(S_G) + x$. But since the facet F is maximal among the facets that meet $\mathcal{B}$, A^{Θ} $(= A \cap \mathcal{B})$ is contained in the affine subspace of A spanned by F^{Θ}. Therefore, $A^{\Theta} = V(S_G) + x$. This implies that $V(S_G) + x = V(T_G) + x$ and hence $S_G = T_G$. We will now show that S_G is a maximal K-split torus of G.

Let S' be a maximal K-split torus of G containing S_G. Then the centralizer $M := Z_H(S')$ of S' in H is stable under Θ. The enlarged Bruhat-Tits building $\mathcal{B}(M/K)$ of $M(K)$ is identified with the union of apartments of $\mathcal{B}(H/K)$ that correspond to maximal K-split tori of M (these are precisely the maximal K-split tori of H that contain S'), cf. [P2, 3.11]. Let z be a point of $\mathcal{B}(M/K)^{\Theta}$ and T' be a maximal K-split torus of M such that the corresponding apartment A' of $\mathcal{B}(M/K)$ contains z. Then $A' = V(T') + z$ and hence ${A'}^{\Theta} = A' \cap \mathcal{B} = V(T')^{\Theta} + z = V(S') + z$ is an affine subspace of A' of dimension $\dim(S')$. Let F' be a facet of A' that contains the point z in its closure and is maximal among the facets of A' meeting $\mathcal{B}$. Then ${A'}^{\Theta}$ is contained in the affine subspace of A' spanned by ${F'}^{\Theta}$, so $\dim({F'}^{\Theta}) = \dim(S') \geqslant \dim(S_G)$. But $\dim(F^{\Theta}) = \dim(S_G) \geqslant \dim({F'}^{\Theta})$. This implies that $\dim(S_G) = \dim(S')$ and hence $S' = S_G$. So S_G is a maximal K-split torus of G.

Thus we have established the following proposition:

Proposition 2.9. *Given points x, $y \in \mathcal{B}$, there exists a maximal K-split torus S_G of G, and a maximal K-split torus T of H containing S_G and hence contained in $Z_H(S_G)$, such that the apartment A of $\mathcal{B}(Z_H(S_G)/K)$ corresponding to T contains x and y. Moreover, $A^{\Theta} = A \cap \mathcal{B}$ is the affine subspace $V(S_G) + x$ of A of dimension* $\dim(S_G)$.

We will now derive the following proposition which will give us apartments in the Bruhat-Tits building of $G(K)$. In the sequel, we will use S, instead of S_G, to denote a maximal K-split torus of G. As $M := Z_H(S)$ is stable under Θ, the enlarged Bruhat-Tits building $\mathcal{B}(M/K)$ of $M(K)$ contains a Θ-fixed point.

Proposition 2.10. *Let S be a maximal K-split torus of G and let T be a maximal K-split torus of H containing S such that the apartment A of $\mathcal{B}(H/K)$ corresponding to T contains a Θ-fixed point x. Then $\mathcal{B}(Z_H(S)/K)^{\Theta} = V(S) + x = A^{\Theta}$. So $\mathcal{B}(Z_H(S)/K)^{\Theta}$ is an affine space under the $\mathbb{R}$-vector space $V(S)$.*

Proof. Let C be the central torus of $Z_H(S)$ and $Z_H(S)'$ the derived subgroup. Then C, $Z_H(S)$ and $Z_H(S)'$ are stable under Θ; $G' := ({Z_H(S)'}^{\Theta})^{\circ}$ is anisotropic over K since S is a maximal K-split torus of G, and so also of $(Z_H(S)^{\Theta})^{\circ}$ $(\subset G)$. Now applying Proposition 2.9 to $Z_H(S)'$ in place of H, we see that the Bruhat-Tits building $\mathcal{B}(Z_H(S)'/K)$ of $Z_H(S)'(K)$ contains only one point fixed under Θ. For if y, $z \in \mathcal{B}(Z_H(S)'/K)^{\Theta}$, then there is an apartment A' of $\mathcal{B}(Z_H(S)'/K)$ that contains these points. Moreover, the dimension of the affine subspace ${A'}^{\Theta}$ of A' is 0 as G' is anisotropic over K. Therefore, $y = z$. This proves that $\mathcal{B}(Z_H(S)'/K)^{\Theta}$ consists of a single point. Hence, $\mathcal{B}(Z_H(S)/K)^{\Theta} = V(C)^{\Theta} + x = V(S) + x$, and so it is an affine space under $V(S)$. □

2.11. Let S be a maximal K-split torus of G. Let $N := N_G(S)$ and $Z := Z_G(S)$ be respectively the normalizer and the centralizer of S in G. As N (in fact, the normalizer $N_H(S)$ of S in H) normalizes the centralizer $Z_H(S)$ of S in H, there is a natural action of $N(K)$ on $\mathcal{B}(Z_H(S)/K)$ and $N(K)$ stabilizes $\mathcal{B}(Z_H(S)/K)^{\Theta}$ under this action. For $n \in N(K)$, the action of n carries an apartment A of $\mathcal{B}(Z_H(S)/K)$ to the apartment $n \cdot A$ by an affine transformation.

Now let T be a maximal K-split torus of $Z_H(S)$ such that the corresponding apartment $A := A_T$ of $\mathcal{B}(Z_H(S)/K)$ contains a Θ-fixed point x. According to the previous proposition, $\mathcal{B}(Z_H(S)/K)^{\Theta} = V(S) + x = A^{\Theta}$. So we can view $\mathcal{B}(Z_H(S)/K)^{\Theta}$ as an affine space under $V(S)$. We will now show, using the proof of the lemma in 1.6 of [PY1], that $\mathcal{B}(Z_H(S)/K)^{\Theta}$ has the properties required of an apartment corresponding to the maximal K-split torus S in the Bruhat-Tits building of $G(K)$ if such a building exists. We need to check the following three conditions.

A1: *The action of $N(K)$ on $\mathcal{B}(Z_H(S)/K)^{\Theta} = A^{\Theta}$ is by affine transformations and the maximal bounded subgroup $Z(K)_b$ of $Z(K)$ acts trivially.*

Let $\mathrm{Aff}(A^{\Theta})$ be the group of affine automorphisms of A^{Θ} and $\varphi : N(K) \to \mathrm{Aff}(A^{\Theta})$ be the action map.

A2: *The group $Z(K)$ acts by translations, and the action is characterized by the following formula: for $z \in Z(K)$,*

$$\chi(\varphi(z)) = -\omega(\chi(z)) \text{ for all } \chi \in \mathrm{X}^*_K(Z)\,(\hookrightarrow \mathrm{X}^*_K(S)),$$

here we regard the translation $\varphi(z)$ as an element of $V(S)$.

A3: *For $g \in \mathrm{Aff}(A^{\Theta})$, denote by $dg \in \mathrm{GL}(V(S))$ the derivative of g. Then the map $N(K) \to \mathrm{GL}(V(S))$, $n \mapsto d\varphi(n)$, is induced from the action of $N(K)$ on $\mathrm{X}_*(S)$ (i.e., it is the Weyl group action).*

Moreover, as the central torus of G is K-anisotropic, these three conditions determine the affine structure on $\mathcal{B}(Z_H(S)/K)^{\Theta}$ uniquely; see [T, 1.2].

Proposition 2.12. *Conditions* A1, A2 *and* A3 *hold.*

Proof. The action of $n \in N(K)$ on $\mathcal{B}(Z_H(S)/K)$ carries the apartment $A = A_T$ via an affine isomorphism $f(n) : A \to A_{nTn^{-1}}$ to the apartment $A_{nTn^{-1}}$ corresponding to the torus nTn^{-1} containing S. As $(A_{nTn^{-1}})^{\Theta} = \mathcal{B}(Z_H(S)/K)^{\Theta} = A^{\Theta}$, we see that $f(n)$ keeps A^{Θ} stable and so $\varphi(n) := f(n)|_{A^{\Theta}}$ is an affine automorphism of A^{Θ}.

The derivative $df(n) : V(T) \to V(nTn^{-1})$ is induced from the map

$$\mathrm{Hom}_K(\mathrm{GL}_1, T) = \mathrm{X}_*(T) \to \mathrm{X}_*(nTn^{-1}) = \mathrm{Hom}_K(\mathrm{GL}_1, nTn^{-1}),$$

$\lambda \mapsto \mathrm{Int}\, n \cdot \lambda$, where $\mathrm{Int}\, n$ is the inner automorphism of H determined by $n \in N(K) \subset H(K)$. So, the restriction $d\varphi(n) : V(S) \to V(S)$ is induced from the homomorphism $\mathrm{X}_*(S) \to \mathrm{X}_*(S)$, $\lambda \mapsto \mathrm{Int}\, n \cdot \lambda$. This proves A3.

Condition A3 implies that $d\varphi$ is trivial on $Z(K)$. Therefore, $Z(K)$ acts by translations. The action of the bounded subgroup $Z(K)_b$ on A^{Θ} admits a fixed point by the fixed point theorem of Bruhat-Tits. Therefore, $Z(K)_b$ acts by the trivial translation. This proves A1.

Since the image of $S(K)$ in $Z(K)/Z(K)_b \simeq \mathbb{Z}^{\dim(S)}$ is a subgroup of finite index, to prove the formula in A2, it suffices to prove it for $z \in S(K)$. But for $z \in S(K)$, $zTz^{-1} = T$, and $f(z)$ is a translation of the apartment A ($\varphi(z)$ is regarded as an element of $V(T)$) which satisfies (see 1.9 of [P2]):

$$\chi(f(z)) = -\omega(\chi(z)) \text{ for all } \chi \in \mathrm{X}^*_K(T).$$

This implies the formula in A2, since the restriction map $\mathrm{X}^*_K(T) \to \mathrm{X}^*_K(S)$ is surjective and the image of the restriction map $\mathrm{X}^*_K(Z) \to \mathrm{X}^*_K(S)$ is of finite index in $\mathrm{X}^*_K(S)$. □

2.13. Apartments of $\mathcal{B}$. By definition, the *apartments* of $\mathcal{B}$ are the affine spaces $\mathcal{B}(Z_H(S)/K)^\Theta$ under the $\mathbb{R}$-vector space $V(S)$ (of dimension $= K$-rank G) for maximal K-split tori S of G. For any apartment A of $\mathcal{B}(Z_H(S)/K)$ that contains a Θ-fixed point, $\mathcal{B}(Z_H(S)/K)^\Theta = A^\Theta$ (Proposition 2.10). The subgroup $N_G(S)(K)$ of $G(K)$ acts by affine transformations on the apartment $\mathcal{B}(Z_H(S)/K)^\Theta$ and $Z_G(S)(K)$ acts on it by translations (Proposition 2.12). Conjugacy of maximal K-split tori of G under $G(K)$ implies that this group acts transitively on the set of apartments of $\mathcal{B}$.

Propositions 2.9 and 2.10 imply the following proposition at once:

Proposition 2.14. *Given any two points of $\mathcal{B}$, there is a maximal K-split torus S of G such that the corresponding apartment of $\mathcal{B}$ contains these two points.*

Proposition 2.15. *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$. Then there is a unique maximal K-split torus S of G such that $\mathcal{A} = \mathcal{B}(Z_H(S)/K)^\Theta$. So the stabilizer of $\mathcal{A}$ in $G(K)$ is $N_G(S)(K)$.*

Proof. We fix a maximal K-split torus S of G such that $\mathcal{A} = \mathcal{B}(Z_H(S)/K)^\Theta$. We will show that S is uniquely determined by $\mathcal{A}$. For this purpose, we observe that the subgroup $N_G(S)(K)$ of $G(K)$ acts on $\mathcal{A}$ and the maximal bounded subgroup $Z_G(S)(K)_b$ of $Z_G(S)(K)$ acts trivially (Proposition 2.12). So the subgroup $\mathcal{Z}$ of $G(K)$ consisting of elements that fix $\mathcal{A}$ pointwise is a bounded subgroup of $G(K)$, normalized by $N_G(S)(K)$, and it contains $Z_G(S)(K)_b$. Now, using the Bruhat decomposition of $G(K)$ with respect to S, we see that every bounded subgroup of $G(K)$ that is normalized by $N_G(S)(K)$ is a normal subgroup of the latter. Hence the identity component of the Zariski-closure of $\mathcal{Z}$ is $Z_G(S)$. As S is the unique maximal K-split torus of G contained in $Z_G(S)$, both the assertions follow. □

2.16. The affine Weyl group of G. Let $G(K)^+$ denote the (normal) subgroup of $G(K)$ generated by K-rational elements of the unipotent radicals of parabolic K-subgroups of G. Let S be a maximal K-split torus of G, N and Z respectively be the normalizer and centralizer of S in G. Let $N(K)^+ := N(K) \cap G(K)^+$. Then $N(K)^+$ maps onto the Weyl group $W := N(K)/Z(K)$ of G (this can be seen using, for example, [CGP, Prop. C.2.24(i)]).

