A NEW APPROACH TO UNRAMIFIED DESCENT IN BRUHAT-TITS THEORY

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Dedicated to my wife Indu Prasad with gratitude

Abstract. We present a new approach to unramified descent ("descente étale") in Bruhat-Tits theory of reductive groups over a discretely valued field k with Henselian valuation ring which appears to be conceptually simpler, and more geometric, than the original approach of Bruhat and Tits. We are able to derive the main results of the theory over k from the theory over the maximal unramified extension K of k. Even in the most interesting case for number theory and representation theory, where k is a locally compact nonarchimedean field, the geometric approach described in this paper appears to be considerably simpler than the original approach.

Introduction. Let k be a field given with a nontrivial $\mathbb{R}$-valued nonarchimedean valuation ω. We will assume throughout this paper that the valuation ring

$$\mathfrak{o} := \{x \in k^{\times} \mid \omega(x) \geqslant 0\} \cup \{0\}$$

of this valuation is Henselian. Let K be the maximal unramified extension of k (the term "unramified extension" includes the condition that the residue field extension is separable, so the residue field of K is the separable closure κ_s of the residue field κ of k). While discussing Bruhat-Tits theory in this introduction, and beginning with 1.6 everywhere in the paper, we will assume that ω is a discrete valuation. Bruhat-Tits theory of connected reductive k-groups G that are quasi-split over K (i.e., G_K contains a Borel subgroup, or, equivalently, the centralizer of a maximal K-split torus of G_K is a torus) has two parts. The first part of the theory, developed in [BrT2, §4], is called the "quasi-split descent" ("descente quasi-déployée" in French), and is due to Iwahori-Matsumoto, Hijikata, and Bruhat-Tits. It is the theory over K assuming that G is quasi-split over K. When G is quasi-split over K, the group $G(K)$ has a rather simple structure. In particular, it admits a "Chevalley-Steinberg system", which is used in [BrT2, §4] to get a valuation of root datum, that in turn is used there to construct the *enlarged* Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$. In [BrT2, Intro.] Bruhat and Tits say that "La descente quasi-déployée est la plus facile.". The second part, called the "unramified descent" ("descente étale" in French), is due to Bruhat and Tits. This part derives Bruhat-Tits theory of G (over k), and also the enlarged Bruhat-Tits building of $G(k)$, from Bruhat-Tits theory of G over K and the enlarged Bruhat-Tits building of $G(K)$, using descent of valuation of root datum from K to k. This second part is quite technical; see [BrT1, §9] and [BrT2, §5].

The purpose of this paper is to present an alternative approach to unramified descent which appears to be conceptually simpler, and more geometric, than the approach in [BrT1], [BrT2], in that it does not use descent of valuation of root datum from K to k to show that $\mathcal{B}(G/K)^{\Gamma}$, where Γ is the Galois group of K/k, is an enlarged affine building. In this approach, we will use Bruhat-Tits theory, and the buildings, only over the maximal unramified extension K of k and derive the main results of the theory for reductive groups over k. In §4, we discuss the notions of hyperspecial points and hyperspecial parahoric subgroups and describe conditions under which they exist. In §5, we define a natural filtration of the root groups $U_a(k)$ and also describe a valuation of the root datum of G/k relative to a maximal k-split torus S, using the geometric results of §§2, 3 that provide the Bruhat-Tits building of $G(k)$. The approach described here appears to be considerably simpler than the original approach even for reductive groups over locally compact nonarchimedean fields (i.e., discretely valued complete fields with finite residue field). In §6, we prove results over discretely valued fields with Henselian valuation ring and perfect residue field of dimension $\leqslant 1$; of these, Theorems 6.1 and 6.2 may be new.

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1. Preliminaries

We assume below that k is a field given with a nontrivial $\mathbb{R}$-valued nonarchimedean valuation ω and the valuation ring $\mathfrak{o}$ of ω is Henselian. It is known that the valuation ring $\mathfrak{o}$ is Henselian if and only if the valuation ω extends uniquely to any algebraic extension of k, and the valuation ring of a discretely valued complete field is always Henselian. (For various equivalent definitions of the Henselian property, see [Be, §§2.3-2.4].) Let K be the maximal unramified extension of k contained in a fixed algebraic closure $\overline{k}$ of k. We will denote the unique valuation of the algebraic closure $\overline{k}\,(\supset K)$, extending the given valuation of k, also by ω. The residue field of k will be denoted by κ and the valuation ring of K by $\mathcal{O}$. The residue field of K is the separable closure κ_s of κ. We will denote the Galois group $\mathrm{Gal}(K/k) = \mathrm{Gal}(\kappa_s/\kappa)$ by Γ. For a subset Y of a set Z given with an action of Γ, we will denote by Y^{Γ} the set of elements of Y fixed under Γ.

Bounded subsets. Let X be an affine k-variety. A subset B of $X(\overline{k})$ is said to be *bounded* if for every $f \in k[X]$, the set $\{\omega(f(b)) \mid b \in B\}$ is bounded below. If X is an affine algebraic k-group and B, B' are two nonempty subsets of $X(\overline{k})$, then $BB' := \{bb' \mid b \in B, b' \in B'\}$ is bounded if and only if both B and B' are bounded.

Let G be a connected reductive group defined over k. The group of k-rational characters on G will be denoted by $\mathrm{X}_k^*(G)$. The following theorem is due to Bruhat, Tits and Rousseau. An elementary proof was given in [P] which we recall here for the reader's convenience.

Theorem 1.1. *$G(k)$ is bounded if and only if G is anisotropic over k.*

Remark 1.2. Thus if k is a nondiscrete locally compact field, then $G(k)$ is compact if and only if G is k-anisotropic.

We fix a faithful k-rational representation of G on a finite dimensional k-vector space V and view G as a k-subgroup of $\mathrm{GL}(V)$. To prove the above theorem we will use the following two lemmas:

Lemma 1.3. *If $f : X \to Y$ is a finite $\overline{k}$-morphism between affine $\overline{k}$-schemes of finite type and B is a bounded subset of $Y(\overline{k})$, then the subset $f^{-1}(B)$ of $X(\overline{k})$ is bounded.*

Proof. Since $\overline{k}[X]$ is module-finite over $\overline{k}[Y]$, we can pick a finite set of generators of $\overline{k}[X]$ as a $\overline{k}[Y]$-module (so also as a $\overline{k}[Y]$-algebra), and each satisfies a monic polynomial over $\overline{k}[Y]$. Hence, this realizes X as a closed subscheme of the closed subscheme $Z \subset Y \times \mathbb{A}^n$ defined by n monic 1-variable polynomials $f_1(t_1), \dots , f_n(t_n)$ over $\overline{k}[Y]$, so it remains to observe that when one has a bound on the coefficients of a monic 1-variable polynomial over $\overline{k}$ of known degree (e.g., specializing any f_j at a $\overline{k}$-point of Y) then one gets a bound on its possible $\overline{k}$-rational roots depending only on the given coefficient bound and the degree of the monic polynomial. □

Lemma 1.4. *Let $\mathcal{G}$ be an unbounded subgroup of $G(k)$ which is dense in G in the Zariski-topology. Then $\mathcal{G}$ contains an element g which has an eigenvalue α (for the action on V) with $\omega(\alpha) < 0$.*

Proof. Let

$$\overline{k} \otimes_k V =: V_0 \supset V_1 \supset \cdots \supset V_s \supset V_{s+1} = \{0\}$$

be a flag of $G_{\overline{k}}$-invariant subspaces such that for $0 \leqslant i \leqslant s$, the natural representation ϱ_i of $G_{\overline{k}}$ on $W_i := V_i/V_{i+1}$ is irreducible. Let $\varrho = \bigoplus_i \varrho_i$ be the representation of $G_{\overline{k}}$ on $\bigoplus_i W_i$. The kernel of ϱ is obviously a unipotent normal $\overline{k}$-subgroup scheme of the reductive group $G_{\overline{k}}$, and hence it is finite. Now as $\mathcal{G}$ is an unbounded subgroup of $G(k)$, Lemma 1.3 implies that $\varrho(\mathcal{G})$ is an unbounded subgroup of $\varrho(G(\overline{k}))$. Hence, there is a non-negative integer $a \leqslant s$ such that $\varrho_a(\mathcal{G})$ is unbounded.