Let $\mathcal{A}$ be the apartment of $\mathcal{B}$ corresponding to S. As in 2.11, let $\varphi : N(K) \to \mathrm{Aff}(\mathcal{A})$ be the action map, then the affine Weyl group W_{aff} of G/K is by definition the subgroup $\varphi(N(K)^+)$ of $\mathrm{Aff}(\mathcal{A})$.

3. Bruhat-Tits theory for G over K

3.1. Bruhat-Tits group schemes $\mathscr{G}_\Omega^\circ$. Let Ω be a nonempty Θ-stable bounded subset of an apartment of $\mathcal{B}(H/K)$. Let $\mathscr{H}_\Omega$ be the smooth affine $\mathcal{O}$-group scheme associated to Ω in 2.1. There is a natural action of Θ on $\mathscr{H}_\Omega$ by $\mathcal{O}$-group scheme automorphisms (2.4). Define the functor $\mathscr{H}_\Omega^\Theta$ of Θ-fixed points that associates to a commutative $\mathcal{O}$-algebra C the subgroup $\mathscr{H}_\Omega(C)^\Theta$ of $\mathscr{H}_\Omega(C)$ consisting of elements fixed under Θ. The functor $\mathscr{H}_\Omega^\Theta$ is represented by a closed smooth $\mathcal{O}$-subgroup scheme of $\mathscr{H}_\Omega$ (see Propositions 3.1 and 3.4 of [E], or Proposition A.8.10 of [CGP]); we will denote this closed smooth $\mathcal{O}$-subgroup scheme also by $\mathscr{H}_\Omega^\Theta$. Its generic fiber is H^Θ, and so the identity component of the generic fiber is G. The *neutral component* $(\mathscr{H}_\Omega^\Theta)^\circ$ of $\mathscr{H}_\Omega^\Theta$ is by definition the union of the identity components of its generic and special fibers; it is an open (so smooth) affine $\mathcal{O}$-subgroup scheme [PY2, §3.5] with generic fiber G. The index of the subgroup $(\mathscr{H}_\Omega^\Theta)^\circ(\mathcal{O})$ in $\mathscr{H}_\Omega^\Theta(\mathcal{O})$ is known to be finite [EGA IV$_3$, Cor. 15.6.5]. It is obvious that $(\mathscr{H}_\Omega^\Theta)^\circ = ((\mathscr{H}_\Omega^\circ)^\Theta)^\circ$. We will denote $(\mathscr{H}_\Omega^\Theta)^\circ$ by $\mathscr{G}_\Omega^\circ$ in the sequel and call it *the Bruhat-Tits $\mathcal{O}$-group scheme associated to G and* Ω. The special fiber of $\mathscr{G}_\Omega^\circ$ will be denoted $\overline{\mathscr{G}}_\Omega^\circ$. As $\mathscr{G}_\Omega^\circ(\mathcal{O}) \subset \mathscr{H}_\Omega(\mathcal{O})$, $\mathscr{G}_\Omega^\circ(\mathcal{O})$ fixes Ω pointwise.

3.2. Let $\Omega' \prec \Omega$ be nonempty bounded subsets of an apartment of $\mathcal{B}(H/K)$. We assume that both Ω and Ω' are stable under the action of Θ on $\mathcal{B}(H/K)$. The $\mathcal{O}$-group scheme homomorphism $\mathscr{H}_\Omega \to \mathscr{H}_{\Omega'}$ of [P2, 1.10] restricts to a homomorphism $\rho_{\Omega',\Omega} : \mathscr{H}_\Omega^\circ \to \mathscr{H}_{\Omega'}^\circ$, and by [E, Prop. 3.5], or [CGP, Prop. A.8.10(2)], it induces a $\mathcal{O}$-group scheme homomorphism $\mathscr{H}_\Omega^\Theta \to \mathscr{H}_{\Omega'}^\Theta$. The last homomorphism gives a $\mathcal{O}$-group scheme homomorphism $\rho^G_{\Omega',\Omega} : (\mathscr{H}_\Omega^\Theta)^\circ = \mathscr{G}_\Omega^\circ \to \mathscr{G}_{\Omega'}^\circ = (\mathscr{H}_{\Omega'}^\Theta)^\circ$ that is the identity homomorphism on the generic fiber G.

3.3. Let $\mathcal{A}$ be the apartment of $\mathcal{B}$ corresponding to a maximal K-split torus S of G and Ω be a nonempty bounded subset of $\mathcal{A}$. The apartment $\mathcal{A}$ is contained in an apartment A of $\mathcal{B}(H/K)$ that corresponds to a maximal K-split torus T of H containing S and $\mathcal{A} = A \cap \mathcal{B} = A^\Theta$ (2.13). So Ω is a bounded subset of A. The group scheme $\mathscr{H}_\Omega$ contains a closed split $\mathcal{O}$-torus $\mathscr{T}$ with generic fiber T, see [P2, 1.9]. Let $\mathscr{S}$ be the $\mathcal{O}$-subtorus of $\mathscr{T}$ whose generic fiber is S ($\mathscr{S}$ is the schematic closure of S in $\mathscr{T}$). The automorphism group Θ of $\mathscr{H}_\Omega$ acts trivially on the $\mathcal{O}$-torus $\mathscr{S}$ (since $S \subset G \subset H^\Theta$) and hence this torus is contained in $\mathscr{G}_\Omega^\circ$. The special fiber $\overline{\mathscr{S}}$ of $\mathscr{S}$ is a maximal torus of $\overline{\mathscr{G}}_\Omega^\circ$ since S is a maximal K-split torus of G.

Proposition 3.4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be apartments of $\mathcal{B}$ and Ω a nonempty bounded subset of $\mathcal{A} \cap \mathcal{A}'$. Then there exists an element $g \in \mathscr{G}_\Omega^\circ(\mathcal{O})$ that maps $\mathcal{A}$ onto $\mathcal{A}'$. Any such element fixes Ω pointwise.*

Proof. We will use Proposition 2.1(ii) of [P2], with $\mathcal{O}$ in place of $\mathfrak{o}$, and denote $\mathscr{G}_\Omega^\circ$ by $\mathscr{G}$, and its special fiber by $\overline{\mathscr{G}}$, in this proof. Let S and S' be the maximal K-split tori of G corresponding to the apartments $\mathcal{A}$ and $\mathcal{A}'$ respectively and $\mathscr{S}$ and $\mathscr{S}'$ be the $\mathcal{O}$-tori of $\mathscr{G}$ with generic fibers S and S' respectively. The special fibers $\overline{\mathscr{S}}$ and $\overline{\mathscr{S}}'$ of $\mathscr{S}$ and $\mathscr{S}'$ are maximal split tori of $\overline{\mathscr{G}}$, and hence according to a result of Borel and Tits there is an element $\overline{g}$ of $\overline{\mathscr{G}}(\kappa)$ which conjugates $\overline{\mathscr{S}}$ onto $\overline{\mathscr{S}}'$ [CGP, Thm. C.2.3]. Now [P2, Prop. 2.1(ii)] implies that there exists a $g \in \mathscr{G}(\mathcal{O})$ lying over $\overline{g}$ that conjugates $\mathscr{S}$ onto $\mathscr{S}'$. This element fixes Ω pointwise and conjugates S onto S' and hence maps $\mathcal{A}$ onto $\mathcal{A}'$. □

3.5. Given a point $x \in \mathcal{B}$, for simplicity we will denote $\mathscr{G}_{\{x\}}^\circ$, $\mathscr{H}_{\{x\}}$, $\mathscr{H}_{\{x\}}^\circ$ and $\mathscr{H}_{\{x\}}^\Theta$ by $\mathscr{G}_x^\circ$, $\mathscr{H}_x$, $\mathscr{H}_x^\circ$ and $\mathscr{H}_x^\Theta$ respectively, and the special fibers of these group schemes will be denoted by $\overline{\mathscr{G}}_x^\circ$, $\overline{\mathscr{H}}_x$, $\overline{\mathscr{H}}_x^\circ$ and $\overline{\mathscr{H}}_x^\Theta$ respectively. The subgroup of $H(K)$ (resp. $G(K)$) consisting of elements that fix x will be denoted by $H(K)^x$ (resp. $G(K)^x$). The subgroup $\mathscr{G}_x^\circ(\mathcal{O})$ ($\subset G(K)^x$) is of finite index in $G(K)^x$.

3.6. Parahoric subgroups of $G(K)$. For $x \in \mathcal{B}$, $\mathscr{G}_x^\circ$ and $P_x := \mathscr{G}_x^\circ(\mathcal{O})$ will respectively be called the *Bruhat-Tits parahoric $\mathcal{O}$-group scheme* and the *parahoric subgroup* of $G(K)$ associated with the point x. Let S be a maximal K-split torus of G such that x lies in the apartment $\mathcal{A}$ of $\mathcal{B}$ corresponding to S. Then the group scheme $\mathscr{G}_x^\circ$ contains a closed split $\mathcal{O}$-torus $\mathscr{S}$ whose generic fiber is S (3.3). The parahoric subgroups of $G(K)$ are by definition the subgroups P_x for $x \in \mathcal{B}$. For a given parahoric subgroup P_x, the associated Bruhat-Tits parahoric $\mathcal{O}$-group scheme is $\mathscr{G}_x^\circ$.

(i) Let P be a parahoric subgroup of $G(K)$, $\mathscr{G}^\circ$ the associated Bruhat-Tits parahoric $\mathcal{O}$-group scheme, $\overline{\mathscr{G}}^\circ$ the special fiber of $\mathscr{G}^\circ$, and $\mathcal{P}$ be a subgroup of P of finite index. Then the image of $\mathcal{P}$ in $\overline{\mathscr{G}}^\circ(\kappa)$ is Zariski-dense in the connected group $\overline{\mathscr{G}}^\circ$, so the affine ring of $\mathscr{G}^\circ$ is:

$$\mathcal{O}[\mathscr{G}^\circ] = \{f \in K[G] \mid f(\mathcal{P}) \subset \mathcal{O}\}.$$

Thus the subgroup $\mathcal{P}$ "determines" the group scheme $\mathscr{G}^\circ$, and hence P is the unique parahoric subgroup of $G(K)$ containing $\mathcal{P}$ as a subgroup of finite index.

(ii) Let P and $\mathscr{G}^\circ$ be as in the preceding paragraph. Let Ω be a nonempty Θ-stable bounded subset of an apartment of $\mathcal{B}(H/K)$ and $\mathscr{G}_\Omega^\circ$ be as in 3.1. We assume that Ω is fixed pointwise by P. Then the inclusion of P in $H(K)^\Omega$ ($= \mathscr{H}_\Omega(\mathcal{O})$) gives a $\mathcal{O}$-group scheme homomorphism $\mathscr{G}^\circ \to \mathscr{H}_\Omega^\circ$ (Proposition 2.2). This homomorphism obviously factors through $\mathscr{G}_\Omega^\circ$ to give a $\mathcal{O}$-group scheme homomorphism $\mathscr{G}^\circ \to \mathscr{G}_\Omega^\circ$ that is the identity on the generic fiber G.

Suppose x, $y \in \mathcal{B}(H/K)$ are fixed by P, and $[xy]$ is the geodesic joining x and y. Then P fixes every point z of $[xy]$. Let $\mathscr{G}_{[xy]}^\circ$ be as in 3.1 (for $\Omega = [xy]$). There are $\mathcal{O}$-group scheme homomorphisms $\mathscr{G}^\circ \to \mathscr{G}_{[xy]}^\circ$ and $\mathscr{G}^\circ \to \mathscr{G}_z^\circ$ that are the identity on the generic fiber G.

3.7. Polysimplicial structure on $\mathcal{B}$. Let P be a parahoric subgroup of $G(K)$ and $\mathscr{G}^\circ$ be the Bruhat-Tits parahoric $\mathcal{O}$-group scheme associated with P (3.6). Let $\mathcal{B}(H/K)^P$ denote the set of points of $\mathcal{B}(H/K)$ fixed by P. According to Corollary 2.3, $\mathcal{B}(H/K)^P$ is the union of facets pointwise fixed by P. Let $\overline{\mathcal{F}}_P := \mathcal{B}(H/K)^P \cap \mathcal{B}$. This closed convex subset is by definition the *closed facet* of $\mathcal{B}$ associated with the parahoric subgroup P. The $\mathcal{O}$-group scheme $\mathscr{G}^\circ$ contains a closed split $\mathcal{O}$-torus $\mathscr{S}$ whose generic fiber S is a maximal K-split torus of G (3.3). The subgroup $\mathscr{S}(\mathcal{O})$ (of $S(K)$) is the maximal bounded subgroup of $S(K)$ and it is contained in P $(= \mathscr{G}^\circ(\mathcal{O}))$, so, according to Corollary 2.3, $\overline{\mathcal{F}}_P$ is contained in the enlarged building $\mathcal{B}(Z_H(S)/K)$ of $Z_H(S)(K)$. This implies that the closed facet $\overline{\mathcal{F}}_P$ is contained in the apartment $\mathcal{A} := \mathcal{B}(Z_H(S)/K)^\Theta$ $(= \mathcal{B}(Z_H(S)/K) \cap \mathcal{B})$ of $\mathcal{B}$ corresponding to the maximal K-split torus S of G.

Let $\mathcal{F}_P$ be the subset of points of $\overline{\mathcal{F}}_P$ that are not fixed by any parahoric subgroup of $G(K)$ larger than P. Then $\mathcal{F}_P = \overline{\mathcal{F}}_P - \bigcup_{Q \supsetneq P} \overline{\mathcal{F}}_Q$. Given another parahoric subgroup subgroup Q of $G(K)$, if $\overline{\mathcal{F}}_Q = \overline{\mathcal{F}}_P$, then $Q = P$. (To see this, we choose points x, $y \in \mathcal{B}$ such that $\mathscr{G}_x^\circ(\mathcal{O}) = P$ and $\mathscr{G}_y^\circ(\mathcal{O}) = Q$. Then $y \in \overline{\mathcal{F}}_Q = \overline{\mathcal{F}}_P$. So P fixes y. Now using 3.6 (ii) we see that $P \subset Q$. We similarly see that $Q \subset P$.) Hence if $Q \supsetneq P$, then $\overline{\mathcal{F}}_Q$ is properly contained in $\overline{\mathcal{F}}_P$. By definition, $\mathcal{F}_P$ is the *facet* of $\mathcal{B}$ associated with the parahoric subgroup P of $G(K)$, and as P varies over the set of parahoric subgroups of $G(K)$, these are are all the facets of $\mathcal{B}$. We will show below (Propositions 3.11 and 3.13) that $\mathcal{F}_P$ is convex and bounded.

For a parahoric subgroup Q of $G(K)$ containing P, obviously, $\mathcal{F}_Q \subset \overline{\mathcal{F}}_Q \subset \overline{\mathcal{F}}_P$, thus $\mathcal{F}_Q \prec \mathcal{F}_P$ and hence $\mathcal{F}_P$ is a maximal facet if and only if P is a minimal parahoric subgroup of $G(K)$. The maximal facets of $\mathcal{B}$ are called the *chambers* of $\mathcal{B}$. It is easily seen using the observations contained in 2.5 that all the chambers are of equal dimension. We say that a facet $\mathcal{F}'$ of $\mathcal{B}$ is a *face* of a facet $\mathcal{F}$ if $\mathcal{F}' \prec \mathcal{F}$, i.e., if $\mathcal{F}'$ is contained in the closure of $\mathcal{F}$.

In the following three lemmas (3.8, 3.9 and 3.10), k is any field of characteristic $p \geqslant 0$. We will use the notation introduced in [CGP, §2.1].

Lemma 3.8. *Let $\mathcal{H}$ be a smooth connected affine algebraic k-group and $\mathcal{Q}$ be a pseudo-parabolic k-subgroup of $\mathcal{H}$. Let $\mathcal{S}$ be a k-torus of $\mathcal{Q}$ whose image in the maximal pseudo-reductive quotient $\mathcal{M} := \mathcal{Q}/\mathscr{R}_{u,k}(\mathcal{Q})$ of $\mathcal{Q}$ contains the maximal central torus of $\mathcal{M}$. Then any 1-parameter subgroup $\lambda : \mathrm{GL}_1 \to \mathcal{H}$ such that $\mathcal{Q} = P_{\mathcal{H}}(\lambda)\mathscr{R}_{u,k}(\mathcal{H})$ has a conjugate under $\mathscr{R}_{u,k}(\mathcal{Q})(k)$ with image in $\mathcal{S}$.*

Proof. Let $\lambda : \mathrm{GL}_1 \to \mathcal{H}$ be a 1-parameter subgroup such that $\mathcal{Q} = P_{\mathcal{H}}(\lambda)\mathscr{R}_{u,k}(\mathcal{H})$. The image $\mathcal{T}$ of λ is contained in $\mathcal{Q}$ and it maps into the central torus of $\mathcal{M}$. Therefore, $\mathcal{T}$ is contained in the solvable subgroup $\mathcal{S}\mathscr{R}_{u,k}(\mathcal{Q})$ of $\mathcal{Q}$. Note that as $\mathcal{S}$ is commutative, the derived subgroup of $\mathcal{S}\mathscr{R}_{u,k}(\mathcal{Q})$ is contained in $\mathscr{R}_{u,k}(\mathcal{Q})$, so the maximal k-tori of $\mathcal{S}\mathscr{R}_{u,k}(\mathcal{Q})$ are conjugate to each other under $\mathscr{R}_{u,k}(\mathcal{Q})(k)$ [Bo, Thm. 19.2]. Hence, there is a $u \in \mathscr{R}_{u,k}(\mathcal{Q})(k)$ such that $u\mathcal{T}u^{-1} \subset \mathcal{S}$. Then the image of the 1-parameter subgroup $\mu : \mathrm{GL}_1 \to \mathcal{S}$, defined as $\mu(t) = u\lambda(t)u^{-1}$, is contained in $\mathcal{S}$. □

Lemma 3.9. *Let $\mathcal{H}$ be a smooth connected affine algebraic k-group given with an action by a finite group Θ and $\mathcal{U}$ be a smooth connected Θ-stable unipotent normal k-subgroup of $\mathcal{H}$. We assume that p does not divide the order of Θ. Let $\overline{\mathcal{S}}$ be a Θ-stable k-torus of $\overline{\mathcal{H}} := \mathcal{H}/\mathcal{U}$. Then there exists a Θ-stable k-torus $\mathcal{S}$ in $\mathcal{H}$ that maps isomorphically onto $\overline{\mathcal{S}}$. In particular, there exists a Θ-stable k-torus in $\mathcal{H}$ that maps isomorphically onto the maximal central torus of $\overline{\mathcal{H}}$.*

Proof. Let $\mathcal{T}$ be a k-torus of $\mathcal{H}$ that maps isomorphically onto $\overline{\mathcal{S}}\,(\subset \overline{\mathcal{H}})$. Considering the Θ-stable solvable subgroup $\mathcal{T}\,\mathcal{U}$; using conjugacy under $\mathcal{U}(k)$ of maximal k-tori of this solvable group [Bo, Thm. 19.2], we see that for $\theta \in \Theta$, $\theta(\mathcal{T}) = u(\theta)^{-1}\mathcal{T}u(\theta)$ for some $u(\theta) \in \mathcal{U}(k)$. Let $\mathcal{U}(k) =: \mathcal{U}_0 \supset \mathcal{U}_1 \supset \mathcal{U}_2 \cdots \supset \mathcal{U}_n = \{1\}$ be the descending central series of the nilpotent group $\mathcal{U}(k)$. Each subgroup $\mathcal{U}_i$ is Θ-stable and $\mathcal{U}_i/\mathcal{U}_{i+1}$ is a commutative p-group if $p \neq 0$, and a $\mathbb{Q}$-vector space if $p = 0$. Now let $i \leqslant n$, be the largest integer such that there exists a k-torus $\mathcal{S}$ in $\mathcal{T}\,\mathcal{U}$ that maps onto $\overline{\mathcal{S}}$, and for every $\theta \in \Theta$, there is a $u(\theta) \in \mathcal{U}_i$ such that $\theta(\mathcal{S}) = u(\theta)^{-1}\mathcal{S}u(\theta)$. Let $\mathcal{N}_i$ be the normalizer of $\mathcal{S}$ in $\mathcal{U}_i$. Then, for $\theta \in \Theta$, $\theta(\mathcal{N}_i) = u(\theta)^{-1}\mathcal{N}_i u(\theta)$ and hence as $\mathcal{U}_i/\mathcal{U}_{i+1}$ is commutative, we see that $\theta(\mathcal{N}_i\mathcal{U}_{i+1}) = \mathcal{N}_i\mathcal{U}_{i+1}$, i.e., $\mathcal{N}_i\mathcal{U}_{i+1}$ is Θ-stable. It is easy to see that $\theta \mapsto u(\theta) \bmod (\mathcal{N}_i\mathcal{U}_{i+1})$ is a 1-cocycle on Θ with values in $\mathcal{U}_i/\mathcal{N}_i\mathcal{U}_{i+1}$. But $\mathrm{H}^1(\Theta, \mathcal{U}_i/\mathcal{N}_i\mathcal{U}_{i+1})$ is trivial since the finite group Θ is of order prime to p if $p \neq 0$, and $\mathcal{U}_i/\mathcal{N}_i\mathcal{U}_{i+1}$ is divisible if $p = 0$. So there exits a $u \in \mathcal{U}_i$ such that for all $\theta \in \Theta$, $u^{-1}u(\theta)\theta(u)$ lies in $\mathcal{N}_i\mathcal{U}_{i+1}$. Now let $\mathcal{S}' = u^{-1}\mathcal{S}u$. Then the normalizer of $\mathcal{S}'$ in $\mathcal{U}_i$ is $u^{-1}\mathcal{N}_i u$ and again as $\mathcal{U}_i/\mathcal{U}_{i+1}$ is commutative, $u^{-1}\mathcal{N}_i u \cdot \mathcal{U}_{i+1} = \mathcal{N}_i\mathcal{U}_{i+1}$. For $\theta \in \Theta$, we choose $u'(\theta) \in \mathcal{U}_{i+1}$ such that $u^{-1}u(\theta)\theta(u) \in u^{-1}\mathcal{N}_i u \cdot u'(\theta)$. Then $\theta(\mathcal{S}') = u'(\theta)^{-1}\mathcal{S}'u'(\theta)$ for all $\theta \in \Theta$. This contradicts the maximality of i unless $i = n$. □

Lemma 3.10. *Let $\mathcal{H}$ be a smooth connected affine algebraic k-group given with an action by a finite group Θ. We assume that p does not divide the order of Θ. Let $\mathcal{G} = (\mathcal{H}^{\Theta})^{\circ}$. Then*

(i) *$\mathscr{R}_{u,k}(\mathcal{G}) = (\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}))^{\circ} = (\mathscr{R}_{u,k}(\mathcal{H})^{\Theta})^{\circ}$; moreover, $\mathcal{G}/(\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}))$ is pseudo-reductive, and if k is perfect then $\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}) = \mathscr{R}_{u,k}(\mathcal{G})$.*

(ii) *Given a Θ-stable pseudo-parabolic k-subgroup $\mathcal{Q}$ of $\mathcal{H}$, $\mathcal{P} := \mathcal{G} \cap \mathcal{Q}$ is a pseudo-parabolic k-subgroup of $\mathcal{G}$, so $\mathcal{P}$ is connected and it equals $(\mathcal{Q}^{\Theta})^{\circ}$.*

(iii) *Conversely, given a pseudo-parabolic k-subgroup $\mathcal{P}$ of $\mathcal{G}$, and a maximal k-torus $\mathcal{S} \subset \mathcal{P}$, there is a Θ-stable pseudo-parabolic k-subgroup $\mathcal{Q}$ of $\mathcal{H}$, $\mathcal{Q}$ containing the centralizer $Z_{\mathcal{H}}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{H}$, such that $\mathcal{P} = \mathcal{G} \cap \mathcal{Q} = (\mathcal{Q}^{\Theta})^{\circ}$.*

Proof. The first assertion of (i) immediately follows from [CGP, Prop. A.8.14(2)]. Now we observe that as $\mathscr{R}_{u,k}(\mathcal{G}) = (\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}))^{\circ}$, $(\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}))/\mathscr{R}_{u,k}(\mathcal{G})$ is a finite étale (unipotent) normal subgroup of the pseudo-reductive quotient $\mathcal{G}/\mathscr{R}_{u,k}(\mathcal{G})$ of $\mathcal{G}$ so it is central. Thus the kernel of the quotient map $\pi : \mathcal{G}/\mathscr{R}_{u,k}(\mathcal{G}) \to \mathcal{G}/(\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}))$ is an étale unipotent central subgroup. Hence, $\mathcal{G}/(\mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}))$ is pseudo-reductive as $\mathcal{G}/\mathscr{R}_{u,k}(\mathcal{G})$ is. Moreover, if k is perfect then every pseudo-reductive k-group is reductive and such a group does not contain a nontrivial étale unipotent normal subgroup. This implies that if k is perfect, then $\mathscr{R}_{u,k}(\mathcal{G}) = \mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H})$.