Since W_a is an irreducible $G_{\overline{k}}$-module, and $\mathcal{G}$ is dense in G in the Zariski-toplogy, $\varrho_a(\mathcal{G})$ spans $\mathrm{End}_{\overline{k}}(W_a)$. We fix $\{g_i\} \subset \mathcal{G}$ so that $\{\varrho_a(g_i)\}$ is a basis of $\mathrm{End}_{\overline{k}}(W_a)$. Let $\{f_i\} \subset \mathrm{End}_{\overline{k}}(W_a)$ be the basis which is dual to the basis $\{\varrho_a(g_i)\}$ with respect to the trace-form. Then $\mathrm{Tr}(f_i \cdot \varrho_a(g_j)) = \delta_{ij}$, where δ_{ij} is the Kronecker's delta. Now assume that the eigenvalues of all the elements of $\mathcal{G}$ lie in the valuation ring $\mathfrak{o}_{\overline{k}}$ of $\overline{k}$. Then for all $x \in \mathcal{G}$, $\mathrm{Tr}(\varrho_a(x))$ is contained in $\mathfrak{o}_{\overline{k}}$. For $g \in \mathcal{G}$, if $\varrho_a(g) = \sum_i c_i f_i$, with $c_i \in \overline{k}$, then $\mathrm{Tr}(\varrho_a(g \cdot g_j)) = \sum_i c_i \mathrm{Tr}(f_i \cdot \varrho_a(g_j)) = c_j$. As $\mathrm{Tr}(\varrho_a(g \cdot g_j)) \in \mathfrak{o}_{\overline{k}}$, we conclude that c_j belongs to the ring of integers $\mathfrak{o}_{\overline{k}}$ for all j (and all $g \in \mathcal{G}$). This implies that $\varrho_a(\mathcal{G})$ is bounded, a contradiction. □

Proof of Theorem 1.1. As $\mathrm{GL}_1(k) = k^\times$ is unbounded, we see that if G is k-isotropic, then $G(k)$ is unbounded. We will now assume that $G(k)$ is unbounded and prove the converse.

It is well known that $G(k)$ is dense in G in the Zariski-topology [Bo, 18.3], hence according to Lemma 1.4, there is an element $g \in G(k)$ which has an eigenvalue α with $\omega(\alpha) \neq 0$. Now in case k is of positive characteristic, after replacing g by a suitable positive integral power, we assume that g is semi-simple. On the other hand, in case k is of characteristic zero, let $g = s \cdot u = u \cdot s$ be the Jordan decomposition of g with $s \in G(k)$ semi-simple and $u \in G(k)$ unipotent. Then the eigenvalues of g are same as that of s. So, after replacing g with s, we may (and do) again assume that g is semi-simple. There is a maximal k-torus T of G such that $g \in T(k)$ (see [BoT], Proposition 10.3 and Theorem 2.14(a); note that according to Theorem 11.10 of [Bo], g is contained in a maximal torus of G). Since any absolutely irreducible representation of a torus is 1-dimensional, there exists a finite Galois extension $\mathfrak{K}$ of k and a character χ of $T_{\mathfrak{K}}$ such that $\chi(g) = \alpha$. Then

$$\omega\big(\big(\sum_{\gamma \in \mathrm{Gal}(\mathfrak{K}/k)} {}^{\gamma}\chi\big)(g)\big) = m\omega(\chi(g)) = m\omega(\alpha) \neq 0;$$

where $m = [\mathfrak{K} : k]$. Thus the character $\sum_{\gamma \in \mathrm{Gal}(\mathfrak{K}/k)} {}^{\gamma}\chi$ is nontrivial. On the other hand, this character is obviously defined over k. Hence, T admits a nontrivial character defined over k and therefore it contains a nontrivial k-split subtorus. This proves that if $G(k)$ is unbounded, then G is isotropic over k. □

Proposition 1.5. *We assume that the derived subgroup $G' := (G, G)$ of G is k-anisotropic. Then $G(k)$ contains a unique maximal bounded subgroup $G(k)_b$; it has the following description:*

$$G(k)_b = \{\, g \in G(k) \mid \chi(g) \in \mathfrak{o}^\times \text{ for all } \chi \in \mathrm{X}_k^*(G)\}.$$

Proof. Let G_a be the inverse image of the maximal k-anisotropic subtorus of the k-torus G/G' under the natural homomorphism $G \to G/G'$. Then G_a is the maximal connected normal k-anisotropic subgroup of G. Let S be the maximal k-split central torus of G. Then $G = S \cdot G_a$ (almost direct product). Let $C = S \cap G_a$; C is a finite central k-subgroup scheme, so G_a/C is k-anisotropic. Let $f : G \to G/C = (S/C) \times (G_a/C)$ be the natural homomorphism. The image of the induced homomorphism $f^* : \mathrm{X}_k^*((S/C) \times (G_a/C)) \to \mathrm{X}_k^*(G)$ is of finite index. It is obvious that as $(G_a/C)(k)$ is bounded (by Theorem 1.1), the proposition is true for the direct product $(S/C) \times (G_a/C)$. Now using Lemma 1.3 we conclude that the proposition holds for G. □

We shall henceforth assume that the valuation ω on k is discrete.

1.6. Let S be a maximal k-split torus of G, $Z(S)$ its centralizer in G and $Z(S)' = (Z(S), Z(S))$ the derived subgroup of $Z(S)$. Then $Z(S)'$ is a connected semi-simple group which is anisotropic over k since S is a maximal k-split torus of G. Hence, by Theorem 1.1, $Z(S)'(k)$ is bounded, and so according to Proposition 1.5, $Z(S)(k)$ contains a unique maximal bounded subgroup $Z(S)(k)_b$. This maximal bounded subgroup admits the following description:

$$Z(S)(k)_b = \{z \in Z(S)(k) \mid \chi(z) \in \mathfrak{o}^\times \ \text{ for all } \ \chi \in \mathrm{X}_k^*(Z(S))\}.$$

The restriction map $\mathrm{X}_k^*(Z(S)) \to \mathrm{X}_k^*(S)$ is injective and its image is of finite index in $\mathrm{X}_k^*(S)$. Let $\mathrm{X}_*(S) = \mathrm{Hom}_k(\mathrm{GL}_1, S)$ and $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$. Let the homomorphism $\nu : Z(S)(k) \to V(S)$ be defined by:

$$\chi(\nu(z)) = -\omega(\chi(z)) \ \text{ for } z \in Z(S)(k) \text{ and } \chi \in \mathrm{X}_k^*(Z(S)) \, (\hookrightarrow \mathrm{X}_k^*(S)).$$

Then $Z(S)(k)_b$ is the kernel of ν. As the image of ν is isomorphic to $\mathbb{Z}^r$, $r = \dim S$, we conclude that $Z(S)(k)/Z(S)(k)_b$ is isomorphic to $\mathbb{Z}^r$.

1.7. Fields of dimension $\leqslant 1$ and a theorem of Steinberg. A field F is said to be of dimension $\leqslant 1$ if finite dimensional central simple algebras with center a finite separable extension of F are matrix algebras [S, Ch. II, §3.1]. For example, every finite field is of dimension $\leqslant 1$.

We now recall the following theorem of Steinberg: *For a smooth connected linear algebraic group $\mathfrak{G}$ defined over a field F of dimension $\leqslant 1$, the Galois cohomology $\mathrm{H}^1(\mathsf{F}, \mathfrak{G})$ is trivial if either F is perfect or $\mathfrak{G}$ is reductive* [S, Ch. III, Thm. 1$'$ and Remark (1) in §2.3]. This vanishing theorem implies that *if F is of dimension $\leqslant 1$, then a connected linear algebraic F-group $\mathfrak{G}$ is quasi-split, i.e., it contains a Borel subgroup defined over F*, assuming that $\mathfrak{G}$ is reductive when F is not perfect; note that the proof of the fact that assertion (i$'$) of Theorem 1 in [S, Ch. III, §2.2] implies that the semi-simple group L contains a Borel subgroup defined over the base field does not require the base field to be perfect.