Since $\mathscr{R}_{u,k}(\mathcal{G}) \subset \mathcal{G} \cap \mathscr{R}_{u,k}(\mathcal{H}) \subset \mathcal{G} \cap \mathcal{Q}$, to prove (ii), we can replace $\mathcal{H}$ by its pseudo-reductive quotient $\mathcal{H}/\mathscr{R}_{u,k}(\mathcal{H})$ and assume that $\mathcal{H}$ is pseudo-reductive. Then $\mathcal{G}$ is also pseudo-reductive by (i). Let $\mathcal{U} = \mathscr{R}_{u,k}(\mathcal{Q})$ be the k-unipotent radical of $\mathcal{Q}$; $\mathcal{U}$ is Θ-stable. Let $\mathcal{S}$ be a Θ-stable k-torus in $\mathcal{Q}$ that maps isomorphically onto the maximal central torus of the pseudo-reductive quotient $\overline{\mathcal{Q}} := \mathcal{Q}/\mathcal{U}$ (Lemma 3.9). By Lemma 3.8, there exists a 1-parameter subgroup $\lambda : \mathrm{GL}_1 \to \mathcal{S}$ such that $\mathcal{Q} = P_{\mathcal{H}}(\lambda)$. Let $\mu = \sum_{\theta \in \Theta} \theta \cdot \lambda$. Then μ is invariant under Θ and so it is a 1-parameter subgroup of $\mathcal{G}$. We will now show that $\mathcal{Q} = P_{\mathcal{H}}(\mu)$. Let Φ (resp. Ψ) be the set of weights in the Lie algebra of $\mathcal{Q}$ (resp. $P_{\mathcal{H}}(\mu)$) with respect to the adjoint action of $\mathcal{S}$. Then since $\mathcal{Q}$, $P_{\mathcal{H}}(\mu)$ and $\mathcal{S}$ are Θ-stable, the subsets Φ and Ψ (of $\mathrm{X}(\mathcal{S})$) are stable under the action of Θ on $\mathrm{X}(\mathcal{S})$. Hence, for all $a \in \Phi$, as $\langle a, \lambda \rangle \geqslant 0$, we conclude that $\langle a, \mu \rangle \geqslant 0$. Therefore, $\Phi \subset \Psi$. On the other hand, for $b \in \Psi$, $\langle b, \mu \rangle \geqslant 0$. If $b\,(\in \Psi)$ does not belong to Φ, then for $\theta \in \Theta$, $\theta \cdot b \notin \Phi$, so for all $\theta \in \Theta$, $\langle \theta \cdot b, \lambda \rangle < 0$, which implies that $\langle b, \mu \rangle < 0$. This is a contradiction. Therefore, $\Phi = \Psi$ and so $\mathcal{Q} = P_{\mathcal{H}}(\mu)$. Now observe that $(\mathcal{Q}^{\Theta})^{\circ} \subset \mathcal{G} \cap \mathcal{Q} \subset \mathcal{Q}^{\Theta}$. As $\mathcal{Q}^{\Theta}$ is a smooth subgroup ([E, Prop. 3.4] or [CGP, Prop. A.8.10(2)]), $\mathcal{G} \cap \mathcal{Q}$ is a smooth k-subgroup, and since it contains the pseudo-parabolic k-subgroup $P_{\mathcal{G}}(\mu)$, it is a pseudo-parabolic k-subgroup of $\mathcal{G}$ [CGP, Prop. 3.5.8], hence in particular it is connected. Therefore, $\mathcal{G} \cap \mathcal{Q} = (\mathcal{Q}^{\Theta})^{\circ}$.

Now we will prove (iii). Let $\lambda : \mathrm{GL}_1 \to \mathcal{S}$ be a 1-parameter subgroup such that $\mathcal{P} = P_{\mathcal{G}}(\lambda)\mathscr{R}_{u,k}(\mathcal{G})$. Then $\mathcal{Q} := P_{\mathcal{H}}(\lambda)\mathscr{R}_{u,k}(\mathcal{H})$ is a pseudo-parabolic k-subgroup of $\mathcal{H}$ that is Θ-stable (since λ is Θ-invariant) and it contains $\mathcal{P}$ as well as $Z_{\mathcal{H}}(\mathcal{S})$. According to (ii), $\mathcal{G} \cap \mathcal{Q} = (\mathcal{Q}^{\Theta})^{\circ}$ is a pseudo-parabolic k-subgroup of $\mathcal{G}$ containing $\mathcal{P}$. The Lie algebras of $\mathcal{P}$ and $(\mathcal{Q}^{\Theta})^{\circ}$ are clearly equal. This implies that $\mathcal{P} = \mathcal{G} \cap \mathcal{Q} = (\mathcal{Q}^{\Theta})^{\circ}$ and we have proved (iii). □

Proposition 3.11. *Let P be a parahoric subgroup of $G(K)$ and $\mathcal{F}_P$ and $\overline{\mathcal{F}}_P$ be as in* 3.7*.*

(i) *Given $x \in \mathcal{F}_P$ and $y \in \overline{\mathcal{F}}_P$, for every point z of the geodesic $[xy]$, except possibly for $z = y$, $\mathscr{G}_z^{\circ}(\mathcal{O}) = P$.*

(ii) *Let F be a facet of $\mathcal{B}(H/K)$ that meets $\overline{\mathcal{F}}_P$ and is maximal among such facets. Then $\mathscr{G}_F^{\circ}(\mathcal{O}) = P$. Thus $F \cap \mathcal{B} \subset \mathcal{F}_P$.*

The first assertion of this proposition implies that $\mathcal{F}_P$ is convex. The second assertion implies that $\mathcal{F}_P$ is an open-dense subset of $\overline{\mathcal{F}}_P$, hence the closure of $\mathcal{F}_P$ is $\overline{\mathcal{F}}_P$.

Proof. To prove the first assertion, let $[xy]$ be the geodesic joining x and y. Let $F_0, F_1, \ldots, F_n$ be the facets of $\mathcal{B}(H/K)$ containing a segment of positive length of the geodesic $[xy]$ (so each F_i is Θ-stable and is fixed pointwise by P, hence $P \subset \mathscr{G}_{F_i}^{\circ}(\mathcal{O})$, cf. 3.6(ii)). Then $[xy] \subset \bigcup_i \overline{F}_i$. We assume the facets $\{F_i\}$ indexed so that x lies in $\overline{F}_0$, y lies in $\overline{F}_n$, and for each $i < n$, $\overline{F}_i \cap \overline{F}_{i+1}$ is nonempty. Let $z_0 = x$. For every positive integer $i\,(\leqslant n)$, $\overline{F}_{i-1}\cap\overline{F}_i$ contains a unique point of $[xy]$; we will denote this point by z_i.

To prove the second assertion of the proposition along with the first, we take x to be a point of $\mathcal{B}$ such that $\mathscr{G}_x^\circ(\mathcal{O}) = P$ (so $x\in\mathcal{F}_P$) and take y to be any point of $F\cap\mathcal{B}$. Let $[xy]$, and for $i\leqslant n$, F_i and z_i be as in the preceding paragraph. Then $F_n = F$.

Since $x\in\overline{F}_0$, there is a $\mathcal{O}$-group scheme homomorphism $\mathscr{G}_{F_0}^\circ\to\mathscr{G}_x^\circ$ that is the identity on the generic fiber G. Thus, $\mathscr{G}_{F_0}^\circ(\mathcal{O})\subset P$. But $P\subset\mathscr{G}_{F_0}^\circ(\mathcal{O})$, so $\mathscr{G}_{z_0}^\circ(\mathcal{O}) = \mathscr{G}_{F_0}^\circ(\mathcal{O}) = P$. Let $j\,(\leqslant n)$ be a positive integer such that for all $i<j$, $\mathscr{G}_{z_i}^\circ(\mathcal{O}) = \mathscr{G}_{F_i}^\circ(\mathcal{O}) = P$. The inclusion of $\{z_j\}$ in $\overline{F}_{j-1}\cap\overline{F}_j$ gives rise to $\mathcal{O}$-group scheme homomorphisms $\mathscr{H}_{F_{j-1}}\xrightarrow{\sigma_j}\mathscr{H}_{z_j}\xleftarrow{\rho_j}\mathscr{H}_{F_j}$ that are the identity on the generic fiber H. The images of the induced homomorphisms $\overline{\mathscr{H}}^\circ_{F_{j-1}}\xrightarrow{\overline{\sigma}_j}\overline{\mathscr{H}}^\circ_{z_j}\xleftarrow{\overline{\rho}_j}\overline{\mathscr{H}}^\circ_{F_j}$ are pseudo-parabolic κ_s-subgroups of $\overline{\mathscr{H}}^\circ_{z_j}$ ([P2, 1.10(2)]). We conclude by Lie algebra consideration that $\overline{\sigma}_j(\overline{\mathscr{G}}^\circ_{F_{j-1}}) = (\overline{\sigma}_j(\overline{\mathscr{H}}^\circ_{F_{j-1}})^\Theta)^\circ$ and $\overline{\rho}_j(\overline{\mathscr{G}}^\circ_{F_j}) = (\overline{\rho}_j(\overline{\mathscr{H}}^\circ_{F_j})^\Theta)^\circ$, and Lemma 3.10(ii) implies that both of these subgroups are pseudo-parabolic subgroups of $\overline{\mathscr{G}}^\circ_{z_j}$. As $\mathscr{G}^\circ_{F_{j-1}}(\mathcal{O}) = P$, whereas, $P\subset\mathscr{G}^\circ_{F_j}(\mathcal{O})\,(\subset\mathscr{G}^\circ_{z_j}(\mathcal{O}))$, we see that $\overline{\sigma}_j(\overline{\mathscr{G}}^\circ_{F_{j-1}})$ is contained in $\overline{\rho}_j(\overline{\mathscr{G}}^\circ_{F_j})$. Let $\overline{Q}$ and $\overline{Q}'$ respectively be the images of $\overline{\sigma}_j(\overline{\mathscr{G}}^\circ_{F_{j-1}})$ and $\overline{\rho}_j(\overline{\mathscr{G}}^\circ_{F_j})$ in the maximal pseudo-reductive quotient $\overline{G}^{\rm pred}_{z_j} := \overline{\mathscr{G}}^\circ_{z_j}/\mathscr{R}_{u,\kappa_s}(\overline{\mathscr{G}}^\circ_{z_j})$ of $\overline{\mathscr{G}}^\circ_{z_j}$. Then $\overline{Q}\subset\overline{Q}'$, and both of them are pseudo-parabolic subgroups of $\overline{G}^{\rm pred}_{z_j}$.

Now let S be a maximal K-split torus of G such that the apartment of $\mathcal{B}$ corresponding to S contains the geodesic $[xy]$ and let $v\in V(S)$ so that $v+x = y$. Then for all sufficiently small positive real number ϵ, $-\epsilon v+z_j\in F_{j-1}$ and $\epsilon v+z_j\in F_j$. Using [P2, 1.10(3)] we infer that the images of the pseudo-parabolic subgroups $\overline{\sigma}_j(\overline{\mathscr{H}}^\circ_{F_{j-1}})$ and $\overline{\rho}_j(\overline{\mathscr{H}}^\circ_{F_j})$ (of $\overline{\mathscr{H}}^\circ_{z_j}$) in the maximal pseudo-reductive quotient $\overline{H}^{\rm pred}_{z_j} := \overline{\mathscr{H}}^\circ_{z_j}/\mathscr{R}_{u,\kappa_s}(\overline{\mathscr{H}}^\circ_{z_j})$ of $\overline{\mathscr{H}}^\circ_{z_j}$ are opposite pseudo-parabolic subgroups. Therefore, the image $\mathcal{H}$ of $\overline{\sigma}_j(\overline{\mathscr{H}}^\circ_{F_{j-1}})\cap\overline{\rho}_j(\overline{\mathscr{H}}^\circ_{F_j})$ in $\overline{H}^{\rm pred}_{z_j}$ is pseudo-reductive. Proposition A.8.14 (2) of [CGP] implies then that $(\mathcal{H}^\Theta)^\circ$ is pseudo-reductive. It is obvious that under the natural homomorphism $\pi:\overline{G}^{\rm pred}_{z_j}\to\overline{H}^{\rm pred}_{z_j}$, the image of $\overline{Q} = \overline{Q}\cap\overline{Q}'$ is $(\mathcal{H}^\Theta)^\circ$. As the kernel of the homomorphism π is a finite (étale unipotent) subgroup (Lemma 3.10(i)), and $(\mathcal{H}^\Theta)^\circ$ is pseudo-reductive, we see that $\overline{Q}$ is a pseudo-reductive subgroup of $\overline{G}^{\rm pred}_{z_j}$. But since $\overline{Q}$ is a pseudo-parabolic subgroup of the latter, we must have $\overline{Q} = \overline{G}^{\rm pred}_{z_j}$, and hence, $\overline{Q}' = \overline{G}^{\rm pred}_{z_j}$. So, $\overline{\sigma}_j(\overline{\mathscr{G}}^\circ_{F_{j-1}}) = \overline{\mathscr{G}}^\circ_{z_j} = \overline{\rho}_j(\overline{\mathscr{G}}^\circ_{F_j})$.