We assume in this paragraph that the residue field κ of k is perfect. Then the residue field of the maximal unramified extension K is the algebraic closure $\overline{\kappa}$ of κ. Let $\widehat{K}$ denote the completion of K. The discrete valuation of K extends uniquely to the completion $\widehat{K}$ and the residue field $\overline{\kappa}$ of K is also the residue field of $\widehat{K}$. Hence, by Lang's theorem, $\widehat{K}$ is a (C_1)-field [S, Ch. II, Example 3.3(c) in §3.3], so it is of dimension $\leqslant 1$ [S, Ch. II, Corollary in §3.2]. According to a well-known result (see, for example, Proposition 3.5.3(2) of [GGM] whose proof simplifies considerably in the smooth affine case), for any smooth algebraic K-group $\mathfrak{G}$, the natural map $\mathrm{H}^1(K, \mathfrak{G}) \to \mathrm{H}^1(\widehat{K}, \mathfrak{G})$ is bijective. This result, combined with the above theorem of Steinberg, implies that for every connected reductive K-group $\mathfrak{G}$, $\mathrm{H}^1(K, \mathfrak{G})$ is trivial, hence every connected reductive K-group is quasi-split.

Notation. Given a smooth connected linear algebraic group $\mathfrak{G}$ defined over a field F, we will denote its F-unipotent radical, i.e., the maximal smooth connected normal unipotent F-subgroup, by $\mathscr{R}_{u,\mathsf{F}}(\mathfrak{G})$. The quotient $\mathfrak{G}^{\mathrm{pred}} := \mathfrak{G}/\mathscr{R}_{u,\mathsf{F}}(\mathfrak{G})$ is *pseudo-reductive*; it is the maximal pseudo-reductive quotient of $\mathfrak{G}$. If the field F is perfect, then pseudo-reductive groups are actually reductive and pseudo-parabolic subgroups of $\mathfrak{G}$ are parabolic subgroups. (For definition and properties of pseudo-reductive groups and pseudo-parabolic subgroups, see [CGP] or [CP].)

For a k-variety X and a field extension F of k, X_{F} will denote the F-variety obtained from X by base change $k \hookrightarrow \mathsf{F}$.

Let $\mathfrak{Z}$ be the maximal k-torus contained in the center of G that splits over K. There is a natural action of the Galois group Γ of K/k on $\mathrm{Hom}_K(\mathrm{GL}_1, \mathfrak{Z}_K)$ and $\mathrm{Hom}_K(\mathrm{GL}_1, \mathfrak{Z}_K)^{\Gamma} = \mathrm{Hom}_k(\mathrm{GL}_1, \mathfrak{Z})$. Let $V(\mathfrak{Z}_K) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Hom}_K(\mathrm{GL}_1, \mathfrak{Z}_K)$. The action of Γ on $\mathrm{Hom}_K(\mathrm{GL}_1, \mathfrak{Z}_K)$ extends to an $\mathbb{R}$-linear action on $V(\mathfrak{Z}_K)$, and $V(\mathfrak{Z}_K)^{\Gamma} = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Hom}_k(\mathrm{GL}_1, \mathfrak{Z})$. We will denote the derived subgroup (G, G) of G by G' throughout this paper. G' is the maximal connected normal semi-simple subgroup of G and there is a natural bijective correspondence between the set of maximal K-split tori of G'_K and the set of maximal K-split tori of G_K given by $T \mapsto \mathfrak{Z}_K T$.

1.8. Buildings. A polysimplicial complex Δ of dimension r is called a *chamber complex* if every facet of Δ is a face of a chamber (i.e., a facet of dimension r) and any two chambers of Δ can be joined by a *gallery* (see Proposition 3.5 for the definition). A chamber complex is *thin* if any polysimplex of codimension 1 (i.e., of dimension $r-1$) is a face of exactly two chambers.

A polysimplicial complex $\mathcal{B}$ of dimension $r\,(\geqslant 1)$, given with a collection of polysimplicial subcomplexes that are thin chamber complexes of dimension r called the apartments of $\mathcal{B}$, is a *building* if the following conditions (cf. [T1, 3.1], and also, [R, Thm. 3.11]) hold:

(B1) $\mathcal{B}$ is *thick*, that is, any facet of codimension 1 is a face of at least three chambers.

(B2) Any two facets of $\mathcal{B}$ lie on an apartment of $\mathcal{B}$.

(B3) If facets $\mathcal{F}_1$ and $\mathcal{F}_2$ are contained in the intersection of two apartments $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{B}$, then there is a polysimplicial isomorphism $\mathcal{A} \to \mathcal{A}'$ which fixes $\mathcal{F}_1$ and $\mathcal{F}_2$ pointwise.

A building is said to be an *affine building* if its apartments are affine spaces.

In the rest of the paper we will assume that Bruhat-Tits theory is available for G over K, that is, there is an affine building $\mathcal{B}(G'/K)$, called the Bruhat-Tits building of $G(K)$, on which this group acts by isometries, and given a nonempty bounded subset Ω of an apartment of this building, there is a smooth affine $\mathcal{O}$-group scheme $\mathscr{G}_{\Omega}^{\circ}$ with generic fiber G and connected special fiber–the building $\mathcal{B}(G'/K)$ and the group schemes $\mathscr{G}_{\Omega}^{\circ}$ having the properties described in 1.9 and 1.10 below.

When G is quasi-split over K–for example, if the residue field κ of k is perfect (1.7)–then Bruhat-Tits theory is available for G over K; see [BrT2, §4].

1.9. Bruhat-Tits theory for G over K: There exists an affine building called the Bruhat-Tits building of $G(K)$. It carries a natural structure of a polysimplicial complex on which $G(K)$ acts by polysimplicial automorphisms. The Bruhat-Tits building of $G(K)$ carries a $G(K)$-invariant metric. The building is also the Bruhat-Tits building of $G'(K)$, and we will denote it by $\mathcal{B}(G'/K)$. As we are assuming that ω is a discrete valuation, $\mathcal{B}(G'/K)$ is complete [BrT1, Thm. 2.5.12(i)]. The apartments of this building are in bijective correspondence with maximal K-split tori of G_K. If A is the apartment of $\mathcal{B}(G'/K)$ corresponding to a maximal K-split torus T of G_K, then for $g \in G(K)$, $g \cdot A$ is also an apartment and it corresponds to the maximal K-split torus gTg^{-1}. Hence the stabilizer of A in $G(K)$ is $N(T)(K)$, where $N(T)$ denotes the normalizer of T in G_K.

The facets of $\mathcal{B}(G'/K)$ of maximal dimension are called *chambers*. The group $G'(K)$ acts transitively on the set of ordered pairs consisting of an apartment of $\mathcal{B}(G'/K)$ and a chamber in it. In particular, $N(T)(K) \cap G'(K)$ acts transitively on the set of chambers in A.

There is a natural action of $G(K)$ on the Euclidean space $V(\mathfrak{Z}_K)$ by translations, with $G'(K)$ acting trivially. The direct product $V(\mathfrak{Z}_K) \times \mathcal{B}(G'/K)$ carries a $G(K)$-invariant metric extending the metric on $\mathcal{B}(G'/K)$, and there is visibly an action of $V(\mathfrak{Z}_K)$ on this product by translations in the first factor. This direct product is called the *enlarged* Bruhat-Tits building of $G(K)$ and we will denote it by $\mathcal{B}(G/K)$. The apartments of the enlarged building are by definition the subspaces of the form $V(\mathfrak{Z}_K) \times A$, where A is an apartment of $\mathcal{B}(G'/K)$. If G is semi-simple, i.e., $G' = G$, then $\mathcal{B}(G/K) = \mathcal{B}(G'/K)$.

Let T be a maximal K-split torus of G_K and A be the corresponding apartment of $\mathcal{B}(G/K)$. Then A is an affine space under $V(T) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(T)$, where $\mathrm{X}_*(T) = \mathrm{Hom}_K(\mathrm{GL}_1, T)$, and $N(T)(K)$ acts on A by affine transformations which we will describe now. Let $\mathrm{Aff}(A)$ be the group of affine automorphisms of A and $\nu : N(T)(K) \to \mathrm{Aff}(A)$ be the action map. For $n \in N(T)(K)$, the derivative $d\nu(n) : V(T) \to V(T)$ is the map induced by the action of n on $\mathrm{X}_*(T)$ (i.e., the Weyl group action). So, for $z \in Z(T)(K)$, $d\nu(z)$ is the identity, hence $\nu(z)$ is a translation; this translation is described by the following formula:

$$\chi(\nu(z)) = -\omega(\chi(z)) \text{ for all } \chi \in \mathrm{X}^*_K(Z(T)) \, (\hookrightarrow \mathrm{X}^*_K(T)),$$

here we regard the translation $\nu(z)$ as an element of $V(T)$. For z in the maximal bounded subgroup $Z(T)(K)_b$ of $Z(T)(K)$, since $\omega(\chi(z)) = 0$ for all $\chi \in \mathrm{X}^*_K(Z(T))$ (Proposition 1.5), the above formula shows that $Z(T)(K)_b$ acts trivially on A.