Since the natural homomorphism $\mathscr{G}^\circ_{F_{j-1}}(\mathcal{O})\to\overline{\mathscr{G}}^\circ_{F_{j-1}}(\kappa)$ is surjective (as $\mathcal{O}$ is henselian and $\mathscr{G}^\circ_{F_{j-1}}$ is smooth, [EGA IV$_4$, 18.5.17]), and $\overline{\sigma}_j(\overline{\mathscr{G}}^\circ_{F_{j-1}}) = \overline{\mathscr{G}}^\circ_{z_j}$, the image of $\mathscr{G}^\circ_{F_{j-1}}(\mathcal{O})\,(\subset \mathscr{G}^\circ_{z_j}(\mathcal{O}))$ in $\overline{\mathscr{G}}^\circ_{z_j}(\kappa)$ is Zariski-dense in $\overline{\mathscr{G}}^\circ_{z_j}$. From this we see that

$$\mathcal{O}[\mathscr{G}^\circ_{z_j}] = \{f \in K[G] \mid f(\mathscr{G}^\circ_{F_{j-1}}(\mathcal{O})) \subset \mathcal{O}\} = \mathcal{O}[\mathscr{G}^\circ_{F_{j-1}}],$$

cf. [BrT2, 1.7.2] and 2.1. Therefore, $\sigma_j|_{\mathscr{G}^\circ_{F_{j-1}}} : \mathscr{G}^\circ_{F_{j-1}} \to \mathscr{G}^\circ_{z_j}$ is a $\mathcal{O}$-group scheme isomorphism. We similarly see that $\rho_j|_{\mathscr{G}^\circ_{F_j}} : \mathscr{G}^\circ_{F_j} \to \mathscr{G}^\circ_{z_j}$ is a $\mathcal{O}$-group scheme isomorphism. Now since $\mathscr{G}^\circ_{F_{j-1}}(\mathcal{O}) = P$, we conclude that $P = \mathscr{G}^\circ_{z_j}(\mathcal{O}) = \mathscr{G}^\circ_{F_j}(\mathcal{O})$. By induction it follows that $P = \mathscr{G}^\circ_{z_i}(\mathcal{O}) = \mathscr{G}^\circ_{F_i}(\mathcal{O})$ for all $i \leqslant n$. In particular, for all $z \in [xy]$, except possibly for $z = y$, $\mathscr{G}^\circ_z(\mathcal{O}) = P$, and $\mathscr{G}^\circ_{F_n}(\mathcal{O}) = P$. ☐

For parahoric subgroups P and Q of $G(K)$, if $\mathcal{F}_P \cap \mathcal{F}_Q$ is nonempty, then for any z in this intersection, $P = \mathscr{G}^\circ_z(\mathcal{O}) = Q$ (Proposition 3.11). Thus every point of $\mathcal{B}$ is contained in a unique facet.

We will use the following simple lemma in the proof of the next proposition.

Lemma 3.12. *Let S be a maximal K-split torus of G, $\mathcal{A}$ the corresponding apartment of $\mathcal{B}$, and $\mathcal{C}$ be a noncompact closed convex subset of $\mathcal{A}$. Then for any point $x \in \mathcal{C}$, there is an infinite ray originating at x and contained in $\mathcal{C}$.*

Proof. Recall that $\mathcal{A}$ is an affine space under the vector space $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$. We identify $\mathcal{A}$ with $V(S)$ using translations by elements in the latter, with x identified with the origin 0, and use a positive definite inner product on $V(S)$ to get a norm on $\mathcal{A}$. With this identification, $\mathcal{C}$ is a closed convex subset of $V(S)$ containing 0. Since $\mathcal{C}$ is noncompact, there exist unit vectors $v_i \in V(S)$, $i \geqslant 1$, and positive real numbers $s_i \to \infty$ such that $s_i v_i$ lies in $\mathcal{C}$. After replacing $\{v_i\}$ by a subsequence, we may (and do) assume that the sequence $\{v_i\}$ converges to a unit vector v. We will now show that for every nonnegative real number t, tv lies in $\mathcal{C}$, this will prove the lemma. To see that tv lies in $\mathcal{C}$, it suffices to observe that for a given t, the sequence $\{tv_i\}$ converges to tv, and for all sufficiently large i (so that $s_i \geqslant t$), tv_i lies in $\mathcal{C}$. ☐

Proposition 3.13. *For any parahoric subgroup P of $G(K)$, the associated closed facet $\overline{\mathcal{F}}_P$ of $\mathcal{B}$, and so also the associated facet $\mathcal{F}_P\,(\subset \overline{\mathcal{F}}_P)$, is bounded.*

Proof. Let S be a maximal K-split torus of G such that the corresponding apartment of $\mathcal{B}$ contains $\overline{\mathcal{F}}_P$ (3.7). Assume, if possible, that $\overline{\mathcal{F}}_P$ is noncompact and fix a point x of $\mathcal{F}_P$. Then, according to the preceding lemma, there is an infinite ray $\mathcal{R} := \{tv + x \mid t \in \mathbb{R}_{\geqslant 0}\}$, for some $v \in V(S)$, originating at x and contained in $\overline{\mathcal{F}}_P$. It is obvious from Proposition 3.11(i) that this ray is actually contained in $\mathcal{F}_P$. Hence, for every point $z \in \mathcal{R}$, $\mathscr{G}^\circ_z(\mathcal{O}) = P$.

As the central torus of G has been assumed to be K-anisotropic, there is a nondivisible root a of G, with respect to S, such that $\langle a,\, v\rangle > 0$. Let S_a be the identity component of the kernel of a and G_a (resp. H_a) be the derived subgroup of the centralizer of S_a in G (resp. H). Fix $t \in \mathbb{R}_{\geqslant 0}$, and let $y = tv + x \in \mathcal{R}$. Let $\mathscr{S}$ be the closed 1-dimensional $\mathcal{O}$-split torus of $\mathscr{G}^\circ_y$ whose generic fiber is the maximal K-split torus of G_a contained in S and let $\lambda : \mathrm{GL}_1 \to \mathscr{S}\,(\hookrightarrow \mathscr{G}^\circ_y \hookrightarrow \mathscr{H}_y)$ be the $\mathcal{O}$-isomorphism such that $\langle a, \lambda\rangle > 0$. Let $c = \langle a, v\rangle / \langle a, \lambda\rangle$. Then $\langle a, v - c\lambda\rangle = 0$.

Let $\mathscr{U}_y$ be the $\mathcal{O}$-subgroup scheme of $\mathscr{H}_y$ representing the functor

$$R \rightsquigarrow \{h \in \mathscr{H}_y(R) \mid \lim_{t\to 0} \lambda(t)h\lambda(t)^{-1} = 1\},$$

cf. [CGP, Lemma 2.1.5]. Using the last assertion of 2.1.8(3), and the first assertion of 2.1.8(4), of [CGP] (with k, which is an an arbitrary commutative ring in these assertions, replaced by $\mathcal{O}$, and G replaced by $\mathscr{H}_y$), we see that $\mathscr{U}_y$ is a closed smooth unipotent $\mathcal{O}$-subgroup scheme of $\mathscr{H}_y$ with connected fibers; the generic fiber of $\mathscr{U}_y$ is $U_H(\lambda)$, where $U_H(\lambda)$ is as in [CGP, Lemma 2.1.5] with G replaced by H. We consider the smooth closed $\mathcal{O}$-subgroup scheme $\mathscr{U}_y^{\Theta}$ of $\mathscr{U}_y$. As $\mathscr{U}_y^{\Theta}$ is clearly normalized by $\mathscr{S}$, it has connected fibers, and hence it is contained in $(\mathscr{H}_y^{\Theta})^{\circ} = \mathscr{G}_y^{\circ}$. The generic fiber of $\mathscr{U}_y^{\Theta}$ is $U_H(\lambda)^{\Theta}$ that contains the root group U_a $(= U_{G_a}(\lambda))$ of G corresponding to the root a.

As $\bigcup_{z\in\mathcal{R}} \mathscr{U}_z(\mathcal{O}) \supset U_{H_a}(\lambda)(K) \supset U_a(K)$, we see that $\bigcup_{z\in\mathcal{R}} \mathscr{U}_z^{\Theta}(\mathcal{O}) \supset U_a(K)$. Now since $\mathscr{G}_z^{\circ} \supset \mathscr{U}_z^{\Theta}$, we conclude that $\bigcup_{z\in\mathcal{R}} \mathscr{G}_z^{\circ}(\mathcal{O}) \supset U_a(K)$. But for all $z \in \mathcal{R}$, $\mathscr{G}_z^{\circ}(\mathcal{O}) = P$, so the parahoric subgroup P contains the unbounded subgroup $U_a(K)$. This is a contradiction. $\square$

Proposition 3.13 implies that each closed facet of $\mathcal{B}$ is a compact polyhedron. Considering the facets lying on the boundary of a maximal closed facet of $\mathcal{B}$, we see that $\mathcal{B}$ contains facets of every dimension $\leqslant K\text{-rank}\, G$.

3.14. Let P be a parahoric subgroup of $G(K)$ and $\mathcal{F} := \mathcal{F}_P$ be the facet of $\mathcal{B}$ associated to P in 3.7. Then for any $x \in \mathcal{F}$, since $P \subset \mathscr{G}_{\mathcal{F}}^{\circ}(\mathcal{O}) \subset \mathscr{G}_x^{\circ}(\mathcal{O}) = P$ (3.6(ii)), $\mathscr{G}_{\mathcal{F}}^{\circ}(\mathcal{O}) = P$ and hence the natural $\mathcal{O}$-group scheme homomorphism $\mathscr{G}_{\mathcal{F}}^{\circ} \to \mathscr{G}_x^{\circ}$ is an isomorphism. In particular, for any facet F of $\mathcal{B}(H/K)$ that meets $\mathcal{F}$, $\mathscr{G}_{\mathcal{F}}^{\circ} = \mathscr{G}_F^{\circ}$.

Proposition 3.15. *Let $\mathcal{F}$ be a facet of $\mathcal{B}$. Then the κ-unipotent radical $\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}_{\mathcal{F}}^{\circ})$ of $\overline{\mathscr{G}}_{\mathcal{F}}^{\circ}$ equals $(\overline{\mathscr{G}}_{\mathcal{F}}^{\circ} \cap \mathscr{R}_{u,\kappa}(\overline{\mathscr{H}}_{\mathcal{F}}^{\circ}))^{\circ}$.*

Let $\mathcal{F}$ and $\mathcal{F}'$ be two facets of $\mathcal{B}$, with $\mathcal{F}' \prec \mathcal{F}$. Then:

(i) *The kernel of the induced homomorphism $\overline{\rho}_{\mathcal{F}',\mathcal{F}}^{G} : \overline{\mathscr{G}}_{\mathcal{F}}^{\circ} \to \overline{\mathscr{G}}_{\mathcal{F}'}^{\circ}$ between the special fibers is a smooth unipotent κ-subgroup of $\overline{\mathscr{G}}_{\mathcal{F}}^{\circ}$ and the image $\mathfrak{p}(\mathcal{F}'/\mathcal{F})$ is a pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}_{\mathcal{F}'}^{\circ}$.*

(ii) *If F and F' are facets of $\mathcal{B}(H/K)$, $F' \prec F$, that meet $\mathcal{F}$ and $\mathcal{F}'$ respectively, then $\mathfrak{p}(\mathcal{F}'/\mathcal{F}) = (\overline{\mathscr{Q}}^{\Theta})^{\circ}$, where $\overline{\mathscr{Q}}$ is the image of $\overline{\rho}_{F',F} : \overline{\mathscr{H}}_F^{\circ} \to \overline{\mathscr{H}}_{F'}^{\circ}$.*

(iii) *The inverse image of the subgroup $\mathfrak{p}(\mathcal{F}'/\mathcal{F})(\kappa)$ of $\overline{\mathscr{G}}_{\mathcal{F}'}^{\circ}(\kappa)$, under the natural surjective homomorphism $\mathscr{G}_{\mathcal{F}'}^{\circ}(\mathcal{O}) \to \overline{\mathscr{G}}_{\mathcal{F}'}^{\circ}(\kappa)$, is $\rho_{\mathcal{F}',\mathcal{F}}^{G}(\mathscr{G}_{\mathcal{F}}^{\circ}(\mathcal{O}))$ $(\subset \mathscr{G}_{\mathcal{F}'}^{\circ}(\mathcal{O}))$.*

Given a pseudo-parabolic κ-subgroup $\overline{\mathscr{P}}$ of $\overline{\mathscr{G}}_{\mathcal{F}'}^{\circ}$, there is a facet $\mathcal{F}$ of $\mathcal{B}$ with $\mathcal{F}' \prec \mathcal{F}$ such that the image of the homomorphism $\overline{\rho}_{\mathcal{F}',\mathcal{F}}^{G} : \overline{\mathscr{G}}_{\mathcal{F}}^{\circ} \to \overline{\mathscr{G}}_{\mathcal{F}'}^{\circ}$ equals $\overline{\mathscr{P}}$.

Proof. The first assertion of the proposition follows immediately from Lemma 3.10(i).