Given two points x and y of $\mathcal{B}(G/K)$, there is a unique geodesic $[xy]$ joining them and this geodesic lies in every apartment which contains x and y. A subset of the building is called *convex* if for any x, y in the set, the geodesic $[xy]$ is contained in the set. For a subset X of $\mathcal{B}(G/K)$ or $\mathcal{B}(G'/K)$, $\overline{X}$ will denote its closure. If X is convex, then so is $\overline{X}$.

Let $G(K)^\natural$ denote the normal subgroup of $G(K)$ consisting of elements that act trivially on $V(\mathfrak{Z}_K)$. This subgroup has the following description:

$$G(K)^\natural = \{g \in G(K) \mid \chi(g) \in \mathcal{O}^\times \text{ for all } \chi \in \mathrm{X}^*_K(G_K)\}.$$

$G(K)^\natural$ contains $G'(K)$ and also every bounded subgroup of $G(K)$. Given a nonempty bounded subset Ω of an apartment A of $\mathcal{B}(G'/K)$, let $G(K)^\Omega$ denote the subgroup of $G(K)^\natural$ consisting of elements that fix Ω pointwise (note that $G(K)^\Omega$ consists precisely of all elements of $G(K)$ that fix $V(\mathfrak{Z}_K)\times\Omega$ pointwise). There is a smooth affine $\mathcal{O}$-group scheme $\mathscr{G}_\Omega$, with generic fiber G_K, whose group of $\mathcal{O}$-rational points considered as a subgroup of $G(K)$ is $G(K)^\Omega$ (when G_K is quasi-split, these group schemes have been constructed in [BrT2, §4]; for a simpler treatment of the existence and smoothness of these "Bruhat-Tits group schemes", see [Y]).

The subgroup $G(K)^\Omega$ is of finite index in the stabilizer of Ω in $G(K)^\natural$. In fact, any element of $G(K)^\natural$ which stabilizes Ω permutes the facets of the building that meet Ω, and hence a subgroup of finite index of the stabilizer of Ω (in $G(K)^\natural$) keeps each facet that meets Ω stable and fixes every vertex of such a facet, hence it fixes pointwise every facet that meets Ω. Thus a subgroup of finite index of the stabilizer of Ω in $G(K)^\natural$ fixes Ω pointwise, therefore this subgroup is contained in $G(K)^\Omega$. As $G(K)^\Omega$ $(=\mathscr{G}_\Omega(\mathcal{O}))$ is a bounded subgroup of $G(K)$, the stabilizer of Ω in $G(K)^\natural$ is bounded.

The neutral component $\mathscr{G}_\Omega^\circ$ of $\mathscr{G}_\Omega$ is by definition the union of the connected generic fiber G_K and the identity component of the special fiber of $\mathscr{G}_\Omega$. The neutral component $\mathscr{G}_\Omega^\circ$ is an open $\mathcal{O}$-subgroup scheme of $\mathscr{G}_\Omega$, and it is affine [PY2, Lemma in §3.5]. The subgroup $\mathscr{G}_\Omega^\circ(\mathcal{O})$ is of finite index in $\mathscr{G}_\Omega(\mathcal{O})$ [EGA IV$_3$, Cor. 15.6.5]. According to [BrT2, 1.7.1-1.7.2] the $\mathcal{O}$-group schemes $\mathscr{G}_\Omega$ and $\mathscr{G}_\Omega^\circ$ are "étoffé" and hence by (ET) of [BrT2, 1.7.1] their affine rings have the following description:

$$\mathcal{O}[\mathscr{G}_\Omega]=\{f\in K[G]\mid f(\mathscr{G}_\Omega(\mathcal{O}))\subset\mathcal{O}\};\ \ \mathcal{O}[\mathscr{G}_\Omega^\circ]=\{f\in K[G]\mid f(\mathscr{G}_\Omega^\circ(\mathcal{O}))\subset\mathcal{O}\}.$$

If the above apartment A corresponds to the maximal K-split torus T of G_K, then there is a closed $\mathcal{O}$-torus $\mathscr{T}$ in $\mathscr{G}_\Omega^\circ$ with generic fiber T. The special fiber $\overline{\mathscr{T}}$ of $\mathscr{T}$ is a maximal κ_s-torus in the special fiber $\overline{\mathscr{G}}_\Omega^\circ$ of $\mathscr{G}_\Omega^\circ$. Note that $\mathscr{T}(\mathcal{O})$ is the maximal bounded subgroup of $T(K)$. For Ω as above, $\mathscr{G}_\Omega^\circ=\mathscr{G}_{\overline{\Omega}}^\circ$. If Ω is a nonempty subset of a facet F of $\mathcal{B}(G'/K)$, then $\mathscr{G}_\Omega^\circ=\mathscr{G}_F^\circ$, and if moreover, G is semi-simple, simply connected and quasi-split over K, then $\mathscr{G}_\Omega^\circ(\mathcal{O})$ is the stabilizer of Ω in $G(K)$, so $\mathscr{G}_\Omega=\mathscr{G}_\Omega^\circ$, i.e., both the fibers of $\mathscr{G}_\Omega$ are connected.

As usual, $G(K)^+$ will denote the normal subgroup of $G(K)$ generated by the K-rational elements of the unipotent radicals of parabolic K-subgroups of G_K. We assume in this paragraph that G is semi-simple and quasi-split over K. Let $\pi:\widehat{G}\to G$ be the simply connected central cover of G, then $\widehat{G}(K)^+=\widehat{G}(K)$ (Steinberg), and so $\pi(\widehat{G}(K))=G(K)^+$. Hence, if Ω is a subset of a facet of $\mathcal{B}(G/K)$, the stabilizer of Ω in $G(K)^+$ fixes Ω pointwise. Moreover, if an element g of $G(K)^+$ belongs to $\mathscr{G}_\Omega(\mathcal{O})$, then it is actually contained in $\mathscr{G}_\Omega^\circ(\mathcal{O})$.

For a facet F of $\mathcal{B}(G'/K)$, $\mathscr{G}_F^\circ$ and $\mathscr{G}_F^\circ(\mathcal{O})$ are respectively called the *Bruhat-Tits parahoric group scheme* and the *parahoric subgroup* of $G(K)$ associated to F. The subset of points (of $\mathcal{B}(G'/K)$) fixed under $\mathscr{G}_F^\circ(\mathcal{O})$ is precisely $\overline{F}$.

1.10. We introduce the following partial order "$\prec$" on the set of nonempty subsets of $\mathcal{B}(G'/K)$: Given two nonempty subsets Ω and Ω' of $\mathcal{B}(G'/K)$, $\Omega' \prec \Omega$ if the closure $\overline{\Omega}$ of Ω contains Ω'. For facets F and F' of $\mathcal{B}(G'/K)$, if $F' \prec F$, we say that F' is a *face* of F. In a collection $\mathcal{C}$ of facets, thus a facet is *maximal* if it is not a proper face of any facet belonging to $\mathcal{C}$, and a facet is *minimal* if no proper face of it belongs to $\mathcal{C}$.