To prove (i), we fix $x \in \mathcal{F}'$ and let F' be the facet of $\mathcal{B}(H/K)$ containing x. As the closure of $\mathcal{F}$ contains x, there is a facet F of $\mathcal{B}(H/K)$ that meets $\mathcal{F}$ and contains x in its closure. Then $F' \subset \overline{F}$, i.e., $F' \prec F$, and F and F' meet $\mathcal{F}$ and $\mathcal{F}'$ respectively. Hence, $\mathscr{G}_{\mathcal{F}}^{\circ} = \mathscr{G}_{F}^{\circ} = (\mathscr{H}_F^{\Theta})^{\circ}$ and $\mathscr{G}_{\mathcal{F}'}^{\circ} = \mathscr{G}_{F'}^{\circ} = (\mathscr{H}_{F'}^{\Theta})^{\circ}$ (3.14). Now we will prove assertions (i) and (ii) together. The kernel $\overline{\mathscr{K}}$ of the homomorphism $\overline{\rho}_{F',F} : \overline{\mathscr{H}}^{\circ}_F \to \overline{\mathscr{H}}^{\circ}_{F'}$ is a smooth unipotent κ-subgroup, and the image $\overline{\mathscr{Q}}$ is a pseudo-parabolic κ-subgroup of $\overline{\mathscr{H}}^{\circ}_{F'}$ [P2, 1.10 (1), (2)]. The pseudo-parabolic subgroup $\overline{\mathscr{Q}}$ is clearly Θ-stable as the facets F and F' are Θ-stable. The kernel of $\overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}$ is $\overline{\mathscr{K}} \cap \overline{\mathscr{G}}^{\circ}_{\mathcal{F}}$, and its image is contained in $(\overline{\mathscr{Q}}^{\Theta})^{\circ}$. Therefore, the kernel of $\overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}$ contains $(\overline{\mathscr{K}}^{\Theta})^{\circ}$ and is contained in $\overline{\mathscr{K}}^{\Theta}$. As $\overline{\mathscr{K}}^{\Theta}$ is a smooth subgroup of $\overline{\mathscr{K}}$, we see that the kernel of $\overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}$ is smooth.

Since the image of the Lie algebra homomorphism $\mathrm{L}(\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}) \to \mathrm{L}(\overline{\mathscr{G}}^{\circ}_{\mathcal{F}'})$ induced by $\overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}$ is $\mathrm{L}(\overline{\mathscr{Q}})^{\Theta}$, the containment $\mathfrak{p}(\mathcal{F}'/\mathcal{F}) = \overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}(\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}) \subset (\overline{\mathscr{Q}}^{\Theta})^{\circ}$ is equality. According to Lemma 3.10(ii), $(\overline{\mathscr{Q}}^{\Theta})^{\circ}$ is a pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}'}$.

To prove (iii), let $F' \prec F$ be as in the proof of (i) above and $\overline{\mathscr{Q}}$ be the image of $\overline{\rho}_{F',F} : \overline{\mathscr{H}}^{\circ}_F \to \overline{\mathscr{H}}^{\circ}_{F'}$. Then, as we saw above, $\overline{\mathscr{Q}}$ is a Θ-stable pseudo-parabolic κ-subgroup of $\overline{\mathscr{H}}^{\circ}_{F'}$ and $\mathfrak{p}(\mathcal{F}'/\mathcal{F}) = \overline{\mathscr{P}} := (\overline{\mathscr{Q}}^{\Theta})^{\circ}$. The inverse image of the subgroup $\overline{\mathscr{Q}}(\kappa)$ of $\overline{\mathscr{H}}^{\circ}_{F'}(\kappa)$ under the natural surjective homomorphism $\mathscr{H}^{\circ}_{F'}(\mathcal{O}) \to \overline{\mathscr{H}}^{\circ}_{F'}(\kappa)$ equals $\rho_{F',F}(\mathscr{H}^{\circ}_{F}(\mathcal{O}))$ $(\subset \mathscr{H}^{\circ}_{F'}(\mathcal{O}))$, see [P2, 1.10 (4)]. Let $\mathscr{G}_F = (\mathscr{H}^{\circ}_F)^{\Theta}$ and $\mathscr{G}_{F'} = (\mathscr{H}^{\circ}_{F'})^{\Theta}$. We will denote the $\mathcal{O}$-group scheme homomorphism $\mathscr{G}_F \to \mathscr{G}_{F'}$ induced by $\rho_{F',F}$ by $\rho^{\Theta}_{F',F}$; the corresponding homomorphism $\overline{\mathscr{G}}_F \to \overline{\mathscr{G}}_{F'}$ between the special fibers of $\mathscr{G}_F$ and $\mathscr{G}_{F'}$ will be denoted by $\overline{\rho}^{\Theta}_{F',F}$. The neutral components of $\mathscr{G}_F$ and $\mathscr{G}_{F'}$ are $\mathscr{G}^{\circ}_{\mathcal{F}}$ and $\mathscr{G}^{\circ}_{\mathcal{F}'}$ respectively (3.14). Let $\mathscr{G}^{\natural}_F$ $(\supset \mathscr{G}^{\circ}_{\mathcal{F}})$ be the inverse image of $\mathscr{G}^{\circ}_{\mathcal{F}'}$ in $\mathscr{G}_F$ under $\rho^{\Theta}_{F',F}$. Since the homomorphism $\rho_{F',F}$ is the identity on the generic fiber H, we infer that $h \in \mathscr{H}^{\circ}_F(\mathcal{O})$ is fixed under Θ if and only if so is $\rho_{F',F}(h)$, and as the generic fiber of both $\mathscr{G}^{\circ}_{\mathcal{F}}$ and $\mathscr{G}^{\circ}_{\mathcal{F}'}$ is G, the generic fiber of $\mathscr{G}^{\natural}_F$ is also G. It is easily seen now that the inverse image of the subgroup $\mathfrak{p}(\mathcal{F}'/\mathcal{F})(\kappa)$ of $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}'}(\kappa)$, under the natural surjective homomorphism $\mathscr{G}^{\circ}_{\mathcal{F}'}(\mathcal{O}) \to \overline{\mathscr{G}}^{\circ}_{\mathcal{F}'}(\kappa)$, is $\rho^{\Theta}_{F',F}(\mathscr{G}^{\natural}_F(\mathcal{O}))$. We will presently show that the last group equals $\rho^{G}_{\mathcal{F}',\mathcal{F}}(\mathscr{G}^{\circ}_{\mathcal{F}}(\mathcal{O}))$, this will prove (iii).

$\mathscr{G}^{\natural}_F$ is the union of its generic fiber G and its special fiber $\overline{\mathscr{G}}^{\natural}_F$; and the identity component of $\overline{\mathscr{G}}^{\natural}_{\mathcal{F}}$ is clearly $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}$. We have shown above that the image $\overline{\mathscr{P}}$ of $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}$ under the homomorphism $\overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}$ is a pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}'}$ and the kernel of this homomorphism is smooth. Hence, as κ is separably closed, $\overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}(\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}(\kappa)) = \overline{\mathscr{P}}(\kappa)$. So, according to [CGP, Thm. C.2.23], there is a pseudo-parabolic κ-subgroup $\overline{\mathscr{P}}'$ of $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}'}$, that contains $\overline{\mathscr{P}}$, such that $\overline{\rho}^{\Theta}_{F',F}(\overline{\mathscr{G}}^{\natural}_F(\kappa)) = \overline{\mathscr{P}}'(\kappa)$. But since κ is infinite, $\overline{\mathscr{P}}'(\kappa)/\overline{\mathscr{P}}(\kappa)$ is infinite unless $\overline{\mathscr{P}}' = \overline{\mathscr{P}}$. So we conclude that $\overline{\mathscr{P}}' = \overline{\mathscr{P}}$, and then $\overline{\rho}^{\Theta}_{F',F}(\overline{\mathscr{G}}^{\natural}_F(\kappa)) = \overline{\mathscr{P}}(\kappa) = \overline{\rho}^{G}_{\mathcal{F}',\mathcal{F}}(\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}(\kappa))$. Now using this, and the fact that the natural homomorphism $\mathscr{G}^{\circ}_{\mathcal{F}}(\mathcal{O}) \to \overline{\mathscr{G}}^{\circ}_{\mathcal{F}}(\kappa)$ is surjective (since $\mathcal{O}$ is henselian and

$\mathscr{G}_{\mathcal{F}}^{\circ}$ is smooth, [EGA IV$_4$, 18.5.17]) and the kernel of this homomorphism equals the kernel of the natural surjective homomorphism $\mathscr{G}_F^{\natural}(\mathcal{O}) \to \overline{\mathscr{G}}_F^{\natural}(\kappa)$, we see that $\rho^G_{\mathcal{F}',\mathcal{F}}(\mathscr{G}_{\mathcal{F}}^{\circ}(\mathcal{O})) = \rho^{\Theta}_{F',F}(\mathscr{G}_F^{\natural}(\mathcal{O}))$. This proves (iii).

Finally, to prove the last assertion of the proposition, we fix a facet F' of $\mathcal{B}(H/K)$ that meets $\mathcal{F}'$. Then $\mathscr{G}_{\mathcal{F}'}^{\circ} = \mathscr{G}_{F'}^{\circ}$ (3.14). Using Lemma 3.10(iii) for κ in place of k and $\overline{\mathscr{H}}^{\circ}_{F'}$ in place of $\mathcal{H}$, we find a Θ-stable pseudo-parabolic κ-subgroup $\overline{\mathscr{Q}}$ of $\overline{\mathscr{H}}^{\circ}_{F'}$ such that $\overline{\mathscr{P}} = (\overline{\mathscr{Q}}^{\Theta})^{\circ}$. Let $(F' \prec)\, F$ be the facet of $\mathcal{B}(H/K)$ corresponding to the pseudo-parabolic κ-subgroup $\overline{\mathscr{Q}}$ of $\overline{\mathscr{H}}^{\circ}_{F'}$. Then F is stable under Θ-action. As $F' \prec F$, there is a natural $\mathcal{O}$-group scheme homomorphism $\rho_{F',F} : \mathscr{H}_F^{\circ} \to \mathscr{H}_{F'}^{\circ}$ that restricts to a $\mathcal{O}$-group scheme homomorphism $\rho^G_{F',F} : \mathscr{G}_F^{\circ} \to \mathscr{G}_{F'}^{\circ}$. Let $\overline{\mathscr{Q}}$ be the image of the former. Then according to (ii), the image of the latter is $(\overline{\mathscr{Q}}^{\Theta})^{\circ} = \overline{\mathscr{P}}$. Let $P = \mathscr{G}_F^{\circ}(\mathcal{O}) \subset \mathscr{G}_{F'}^{\circ}(\mathcal{O}) =: Q$, and $\mathcal{F} = \mathcal{F}_P$. Then $P \subset Q$ are parahoric subgroups of $G(K)$, $\mathcal{F}' = \mathcal{F}_Q \subset \overline{\mathcal{F}}_Q \subset \overline{\mathcal{F}}_P = \overline{\mathcal{F}}$, thus $\mathcal{F}' \prec \mathcal{F}$. As F and F' meet $\mathcal{F}$ and $\mathcal{F}'$ respectively, $\mathscr{G}_{\mathcal{F}}^{\circ} = \mathscr{G}_F^{\circ}$ and $\mathscr{G}_{\mathcal{F}'}^{\circ} = \mathscr{G}_{F'}^{\circ}$ (3.14), and hence the image of the homomorphism $\overline{\rho}^G_{\mathcal{F}',\mathcal{F}} : \overline{\mathscr{G}}^{\circ}_{\mathcal{F}} \to \overline{\mathscr{G}}^{\circ}_{\mathcal{F}'}$ equals $\overline{\mathscr{P}}$. □

Proposition 3.15 and [CGP, Propositions 2.2.10 and 3.5.1] imply the following. (Recall that the residue field κ of K has been assumed to be separably closed!)

Corollary 3.16. (i) *A facet $\mathcal{F}$ of $\mathcal{B}$ is a chamber (=maximal facet) if and only if $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}$ does not contain a proper pseudo-parabolic κ-subgroup. Equivalently, $\mathcal{F}$ is a chamber if and only if the pseudo-reductive quotient $\overline{G}^{\mathrm{pred}}_{\mathcal{F}}$ is commutative (this is the case if and only if $\overline{G}^{\mathrm{pred}}_{\mathcal{F}}$ contains a unique maximal κ-torus, or, equivalently, every torus of this pseudo-reductive group is central).*

(ii) *The codimension of a facet $\mathcal{F}$ of $\mathcal{B}$ equals the κ-rank of the derived subgroup of the pseudo-split pseudo-reductive quotient $\overline{G}^{\mathrm{pred}}_{\mathcal{F}} := \overline{\mathscr{G}}^{\circ}_{\mathcal{F}}/\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^{\circ}_{\mathcal{F}})$ of $\overline{\mathscr{G}}^{\circ}_{\mathcal{F}}$.*

We will now establish the following analogues of Propositions 3.5–3.7 of [P2].

Proposition 3.17. *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$, and $\mathcal{C}$, $\mathcal{C}'$ two chambers in $\mathcal{A}$. Then there is a gallery joining $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{A}$, i.e., there is a finite sequence*

$$\mathcal{C} = \mathcal{C}_0,\ \mathcal{C}_1,\ \ldots\ ,\ \mathcal{C}_m = \mathcal{C}'$$

of chambers in $\mathcal{A}$ such that for i with $1 \leqslant i \leqslant m$, $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ share a face of codimension 1.