Given nonempty bounded subsets Ω and Ω' of an apartment of $\mathcal{B}(G'/K)$, with $\Omega' \prec \Omega$, the inclusion $G(K)^{\Omega} \subset G(K)^{\Omega'}$ gives rise to a $\mathcal{O}$-group scheme homomorphism $\mathscr{G}_{\Omega} \to \mathscr{G}_{\Omega'}$ that is the identity homomorphism on the generic fiber G_K. This homomorphism restricts to a $\mathcal{O}$-group scheme homomorphism $\rho_{\Omega',\Omega} : \mathscr{G}^{\circ}_{\Omega} \to \mathscr{G}^{\circ}_{\Omega'}$ and induces a κ_s-homomorphism $\overline{\rho}_{\Omega',\Omega} : \overline{\mathscr{G}}^{\circ}_{\Omega} \to \overline{\mathscr{G}}^{\circ}_{\Omega'}$. The restriction of $\overline{\rho}_{\Omega',\Omega}$ to any torus of $\overline{\mathscr{G}}^{\circ}_{\Omega}$ is an isomorphism onto a torus of $\overline{\mathscr{G}}^{\circ}_{\Omega'}$. In particular, if $F' \prec F$ are two facets of $\mathcal{B}(G'/K)$, then there is a $\mathcal{O}$-group scheme homomorphism $\rho_{F',F} : \mathscr{G}^{\circ}_{F} \to \mathscr{G}^{\circ}_{F'}$ that is the identity homomorphism on the generic fiber G_K. We will assume in this paper that:

(1) The kernel of the induced homomorphism $\overline{\rho}_{F',F} : \overline{\mathscr{G}}^{\circ}_{F} \to \overline{\mathscr{G}}^{\circ}_{F'}$ is a smooth unipotent κ_s-subgroup of $\overline{\mathscr{G}}^{\circ}_{F}$.

(2) The image $\mathfrak{p}(F'/F) := \overline{\rho}_{F',F}(\overline{\mathscr{G}}^{\circ}_{F})$ is a pseudo-parabolic κ_s-subgroup of $\overline{\mathscr{G}}^{\circ}_{F'}$.

(3) Let T be a maximal K-split torus of G_K such that the apartment of $\mathcal{B}(G'/K)$ corresponding to T contains F. Let $\mathscr{T}$ be the closed $\mathcal{O}$-torus of $\mathscr{G}^{\circ}_{F}$ with generic fiber T, and let $\overline{\mathscr{T}}$ be the special fiber of $\mathscr{T}$. We consider $\overline{\mathscr{T}}$ to be a maximal κ_s-torus of $\overline{\mathscr{G}}^{\circ}_{F}$, as well as of $\overline{\mathscr{G}}^{\circ}_{F'}$ (under the homomorphism $\overline{\rho}_{F',F}$), and also of the maximal pseudo-reductive quotient $\overline{G}^{\rm pred}_{F'} := \overline{\mathscr{G}}^{\circ}_{F'}/\mathscr{R}_{u,\kappa_s}(\overline{\mathscr{G}}^{\circ}_{F'})$ of $\overline{\mathscr{G}}^{\circ}_{F'}$. Let x be a point of F' and v be a vector in $V(T)$ such that $v+x$ is a point of F. Then the nonzero weights of $\overline{\mathscr{T}}$ in the Lie algebra of the pseudo-parabolic κ_s-subgroup $\mathfrak{p}(F'/F)/\mathscr{R}_{u,\kappa_s}(\overline{\mathscr{G}}^{\circ}_{F'})$ of $\overline{G}^{\rm pred}_{F'}$ are the roots a of $\overline{G}^{\rm pred}_{F'}$ (with respect to $\overline{\mathscr{T}}$) such that $v(a) \geqslant 0$.

(4) The inverse image of the subgroup $\mathfrak{p}(F'/F)(\kappa_s)$ of $\overline{\mathscr{G}}^{\circ}_{F'}(\kappa_s)$, under the natural homomorphism $\pi_{F'} : \mathscr{G}^{\circ}_{F'}(\mathcal{O}) \to \overline{\mathscr{G}}^{\circ}_{F'}(\kappa_s)$ is $\mathscr{G}^{\circ}_{F}(\mathcal{O})$.

(5) $F \mapsto \mathfrak{p}(F'/F)$ is an order-preserving bijective map of the partially-ordered set $\{F \mid F' \prec F\}$ onto the set of pseudo-parabolic κ_s-subgroups of $\overline{\mathscr{G}}^{\circ}_{F'}$ partially-ordered by the opposite of inclusion.

Note that (4) implies that the inverse image $P^{+}_{F'}$ under $\pi_{F'}$ of the normal subgroup $\mathscr{R}_{u,\kappa_s}(\overline{\mathscr{G}}^{\circ}_{F'})(\kappa_s)\,(\subset \mathfrak{p}(F'/F)(\kappa_s))$ of $\overline{\mathscr{G}}^{\circ}_{F'}(\kappa_s)$ is contained in $\mathscr{G}^{\circ}_{F}(\mathcal{O})$. So $P^{+}_{F'}$ fixes every facet F, $F' \prec F$, pointwise. (5) implies that a facet C of $\mathcal{B}(G'/K)$ is a chamber (i.e., it is a maximal facet) if and only if $\overline{\mathscr{G}}^{\circ}_{C}$ does not contain a proper pseudo-parabolic κ_s-subgroup, or, equivalently, the maximal pseudo-reductive quotient of $\overline{\mathscr{G}}^{\circ}_{C}$ is commutative [CGP, Lemma 2.2.3]. (When G_K is quasi-split, the above assertions are proved in [BrT2, Thm. 4.6.33].)

1.11. Bruhat-Tits theory for the derived subgroup G'. For a nonempty bounded subset Ω of an apartment of the building $\mathcal{B}(G'/K)$ of $G'(K)$, let $\mathscr{G}_{\Omega}$ be the smooth affine $\mathcal{O}$-group scheme as in the preceding subsection. Then the neutral component of the canonical smoothening [BLR, 7.1, Thm. 5] (see also [PY2, 3.2]) of the schematic closure of G' in $\mathscr{G}_\Omega$ is by definition the Bruhat-Tits $\mathcal{O}$-group scheme associated to Ω and G'. Its generic fiber is G'. It is easily seen that these $\mathcal{O}$-group schemes have the properties described in 1.9 and 1.10, and hence if Bruhat-Tits theory is available for G over K, then it is also available for the derived subgroup G' over K.

1.12. As before, let $G(K)^\natural$ denote the normal subgroup of $G(K)$ consisting of elements that act trivially on $V(\mathfrak{Z}_K)$; $G(K)^\natural$ contains $G'(K)$ and also all bounded subgroups of $G(K)$. Let T be a maximal K-split torus of G_K, $N(T)$ be its normalizer, and $Z(T)$ be its centralizer, in G_K. Let A be the apartment of $\mathcal{B}(G'/K)$ corresponding to T and C be a chamber in A. Then the stabilizer of A in $G(K)$ is $\mathscr{N} := N(T)(K)$. Let $\mathscr{I}$ be the stabilizer of C in $G(K)^\natural$; $\mathscr{I}$ is a bounded subgroup of $G(K)$. The maximal bounded subgroup $\mathscr{Z}_b$ of $\mathscr{Z} := Z(T)(K)$ fixes C pointwise (1.9) and hence it is contained in $\mathscr{I}$. Let $\mathscr{N}^\natural = \mathscr{N} \cap G(K)^\natural$ and $\mathscr{Z}^\natural = \mathscr{Z} \cap G(K)^\natural$. As $G'(K)$ acts transitively on the set of ordered pairs consisting of an apartment and a chamber in it, and any two chambers of a building lie on an apartment, we conclude that $G(K)^\natural = \mathscr{I}\mathscr{N}^\natural\mathscr{I}$. The Weyl group $\mathscr{N}/\mathscr{Z}$ of G_K is finite. We fix a finite subset $\mathcal{S}$ of $\mathscr{N}^\natural$ that maps onto $\mathscr{N}^\natural/\mathscr{Z}^\natural$. Then $G(K)^\natural = \mathscr{I}\mathcal{S}\mathscr{Z}^\natural\mathscr{I}$. It is obvious that a subset $\mathcal{X}$ of $G(K)^\natural$ is bounded if and only if $\mathscr{I}\mathcal{X}\mathscr{I}$ is bounded, or, equivalently, if and only if there exists a bounded subset $\mathcal{Y}$ of $\mathscr{Z}^\natural$ such that $\mathcal{X} \subset \mathscr{I}\mathcal{S}\mathcal{Y}\mathscr{I}$.