Proof. Let $\mathcal{A}_2$ be the codimension 2-skelton of $\mathcal{A}$, i.e., the union of all facets in $\mathcal{A}$ of codimension at least 2. Then $\mathcal{A}_2$ is a closed subset of $\mathcal{A}$ of codimension 2, so $\mathcal{A} - \mathcal{A}_2$ is a connected open subset of the affine space $\mathcal{A}$. Hence $\mathcal{A} - \mathcal{A}_2$ is arcwise connected. This implies that given points $x \in \mathcal{C}$ and $x' \in \mathcal{C}'$, there is a piecewise linear curve in $\mathcal{A} - \mathcal{A}_2$ joining x and x'. Now the chambers in $\mathcal{A}$ that meet this curve make a gallery joining $\mathcal{C}$ to $\mathcal{C}'$. □

As the central torus of G is K-anisotropic, the dimension of any apartment, or any chamber, in $\mathcal{B}$ is equal to the K-rank of G. A *panel* in $\mathcal{B}$ is by definition a facet of codimension 1.

Proposition 3.18. *$\mathcal{B}$ is thick, that is any panel is a face of at least three chambers, and every apartment of $\mathcal{B}$ is thin, that is any panel lying in an apartment is a face of exactly two chambers of the apartment.*

Proof. Let $\mathcal{F}$ be a facet of $\mathcal{B}$ that is not a chamber, and $\mathcal{C}$ be a chamber of which $\mathcal{F}$ is a face. Then there is an $\mathcal{O}$-group scheme homomorphism $\rho^G_{\mathcal{F},\mathcal{C}} : \mathscr{G}^\circ_{\mathcal{C}} \to \mathscr{G}^\circ_{\mathcal{F}}$ (3.2). The image of $\overline{\mathscr{G}}^\circ_{\mathcal{C}}$ in $\overline{\mathscr{G}}^\circ_{\mathcal{F}}$, under the induced homomorphism of special fibers, is a minimal pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}^\circ_{\mathcal{F}}$, and conversely, any minimal pseudo-parabolic κ-subgroup of the latter determines a chamber of $\mathcal{B}$ with $\mathcal{F}$ as a face (Corollary 3.16). Now as κ is infinite, $\overline{\mathscr{G}}^\circ_{\mathcal{F}}$ contains infinitely many minimal pseudo-parabolic κ-subgroups. We conclude that $\mathcal{F}$ is a face of infinitely many chambers.

The second assertion follows at once from the following well-known result in algebraic topology: In any simplicial complex whose geometric realization is a topological manifold without boundary (such as an apartment $\mathcal{A}$ of $\mathcal{B}$), any simplex of codimension 1 is a face of exactly two chambers (i.e., maximal dimensional simplices). □

Proposition 3.19. *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$ and S be the maximal K-split torus of G corresponding to this apartment. (Then $\mathcal{A} = \mathcal{B}(Z_H(S)/K)^{\Theta}$.) The group $N_G(S)(K)$ acts transitively on the set of chambers of $\mathcal{A}$.*

Proof. According to Proposition 3.17, given any two chambers in $\mathcal{A}$, there exists a minimal gallery in $\mathcal{A}$ joining these two chambers. So to prove the proposition by induction on the length of a minimal gallery joining two chambers, it suffices to prove that given two different chambers $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{A}$ which share a panel $\mathcal{F}$, there is an element $n \in N_G(S)(K)$ such that $n \cdot \mathcal{C} = \mathcal{C}'$. Let $\mathscr{G} := \mathscr{G}^\circ_{\mathcal{F}}$ be the Bruhat-Tits smooth affine $\mathcal{O}$-group scheme associated with the panel $\mathcal{F}$ and $\mathscr{S} \subset \mathscr{G}$ be the closed $\mathcal{O}$-torus with generic fiber S. Let $\overline{\mathscr{G}}$ be the special fiber of $\mathscr{G}$ and $\overline{\mathscr{S}}$ the special fiber of $\mathscr{S}$. Then $\overline{\mathscr{S}}$ is a maximal torus of $\overline{\mathscr{G}}$. The chambers $\mathcal{C}$ and $\mathcal{C}'$ correspond to minimal pseudo-parabolic subgroups $\overline{\mathscr{P}}$ and $\overline{\mathscr{P}}'$ of $\overline{\mathscr{G}}$ (Corollary 3.16). Both of these minimal pseudo-parabolic κ-subgroups contain $\overline{\mathscr{S}}$ since the chambers $\mathcal{C}$ and $\mathcal{C}'$ lie in $\mathcal{A}$. But then by Theorems C.2.5 and C.2.3 of [CGP], there is an element $\overline{n} \in \overline{\mathscr{G}}(\kappa)$ that normalizes $\overline{\mathscr{S}}$ and conjugates $\overline{\mathscr{P}}$ onto $\overline{\mathscr{P}}'$. Now from Proposition 2.1(iii) of [P2] we conclude that there is an element $n \in N_{\mathscr{G}}(\mathscr{S})(\mathcal{O})$ lying over $\overline{n}$. It is clear that n normalizes S and hence it lies in $N_G(S)(K)$; it fixes $\mathcal{F}$ pointwise and $n \cdot \mathcal{C} = \mathcal{C}'$. □

Now in view of Propositions 2.14, 3.4, 3.17 and 3.18, Theorem 3.11 of [Ro] (cf. also [P2, 1.8]) implies that $\mathcal{B}$ is an affine building if for any maximal K-split torus S of G, $\mathcal{B}(Z_H(S)/K)^{\Theta}$ is taken to be the corresponding apartment, and $\mathcal{B}$ is given the polysimplicial structure described in 3.7. Thus we obtain the following:

Theorem 3.20. $\mathcal{B} = \mathcal{B}(H/K)^{\Theta}$ *is an affine building. Its apartments are the affine spaces* $\mathcal{B}(Z_H(S)/K)^{\Theta}$ *under* $V(S) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$ *for maximal* K*-split tori* S *of* G*. Its facets are as in* 3.7*. The group* $G(K)$ *acts on* $\mathcal{B}$ *by polysimplicial isometries.*

From Propositions 2.15 and 3.19 we obtain the following.

Proposition 3.21. $G(K)$ *acts transitively on the set of ordered pairs* $(\mathcal{A}, \mathcal{C})$ *consisting of an apartment* $\mathcal{A}$ *of* $\mathcal{B}$ *and a chamber* $\mathcal{C}$ *of* $\mathcal{A}$*.*

Remark 3.22. (i) As in [P2, 3.16], using the preceding proposition we can obtain Tits systems in suitable subgroups of $G(K)$.

(ii) As in [P2, §5], we can obtain filtration of root groups and a valuation of root datum for G/K.

§4. Tamely-ramified descent

We begin by proving the following proposition:

Proposition 4.1. *Let* k *be a field of characteristic* $p \geqslant 0$*. Let* $\mathcal{H}$ *be a noncommutative pseudo-reductive* k*-group,* θ *a* k*-automorphism of* $\mathcal{H}$ *of finite order not divisible by* p*, and* $\mathcal{G} := (\mathcal{H}^{\langle\theta\rangle})^{\circ}$*. Then*

(i) *No maximal torus of* $\mathcal{G}$ *is central in* $\mathcal{H}$*.*

(ii) *The centralizer in* $\mathcal{H}$ *of any maximal torus of* $\mathcal{G}$ *is commutative.*

(iii) *Given a maximal* k*-torus* $\mathcal{S}$ *of* $\mathcal{G}$*, there is a* θ*-stable maximal* k*-torus of* $\mathcal{H}$ *containing* $\mathcal{S}$*.*

(iv) *If* k *is separably closed, then* $\mathcal{H}$ *contains a* θ*-stable proper pseudo-parabolic* k*-subgroup.*

Proof. We fix an algebraic closure $\overline{k}$ of k. Let $\mathcal{H}'$ be the maximal reductive quotient of $\mathcal{H}_{\overline{k}}$. As $\mathcal{H}$ is noncommutative, $\mathcal{H}'$ is also noncommutative (see [CGP, Prop. 1.2.3]). The automorphism θ induces a $\overline{k}$-automorphism of $\mathcal{H}'$ which we will denote again by θ. According to a theorem of Steinberg [St, Thm. 7.5], $\mathcal{H}_{\overline{k}}$ contains a θ-stable Borel subgroup $\mathcal{B}$, and this Borel subgroup contains a θ-stable maximal torus $\mathcal{T}$. The natural quotient map $\pi : \mathcal{H}_{\overline{k}} \to \mathcal{H}'$ carries $\mathcal{T}$ isomorphically onto a maximal torus of $\mathcal{H}'$. We endow the root system of $\mathcal{H}'$ with respect to the maximal torus $\mathfrak{T}' := \pi(\mathcal{T}) \cap \mathscr{D}(\mathcal{H}')$ of the derived subgroup $\mathscr{D}(\mathcal{H}')$ of $\mathcal{H}'$ with the ordering determined by the Borel subgroup $\pi(\mathcal{B})$. Let a be the sum of all positive roots. Then as $\pi(\mathcal{B})$ is θ-stable, a is fixed under θ acting on the character group $\mathrm{X}(\mathfrak{T}')$ of $\mathfrak{T}'$. Therefore, $\mathrm{X}(\mathfrak{T}')$ admits a nontrivial torsion-free quotient on which θ acts trivially. This implies that $\mathcal{T}$ contains a nontrivial subtorus $\mathscr{T}$ that is fixed pointwise under θ and is mapped by π into $\mathfrak{T}'\, (\subset \mathscr{D}(\mathcal{H}'))$. The subtorus $\mathscr{T}$ is therefore contained in $\mathcal{G}_{\overline{k}}$. Since the center of the semi-simple group $\mathscr{D}(\mathcal{H}')$ does not contain a nontrivial smooth connected subgroup, we infer that $\mathscr{T}$ is not central in $\mathcal{H}_{\overline{k}}$. Thus the subgroup $\mathcal{G}_{\overline{k}}$ contains a noncentral torus of $\mathcal{H}_{\overline{k}}$. Now by conjugacy of maximal tori in $\mathcal{G}_{\overline{k}}$, we see that no maximal torus of this group can be central in $\mathcal{H}_{\overline{k}}$. This proves (i).

To prove (ii), let $\mathcal{S}$ be a maximal torus of $\mathcal{G}$. Then the centralizer $Z_{\mathcal{H}}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{H}$ is a θ-stable pseudo-reductive subgroup of $\mathcal{H}$, and $(Z_{\mathcal{H}}(\mathcal{S})^{\langle\theta\rangle})^{\circ} = Z_{\mathcal{G}}(\mathcal{S})$. As $\mathcal{S}$ is a maximal torus of $Z_{\mathcal{G}}(\mathcal{S})$ that is central in $Z_{\mathcal{H}}(\mathcal{S})$, if $Z_{\mathcal{H}}(\mathcal{S})$ were noncommutative, we could apply (i) to this subgroup in place of $\mathcal{H}$ to get a contradiction.

To prove (iii), we consider the centralizer $Z_{\mathcal{H}}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{H}$. This centralizer is θ-stable and commutative according to (ii). The unique maximal k-torus of it contains $\mathcal{S}$ and is a θ-stable maximal torus of $\mathcal{H}$.

To prove (iv), we assume now that k is separably closed and let $\mathcal{S}$ be a maximal torus of $\mathcal{G}$. Then $\mathcal{S}$ is k-split, and in view of (i), there is a 1-parameter subgroup $\lambda : \mathrm{GL}_1 \to \mathcal{S}$ whose image is not central in $\mathcal{H}$. Then $P_{\mathcal{H}}(\lambda)$ is a θ-stable proper pseudo-parabolic k-subgroup of $\mathcal{H}$. $\square$

In the following proposition we will use the notation introduced in §§1, 2. As in 2.4, we will assume that H is semi-simple and the central torus of G is K-anisotropic. We will further assume that H is K-isotropic, Θ is a finite *cyclic* group of automorphisms of H, and p does not divide the order of Θ.