The subgroup $\mathscr{Z}_b$ of $\mathscr{Z}$ has the following description (Proposition 1.5): An element $z \in \mathscr{Z}$ belongs to $\mathscr{Z}_b$ if and only if for every K-rational character χ of $Z(T)$, $\omega(\chi(z)) = 0$. We fix a basis $\{\chi_j\}_{j=1}^{\dim T}$ of the group of K-rational characters of $Z(T)$. Then the map $z \mapsto (\omega(\chi_j(z)))$ provides an embedding of $\mathscr{Z}/\mathscr{Z}_b$ into $\mathbb{Z}^{\dim T}$ and so a subset of $\mathscr{Z}$ is bounded if and only if its image in $\mathscr{Z}/\mathscr{Z}_b$ is finite, or, equivalently, if and only if it is contained in the union of finitely many cosets of $\mathscr{Z}_b$ in $\mathscr{Z}$. Thus $\mathcal{X}\,(\subset G(K)^\natural)$ is bounded if and only if there exist a finite subset $\{n_i\} \subset \mathscr{N}^\natural$ such that $\mathcal{X} \subset \bigcup_i \mathscr{I} n_i \mathscr{I}$.

Using these observations, we prove the following proposition.

Proposition 1.13. *A subset $\mathcal{X}$ of $G(K)^\natural$ is bounded if and only if for every $x \in \mathcal{B}(G'/K)$ the set $\{g \cdot x \,|\, g \in \mathcal{X}\}$ is of bounded diameter.*

So if a nonempty closed convex subset of $\mathcal{B}(G'/K)$ is stable under the action of a bounded subgroup $\mathcal{G}$ of $G(K)$, then by the Bruhat-Tits fixed point theorem (Proposition 3.2.4 of [BrT1]) it contains a point fixed by $\mathcal{G}$.

Proof. It is easy to see that to prove the proposition it suffices to prove that $\mathcal{X}$ is bounded if and only if for some $x \in \mathcal{B}(G'/K)$, the set $\{g \cdot x \,|\, g \in \mathcal{X}\}$ is of bounded diameter. We will now use the notation introduced in 1.12 and choose a $x_0 \in C$ fixed by $\mathscr{I}$. We have observed in 1.12 that $\mathcal{X}$ is bounded if and only if there is a finite subset $\{n_i\}$ of $\mathscr{N}^\natural$ such that $\mathcal{X} \subset \bigcup_i \mathscr{I} n_i \mathscr{I}$. Let d be a $G(K)$-invariant metric on $\mathcal{B}(G/K)$. Then for every $g \in \mathscr{I} n_i \mathscr{I}$, $d(g \cdot x_0, x_0) = d(n_i \cdot x_0, x_0)$. This implies that $\mathscr{I} n_i \mathscr{I} \cdot x_0$ is a subset of bounded diameter for each i, proving the proposition. $\square$

1.14. There is a natural action of $\Gamma = \mathrm{Gal}(K/k)$ on $\mathcal{B}(G'/K)$ by polysimplicial isometries such that the orbit of every point under this action is finite, and for all $g \in G(K)$, $x \in \mathcal{B}(G'/K)$, $\gamma \in \Gamma$, we have $\gamma(g \cdot x) = \gamma(g) \cdot \gamma(x)$ (cf. [BrT2, 4.2.12]). Thus, there is an action of $\Gamma \ltimes G(K)$ on $\mathcal{B}(G/K)\,(= V(\mathfrak{Z}_K) \times \mathcal{B}(G'/K))$. According to the Bruhat-Tits fixed point theorem, $\mathcal{B}(G/K)$ contains a point fixed under Γ.

For any apartment A of $\mathcal{B}(G/K)$, and $\gamma \in \Gamma$, $\gamma(A)$ is an apartment and the action map $A \to \gamma(A)$ is affine. Therefore, if T is a k-torus of G such that T_K is a maximal K-split torus of G_K, then the apartment A_T of $\mathcal{B}(G/K)$ corresponding to T_K is stable under the action of Γ, and Γ acts on A_T by affine transformations through a finite quotient.

Given a nonempty bounded subset Ω of an apartment of $\mathcal{B}(G'/K)$ that is stable under the action of Γ, $\mathscr{G}_\Omega(\mathcal{O})$ is stable under Γ and hence the affine ring $\mathcal{O}[\mathscr{G}_\Omega]\,(\subset K[G])$ of $\mathscr{G}_\Omega$ is stable under the natural action of Γ on $K[G] = K \otimes_k k[G]$. In such cases (i.e., when Ω is stable under the action of Γ), the $\mathcal{O}$-group scheme $\mathscr{G}_\Omega$, and so also its neutral component, admit unique descents to smooth affine $\mathfrak{o}$-group schemes with generic fiber G; the affine rings of these descents are $(\mathcal{O}[\mathscr{G}_\Omega])^\Gamma = \mathcal{O}[\mathscr{G}_\Omega] \cap k[G]$ and $(\mathcal{O}[\mathscr{G}_\Omega^\circ])^\Gamma = \mathcal{O}[\mathscr{G}_\Omega^\circ] \cap k[G]$; see [BLR, §6.2, Ex. B]. As it is unlikely to cause confusion, in the sequel whenever Ω is stable under Γ, we will use $\mathscr{G}_\Omega$ and $\mathscr{G}_\Omega^\circ$ to denote these smooth affine $\mathfrak{o}$-group schemes, and denote the special fibers of these group schemes by $\overline{\mathscr{G}}_\Omega$ and $\overline{\mathscr{G}}_\Omega^\circ$ respectively. $\overline{\mathscr{G}}_\Omega$ is a smooth affine κ-group and $\overline{\mathscr{G}}_\Omega^\circ$ is its identity component. The maximal pseudo-reductive quotient of $\overline{\mathscr{G}}_\Omega^\circ$ will be denoted by $\overline{G}_\Omega^{\mathrm{pred}}$.

For a point $x \in \mathcal{B}(G'/K)$ fixed under Γ, we will denote $\mathscr{G}_{\{x\}}$, $\mathscr{G}_{\{x\}}^\circ$, $\overline{\mathscr{G}}_{\{x\}}^\circ$ and $\overline{G}_{\{x\}}^{\mathrm{pred}}$ by $\mathscr{G}_x$, $\mathscr{G}_x^\circ$, $\overline{\mathscr{G}}_x^\circ$ and $\overline{G}_x^{\mathrm{pred}}$ respectively. By definition, $\mathscr{G}_x^\circ$ and $\mathscr{G}_x^\circ(\mathfrak{o})$ are respectively the *Bruhat-Tits parahoric group scheme* and the *parahoric subgroup* of $G(k)$ associated to the point x.

Let T be a k-torus of G such that T_K is a maximal K-split torus of G_K. Let Ω be a nonempty bounded subset of the apartment of $\mathcal{B}(G'/K)$ corresponding to T_K. We assume that Ω is stable under the action of Γ. Then the $\mathcal{O}$-torus $\mathscr{T}$ of 1.9 admits a unique descent to a closed $\mathfrak{o}$-torus of $\mathscr{G}_\Omega^\circ$; in the sequel we will denote this $\mathfrak{o}$-torus also by $\mathscr{T}$. The generic fiber of $\mathscr{T}$ is T, its special fiber $\overline{\mathscr{T}}$ is a maximal κ-torus of $\overline{\mathscr{G}}_\Omega^\circ$, and $\mathscr{G}_\Omega^\circ(\mathfrak{o}) \cap T(k)$ is the maximal bounded subgroup of $T(k)$. If the k-torus T contains a maximal k-split torus S of G, then the generic fiber of the maximal $\mathfrak{o}$-split subtorus $\mathscr{S}$ of $\mathscr{T}$ is S and the special fiber $\overline{\mathscr{S}}\,(\subset \overline{\mathscr{T}})$ of $\mathscr{S}$ is a maximal κ-split torus of $\overline{\mathscr{G}}_\Omega^\circ$.

In view of the results on descent of $\mathcal{O}$-group schemes described above, it is obvious that to establish descent of Bruhat-Tits theory from K to k for G it only needs to be shown that $\mathcal{B}(G'/K)^\Gamma$ is an affine building.

1.15. Let $\mathcal{B} = \mathcal{B}(G'/K)^{\Gamma}$; $\mathcal{B}$ is closed and convex and is stable under the action of $G(k)$ on $\mathcal{B}(G'/K)$. We will show that $\mathcal{B}$ carries a natural structure of a polysimplicial complex (3.2), its facets (or polysimplices) being the intersections with $\mathcal{B}$ of facets of $\mathcal{B}(G'/K)$ that are stable under Γ, and it is an affine building. The maximal facets (maximal in the ordering defined in 1.10) will be called *chambers* of $\mathcal{B}$. We will prove that the dimension of $\mathcal{B}$, and so that of any chamber of $\mathcal{B}$, is $r := k$-rank G'. The apartments of $\mathcal{B}$ are, by definition, the polysimplicial subcomplexes which are intersections of *special k-apartments* of $\mathcal{B}(G'/K)$ (see 1.16 below) with $\mathcal{B}$. We will show that the apartments of $\mathcal{B}$ are affine spaces of dimension r and they are in bijective correspondence with maximal k-split tori of G. To show that $\mathcal{B}$, considered as a polysimplicial complex given with the above apartments, is an affine building, we will verify that the conditions defining an affine building (recalled in 1.8) hold.

If a facet of $\mathcal{B}(G'/K)$ is stable under the action of Γ, then its barycenter is fixed under Γ, i.e., the barycenter belongs to $\mathcal{B}$. Conversely, if a facet F contains a point x fixed under Γ, i.e., if F meets $\mathcal{B}$, then being the unique facet containing x, F is stable under the action of Γ. A facet of the building $\mathcal{B}(G'/K)$ that meets $\mathcal{B}$ will be called a *k-facet*.

1.16. Special k-tori and special k-apartments. A *special k-torus* in G is a k-torus $T\,(\subset G)$ that contains a maximal k-split torus of G and T_K is a maximal K-split torus of G_K. The apartment in $\mathcal{B}(G/K)$, or in $\mathcal{B}(G'/K)$, corresponding to T_K, for a special k-torus T, will henceforth be called a *special k-apartment* corresponding to the (special) k-torus T. According to [BrT2, Cor. 5.1.12], if Bruhat-Tits theory is available for G over K (for example, if G is quasi-split over K), then G contains a special k-torus. As this is an important and very useful result, we will give its proof in the next section (see Proposition 2.3).

It is clear from the definition that every special k-apartment is stable under the action of the Galois group Γ. If $x \neq y$ are two points of a Γ-stable apartment A which are fixed under Γ, then the whole straight line in A passing through x and y is pointwise fixed under Γ.

1.17. Let X be a nonempty convex subset of $\mathcal{B}(G'/K)$ and $\mathcal{C}$ be the set of facets of $\mathcal{B}(G'/K)$, or facets lying in a given apartment A of this building, that meet X. Then it is easy to see (Proposition 9.2.5 (i), (ii), of [BrT1]) that all maximal facets in $\mathcal{C}$ are of equal dimension. If F is maximal among the facets lying in A that meet X, then $A \cap X$ is contained in the affine subspace of A spanned by $F \cap X$. So, for any facet F' in A, $\dim(F \cap X) \geqslant \dim(F' \cap X)$.

As $\mathcal{B}$ is a nonempty convex subset of $\mathcal{B}(G'/K)$, the above assertions hold for $X = \mathcal{B}$. Maximal k-facets of $\mathcal{B}(G'/K)$ will be called *k-chambers*. The k-chambers are of equal dimension, and moreover, for any k-chamber C, $\mathcal{C} := C^{\Gamma} = C \cap \mathcal{B}$ is a chamber of $\mathcal{B}$. Conversely, given a chamber $\mathcal{C}$ of $\mathcal{B}$, the unique facet C of $\mathcal{B}(G'/K)$ that contains $\mathcal{C}$ is a k-chamber and $\mathcal{C} = C \cap \mathcal{B}$. Note that a k-chamber may not be a chamber (i.e., it may not be a facet of $\mathcal{B}(G'/K)$ of maximal dimension); see, however, Proposition 2.4.

1.18. Given a nonempty Γ-stable bounded subset Ω of an apartment of $\mathcal{B}(G'/K)$ and a nonempty Γ-stable subset Ω' of $\overline{\Omega}$, the homomorphism $\rho_{\Omega',\Omega}$ described in 1.10 descends to a $\mathfrak{o}$-group scheme homomorphism $\mathscr{G}_{\Omega}^{\circ} \to \mathscr{G}_{\Omega'}^{\circ}$ that is the identity homomorphism on the generic fiber G. We shall denote this $\mathfrak{o}$-homomorphism also by $\rho_{\Omega',\Omega}$; it induces a κ-homomorphism $\overline{\rho}_{\Omega',\Omega} : \overline{\mathscr{G}}_{\Omega}^{\circ} \to \overline{\mathscr{G}}_{\Omega'}^{\circ}$ between the special fibers. In particular, if $F' \prec F$ are two k-facets of $\mathcal{B}(G'/K)$, then there is a $\mathfrak{o}$-group scheme homomorphism $\mathscr{G}_{F}^{\circ} \to \mathscr{G}_{F'}^{\circ}$ that is the identity homomorphism on the generic fiber G. The image of the induced homomorphism $\overline{\mathscr{G}}_{F}^{\circ} \to \overline{\mathscr{G}}_{F'}^{\circ}$ is a pseudo-parabolic κ-subgroup $\mathfrak{p}(F'/F)$ of $\overline{\mathscr{G}}_{F'}^{\circ}$, and $F \mapsto \mathfrak{p}(F'/F)$ is an order-preserving bijective map of the partially-ordered set $\{F \mid F' \prec F\}$ onto the set of pseudo-parabolic κ-subgroups of $\overline{\mathscr{G}}_{F'}^{\circ}$ partially-ordered by opposite of inclusion (1.10).

Thus, F is a maximal k-facet (i.e., it is a k-chamber) if and only if $\mathfrak{p}(F'/F)$ is a minimal pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}_{F'}^{\circ}$. Now note that the projection map $\overline{\mathscr{G}}_{F'}^{\circ} \to \overline{G}_{F'}^{\mathrm{pred}}$ induces an inclusion preserving bijective correspondence between the pseudo-parabolic κ-subgroups of $\overline{\mathscr{G}}_{F'}^{\circ}$ and the pseudo-parabolic κ-subgroups of its maximal pseudo-reductive quotient $\overline{G}_{F'}^{\mathrm{pred}}$ [CGP, Prop. 2.2.10]. Hence, a k-facet C of $\mathcal{B}(G/K)$ is a k-chamber if and only if the pseudo-reductive κ-group $\overline{G}_{C}^{\mathrm{pred}}$ does not contain a proper pseudo-parabolic κ-subgroup, or, equivalently, this pseudo-reductive group contains a unique maximal κ-split torus (this torus is central so it is contained in every maximal torus of $\overline{G}_{C}^{\mathrm{pred}}$) [CGP, Lemma 2.2.3].

2. Ten basic propositions

Proposition 2.1. *Let $\mathscr{G}$ be a smooth affine $\mathfrak{o}$-group scheme and $\overline{\mathscr{G}} := \mathscr{G}_{\kappa}$ be its special fiber.*

(i) *Let $\overline{\mathscr{T}}$ be a κ-torus of $\overline{\mathscr{G}}$. There exists a closed $\mathfrak{o}$-torus $\mathscr{T}$ in $\mathscr{G}$ whose special fiber is $\overline{\mathscr{T}}$.*

(ii) *Let $\mathscr{T}$ and $\mathscr{T}'$ be two closed $\mathfrak{o}$-tori of $\mathscr{G}$ such that there is an element $\overline{g} \in \mathscr{G}(\kappa)\,(= \overline{\mathscr{G}}(\kappa))$ that conjugates $\mathscr{T}_{\kappa}$ onto $\mathscr{T}'_{\kappa}$. There exists a $g \in \mathscr{G}(\mathfrak{o})$ lying over $\overline{g}$ that conjugates $\mathscr{T}$ onto $\mathscr{T}'$.*

(iii) *Let $\mathscr{T}$ be a closed $\mathfrak{o}$-torus of $\mathscr{G}$. Then the normalizer $N_{\mathscr{G}}(\mathscr{T})$ of $\mathscr{T}$ in $\mathscr{G}$ is a closed smooth $\mathfrak{o}$-subgroup scheme of $\mathscr{G}$. In particular, the natural homomorphism $N_{\mathscr{G}}(\mathscr{T})(\mathfrak{o}) \to N_{\mathscr{G}}(\mathscr{T})(\kappa)$ is onto.*

Remark. The proof of assertion (i) of this proposition (and also that of the next proposition) is essentially same as the proof of Proposition 5.1.10 of [BrT2]. In (i), since the special fiber of $\mathscr{T}$ is $\overline{\mathscr{T}}$, the character groups of $\mathscr{T}_{\mathcal{O}}$ and $\overline{\mathscr{T}}_{\kappa_s}$ are isomorphic as Γ-modules, $\Gamma = \mathrm{Gal}(K/k) = \mathrm{Gal}(\kappa_s/\kappa)$. In particular, $\mathscr{T}$ is split if $\overline{\mathscr{T}}$ is split.