Proposition 4.2. *The Bruhat-Tits building $\mathcal{B}(H/K)$ of $H(K)$ contains a Θ-stable chamber.*

Proof. Let F be a Θ-stable facet of $\mathcal{B}(H/K)$ that is maximal among the Θ-stable facets. Let $\mathscr{H} := \mathscr{H}_F^{\circ}$ be the Bruhat-Tits smooth affine $\mathcal{O}$-group scheme with generic fiber H, and connected special fiber $\overline{\mathscr{H}}$, corresponding to F. Let $\mathcal{H} := \overline{\mathscr{H}}/\mathscr{R}_{u,\kappa}(\overline{\mathscr{H}})$ be the maximal pseudo-reductive quotient of $\overline{\mathscr{H}}$. In case $\mathcal{H}$ is commutative, $\overline{\mathscr{H}}$ does not contain a proper pseudo-parabolic κ-subgroup and so F is a chamber of $\mathcal{B}(H/K)$. We assume, if possible, that $\mathcal{H}$ is not commutative. As F is stable under the action of Θ, there is a natural action of this finite cyclic group on $\mathscr{H}$ by $\mathcal{O}$-group scheme automorphisms (2.4). This action induces an action of Θ on $\overline{\mathscr{H}}$, and so also on its pseudo-reductive quotient $\mathcal{H}$. Now taking θ to be a generator of Θ, and using the preceding proposition for $\mathcal{H}/\kappa$, we conclude that $\mathcal{H}$ contains a Θ-stable proper pseudo-parabolic κ-subgroup. The inverse image $\overline{\mathscr{P}}$ in $\overline{\mathscr{H}}$ of any such pseudo-parabolic subgroup of $\mathcal{H}$ is a Θ-stable proper pseudo-parabolic κ-subgroup of $\overline{\mathscr{H}}$. The facet F' corresponding to $\overline{\mathscr{P}}$ is Θ-stable and $F \prec F'$. This contradicts the maximality of F. Hence, $\mathcal{H}$ is commutative and F is a chamber. $\square$

To prove the next theorem (Theorem 4.4), we will use the following:

Proposition 4.3. *Let $\mathfrak{K}$ be a field complete with respect to a discrete valuation and with separably closed residue field. Let $\mathfrak{G}$ be a connected absolutely simple $\mathfrak{K}$-group of inner type A that splits over a finite tamely-ramified field extension $\mathfrak{L}$ of $\mathfrak{K}$. Then $\mathfrak{G}$ is $\mathfrak{K}$-split.*

Proof. We may (and do) assume that $\mathfrak{G}$ is simply connected. Then $\mathfrak{G}$ is $\mathfrak{K}$-isomorphic to $\mathrm{SL}_{n,\mathfrak{D}}$, where $\mathfrak{D}$ is a finite dimensional division algebra with center $\mathfrak{K}$ that splits over the finite tamely-ramified field extension $\mathfrak{L}$ of $\mathfrak{K}$. By Propositions 4 and 12 of [S, Ch. II] the degree of $\mathfrak{D}$ is a power of p, where p is the characteristic of the residue field of $\mathfrak{K}$. But a noncommutative division algebra of degree a power of p cannot split over a field extension of degree prime to p. So, $\mathfrak{D} = \mathfrak{K}$, hence $\mathfrak{G} \simeq \mathrm{SL}_n$ is $\mathfrak{K}$-split. ☐

Theorem 4.4. *A semi-simple K-group G that is quasi-split over a finite tamely-ramified field extension of K is already quasi-split over K.*

This theorem has been proved by Philippe Gille in [Gi] by an entirely different method.

Proof. We assume that all field extensions appearing in this proof are contained in a fixed separable closure of K. To prove the theorem, we may (and do) replace G by its simply-connected central cover and assume that G is simply connected. Let S be a maximal K-split torus of G. Then G is quasi-split over a (separable) extension L of K if and only if the derived subgroup $Z_G(S)'$ of the centralizer $Z_G(S)$ of S is quasi-split over L. Moreover, G is quasi-split over K if and only if $Z_G(S)'$ is trivial. Therefore, to prove the theorem we need to show that any semi-simple simply connected K-*anisotropic* K-group that is quasi-split over a finite tamely-ramified field extension of K is necessarily trivial. Let G be any such group.

There exists a finite indexing set I, and for each $i \in I$, a finite separable field extension K_i of K and an absolutely almost simple simply connected K_i-anisotropic K_i-group G_i such that $G = \prod_{i\in I} \mathrm{R}_{K_i/K}(G_i)$. Now G is quasi-split over a finite separable field extension L of K if and only if for each i, $\mathrm{R}_{K_i/K}(G_i)$ is quasi-split over L. But $\mathrm{R}_{K_i/K}(G_i)$ is quasi-split over L if and only if G_i is quasi-split over the compositum $L_i := K_iL$. For $i \in I$, the finite extension K_i of K is complete and its residue field is separably closed, and if L is a finite tamely-ramified field extension of K, then L_i is a finite tamely-ramified field extension of K_i. So to prove the theorem, we may (and do) replace K by K_i and G by G_i to assume that G is an absolutely almost simple simply connected K-*anisotropic* K-group that is quasi-split over a finite tamely-ramified field extension of K. We will show that such a group G is trivial.

Let L be a finite tamely-ramified field extension of K of minimal degree over which G is quasi-split. Since the residue field κ of K is separably closed, L is a cyclic extension of K. Let Θ be the Galois group of L/K. Then Θ is a finite cyclic group of order not divisible by $p\,(= \mathrm{char}(\kappa))$.

As G_L is quasi-split, Bruhat-Tits theory is available for G over L [BrT2, §4]. The Galois group Θ acts on $G(L)$ by continuous automorphisms and so it acts on the Bruhat-Tits building $\mathcal{B}(G/L)$ of $G(L)$ by polysimplicial isometries. Let $H = \mathrm{R}_{L/K}(G_L)$. Then H is quasi-split over K and hence Bruhat-Tits theory is also available for H over K. Let $\mathcal{B}(H/K)$ be the Bruhat-Tits building of $H(K)\,(= G(L))$. Elements of Θ act by K-automorphisms on H and so on $\mathcal{B}(H/K)$ by polysimplicial isometries; moreover, $G = H^{\Theta}$. There is a natural Θ-equivariant identification of the building $\mathcal{B}(H/K)$ with the building $\mathcal{B}(G/L)$. (Note that K-rank H = L-rank G_L, and there is a natural bijective correspondence between the set of maximal K-split tori of H and the set of maximal L-split tori of G_L, see [CGP, Prop. A.5.15(2)]. This correspondence will be used below.) The results of §3 imply that Bruhat-Tits theory is available for G over K and $\mathcal{B} := \mathcal{B}(H/K)^{\Theta}$ $(= \mathcal{B}(G/L)^{\Theta})$ is the Bruhat-Tits building of $G(K)$.

Since G is K-anisotropic, the building of $G(K)$ consists of a single point, hence Θ fixes a unique point of $\mathcal{B}(G/L)$. Let C be the facet of $\mathcal{B}(G/L)$ that contains this point. Then C is stable under Θ. According to Proposition 4.2, C is a chamber. Let $\mathscr{H} := \mathscr{H}_C^{\circ}$ be the Bruhat-Tits smooth affine $\mathcal{O}$-group scheme associated to C with generic fiber H and connected special fiber $\overline{\mathscr{H}}$. As C is a chamber, the maximal pseudo-reductive quotient $\overline{\mathscr{H}}^{\mathrm{pred}}$ of $\overline{\mathscr{H}}$ is commutative [P2, 1.10]. Now using Proposition 2.6 for $\Omega = C = F$ we obtain a Θ-stable maximal K-split torus T of H such that C lies in the apartment $A(T)$ corresponding to T (and the special fiber of the schematic closure of T in $\mathscr{H}$ maps onto the maximal torus of $\overline{\mathscr{H}}^{\mathrm{pred}}$). Let T' be the image of T_L under the natural surjective homomorphism $q : H_L = \mathrm{R}_{L/K}(G_L)_L \to G_L$. Then T' is a L-torus of G_L and according to [CGP, Prop. A.5.15(2)] it is the unique maximal L-split torus of G_L such that $\mathrm{R}_{L/K}(T')\,(\subset \mathrm{R}_{L/K}(G_L) = H)$ contains the maximal K-split torus T of H.

We identify $H(K)$ with $G(L)$. Then for $x \in H(K)(\subset H(L))$ and $\theta \in \Theta$, we have $q(\theta(x)) = \theta(x)$. Since $T(K)$ is Θ-stable, for $t \in T(K)$ and $\theta \in \Theta$, $\theta(t)$ lies in $T'(L)$. Now as $T(K)$ is Zariski-dense in T, its image in $T'(L)$ is Zariski-dense in T'. Since this image is stable under the action of $\Theta = \mathrm{Gal}(L/K)$ on $G(L)$, from the Galois criterion [Bo, Ch. AG, Thm. 14.4(3)] we infer that T' descends to a K-torus of G, i.e., there is a K-torus $\mathcal{T}$ of G such that $T' = \mathcal{T}_L$. In the natural identification of $\mathcal{B}(H/K)$ with $\mathcal{B}(G/L)$, the apartment $A(T)$ of the former is Θ-equivariantly identified with the apartment $A(T')$ of the latter. We will view the chamber C as a Θ-stable chamber in $A(T')$.

Let Δ be the basis of the affine root system of the absolutely almost simple, simply connected quasi-split L-group G_L with respect to $T'\,(= \mathcal{T}_L)$, determined by the Θ-stable chamber C [BrT2, §4]. Then Δ is stable under the action of Θ on the affine root system of G_L with respect to T'. There is a natural Θ-equivariant bijective correspondence between the set of vertices of C and Δ. Since $\mathcal{B}$, and hence C^{Θ}, consists of a single point, Θ acts transitively on the set of vertices of C so it acts transitively on Δ. Now from the classification of irreducible affine root systems [BrT1, §1.4.6], we see that G_L is a split group of type A_n for some n. Proposition 4.3 implies that G cannot be of inner type A_n over K. On the other hand, if G is of outer type A_n, then over a quadratic Galois extension $K'\,(\subset L)$ of K it is of inner type. Now, according to Proposition 4.3, G splits over K'. We conclude that $L = K'$ and hence $\#\Theta = 2$. As Θ acts transitively on Δ and $\#\Delta = n+1$, we infer that $n + 1 = 2$, i.e., $n = 1$, and then G is of inner type, a contradiction. $\square$

4.5. Now let k be a field endowed with a nonarchimedean discrete valuation. We assume that the valuation ring of k is Henselian. Let K be the maximal unramified extension of k, and L be a finite tamely-ramified field extension of K with Galois group $\Theta := \mathrm{Gal}(L/K)$. Let G be a connected reductive k-group that is quasi-split over K and $H = \mathrm{R}_{L/K}(G_L)$. Then $G = H^{\Theta}$, and by Theorem 3.20, the Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$ can be identified with the subspace of points in the Bruhat-Tits building of $G(L)\,(= H(K))$ that are fixed under Θ (with polysimplicial structure on $\mathcal{B}(G/K)$ as in 3.7). Now by "unramified descent" [P2], *Bruhat-Tits theory is available for G over k and the Bruhat-Tits building of $G(k)$ is* $\mathcal{B}(G/K)^{\mathrm{Gal}(K/k)}$.

References

[Bo] A. Borel, *Linear algebraic groups* (second edition). Springer-Verlag, New York (1991).

[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models.* Springer-Verlag, Heidelberg (1990).

[BrT1] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local.* Publ. Math. IHES **41**(1972).

[BrT2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local, II.* Publ. Math. IHES **60**(1984).

[CGP] B. Conrad, O. Gabber and G. Prasad, *Pseudo-reductive groups* (second edition). Cambridge U. Press, New York (2015).

[E] B. Edixhoven, *Néron models and tame ramification.* Comp. Math. **81**(1992), 291-306.

[GGM] O. Gabber, P. Gille and L. Moret-Bailly, *Fibrés principaux sur les corps valués henséliens.* Algebr. Geom. **1**(2014), 573-612.

[Gi] P. Gille, *Semi-simple groups that are quasi-split over a tamely-ramified extension*, Rendiconti Sem. Mat. Padova, to appear.

[EGA IV$_3$] A. Grothendieck, *Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES **28**(1966), 5-255.

[EGA IV$_4$] A. Grothendieck, *Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES **32**(1967), 5-361.

[P1] G. Prasad, *Galois-fixed points in the Bruhat-Tits building of a reductive group.* Bull. Soc. Math. France **129**(2001), 169-174.

[P2] G. Prasad, *A new approach to unramified descent in Bruhat-Tits theory.* American J. Math. (to appear).

[PY1] G. Prasad and J.-K. Yu, *On finite group actions on reductive groups and buildings.* Invent. Math. **147**(2002), 545–560.

[PY2] G. Prasad and J.-K. Yu, *On quasi-reductive group schemes.* J. Alg. Geom. **15** (2006), 507-549.

[Ri] R. Richardson, *On orbits of algebraic groups and Lie groups*, Bull. Australian Math. Soc. **25**(1982), 1-28.

[Ro] M. Ronan, *Lectures on buildings.* University of Chicago Press, Chicago (2009).

[Rou] G. Rousseau, *Immeubles des groupes réductifs sur les corps locaux*, University of Paris, Orsay, thesis (1977).
(Available at http://www.iecl.univ-lorraine.fr/~Guy.Rousseau/Textes/)

[S] J-P. Serre, *Galois cohomology.* Springer-Verlag, New York (1997).

[St] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the Amer. Math. Soc. **80**(1968).

[T] J. Tits, *Reductive groups over local fields.* Proc. Symp. Pure Math. #**33**, Part I, 29–69, American Math. Soc. (1979).

University of Michigan
Ann Arbor, MI 48109.
e-mail: gprasad@umich.edu