Proof of Proposition 2.1. (i) Let X be the character group of $\overline{\mathscr{T}}_{\kappa_s}$ considered as a Γ-module under the natural action of Γ and $\kappa_s[X]$ (resp. $\mathcal{O}[X]$) be the group ring of X with coefficients in κ_s (resp. $\mathcal{O}$). Then the affine ring of the κ-torus $\overline{\mathscr{T}}$ is $(\kappa_s[X])^{\Gamma}$.

Let $\mathscr{T}$ be the $\mathfrak{o}$-torus whose affine ring is $(\mathcal{O}[X])^{\Gamma}$. Then, clearly, the special fiber $\mathscr{T}_\kappa$ of $\mathscr{T}$ is isomorphic to $\overline{\mathscr{T}}$ and the character group of $\mathscr{T}_{\mathcal{O}}$ is isomorphic as a Γ-module to X. We fix a κ-isomorphism $\bar{\iota} : \mathscr{T}_\kappa \to \overline{\mathscr{T}}\ (\subset \overline{\mathscr{G}})$ and view it as a closed immersion of $\mathscr{T}_\kappa$ into $\overline{\mathscr{G}}$. According to a result of Grothendieck [SGA3$_{\rm II}$, Exp. XI, 4.2], the homomorphism scheme $\underline{\mathrm{Hom}}_{\mathrm{Spec}(\mathfrak{o})-\mathrm{gr}}(\mathscr{T}, \mathscr{G})$ is representable by a smooth $\mathfrak{o}$-scheme $\mathscr{X}$. Clearly, $\bar{\iota} \in \mathscr{X}(\kappa)$. Now since $\mathfrak{o}$ is Henselian, the natural map $\mathscr{X}(\mathfrak{o}) \to \mathscr{X}(\kappa)$ is surjective [EGA IV, 18.5.17], and hence there is a $\mathfrak{o}$-homomorphism $\iota : \mathscr{T} \to \mathscr{G}$ lying over $\bar{\iota}$, i.e., $\iota_\kappa = \bar{\iota}$. As $\bar{\iota}$ is a closed immersion, using [SGA3$_{\rm II}$, Exp. IX, 2.5 and 6.6] we see that ι is also a closed immersion. We identify $\mathscr{T}$ with a closed $\mathfrak{o}$-torus of $\mathscr{G}$ in terms of ι. Then the special fiber of $\mathscr{T}$ is $\overline{\mathscr{T}}$. This proves assertion (i).

(ii) The transporter scheme $\mathfrak{T} := \mathrm{Transp}_{\mathscr{G}}(\mathscr{T}, \mathscr{T}')$, consisting of points of the scheme $\mathscr{G}$ that conjugate $\mathscr{T}$ onto $\mathscr{T}'$, is a closed smooth $\mathfrak{o}$-subscheme of $\mathscr{G}$ (see [C, Prop. 2.1.2] or [SGA3$_{\rm II}$, Exp. XI, 2.4bis]). Let $\overline{\mathfrak{T}}$ be the special fiber of $\mathfrak{T}$. Then $\overline{g}$ belongs to $\overline{\mathfrak{T}}(\kappa)$. Now as $\mathfrak{o}$ is Henselian, the natural map $\mathfrak{T}(\mathfrak{o}) \to \overline{\mathfrak{T}}(\kappa)$ is surjective [EGA IV$_4$, 18.5.17]. Therefore, there exists a $g \in \mathfrak{T}(\mathfrak{o})$ lying over $\overline{g}$. This g will conjugate $\mathscr{T}$ onto $\mathscr{T}'$.

(iii) In the proof of assertion (ii), by taking $\mathscr{T}' = \mathscr{T}$ we conlclude (iii). □

Proposition 2.2. *Let T be a maximal K-split torus of G_K and Ω be a nonempty bounded subset of the apartment A of $\mathcal{B}(G'/K)$ corresponding to T. Let T' be another maximal K-split torus of G_K and A' be the corresponding apartment of $\mathcal{B}(G'/K)$. Then Ω is contained in A' if and only if one of the following three equivalent conditions hold:*

(i) *There is an element $g \in \mathscr{G}_\Omega^\circ(\mathcal{O})$ such that $T' = gTg^{-1}$. This element carries A to A' and fixes Ω pointwise.*

(ii) *$\mathscr{G}_\Omega^\circ$ contains a closed $\mathcal{O}$-torus with generic fiber T'.*

(iii) *$\mathscr{G}_\Omega^\circ(\mathcal{O}) \cap T'(K)$ is the maximal bounded subgroup of $T'(K)$.*

When G_K is quasi-split, the first assertion of this proposition is [BrT2, Prop. 4.6.28(iii)]. The proof given below is different from the proof in [BrT2].

Proof. We will use the preceding proposition, with $\mathcal{O}$ in place of $\mathfrak{o}$, and denote $\mathscr{G}_\Omega^\circ$ by $\mathscr{G}$, and its special fiber by $\overline{\mathscr{G}}$, in this proof. Let $\mathscr{T}$ be the closed $\mathcal{O}$-torus of $\mathscr{G}$ with generic fiber T. If Ω is contained in A', then $\mathscr{G}$ contains a closed $\mathcal{O}$-torus with generic fiber T'. Let us assume now that $\mathscr{G}$ contains a closed $\mathcal{O}$-torus $\mathscr{T}'$ with generic fiber T'. As the residue field κ_s of $\mathcal{O}$ is separably closed, the special fibers $\overline{\mathscr{T}}$ and $\overline{\mathscr{T}}'$ of $\mathscr{T}$ and $\mathscr{T}'$ are maximal split tori of $\overline{\mathscr{G}}$, and hence there is an element $\overline{g}$ of $\overline{\mathscr{G}}(\kappa_s)$ that conjugates $\overline{\mathscr{T}}$ onto $\overline{\mathscr{T}}'$ [CGP, Thm. C.2.3]. Now Proposition 2.1(ii) implies that there exists a $g \in \mathscr{G}(\mathcal{O})$ lying over $\overline{g}$ that conjugates $\mathscr{T}$ onto $\mathscr{T}'$. This element fixes Ω pointwise and conjugates T onto T' and hence carries A to A'. Hence Ω is contained in A'. Conversely, if there is an element $g \in \mathscr{G}(\mathcal{O})$ such that $T' = gTg^{-1}$, then $\mathscr{T}' := g\mathscr{T}g^{-1}$ is a closed $\mathcal{O}$-torus of $\mathscr{G}$ with generic fiber T', and g carries A to A' fixing Ω pointwise.

By Lemma 4.1 of [PY2], $\mathscr{G}(\mathcal{O})\cap T'(K)$ is the maximal bounded subgroup of $T'(K)$ if and only if the schematic closure of T' in $\mathscr{G}$ is a $\mathcal{O}$-torus. $\square$

Proposition 2.3 ([BrT2, Cor. 5.1.12]). *G contains a special k-torus.*

Proof. Let S be a maximal k-split torus of G and $\mathscr{S}$ be the split $\mathfrak{o}$-torus with generic fiber S. The subgroup $\mathscr{S}(\mathcal{O})$ of $S(K)$ is the maximal bounded subgroup of the latter and is clearly stable under the action of the Galois group Γ. Therefore there exists a point, say x, of $\mathcal{B}(G'/K)$ that is fixed under $\mathscr{S}(\mathcal{O})$, as well as, under Γ. In particular, $x\in\mathcal{B}=\mathcal{B}(G'/K)^{\Gamma}$.

Let $\mathscr{G}:=\mathscr{G}_x$ be the smooth affine $\mathfrak{o}$-group scheme with generic fiber G associated to x in 1.14. As $\mathscr{S}(\mathcal{O})$ fixes x, there is a natural inclusion $\mathscr{S}(\mathcal{O})\hookrightarrow\mathscr{G}_x(\mathcal{O})$ which gives a $\mathfrak{o}$-group scheme homomorphism $\varphi:\mathscr{S}\to\mathscr{G}_x$ that is the natural inclusion on generic fibers. We identify $\mathscr{S}$ with a closed $\mathfrak{o}$-torus of $\mathscr{G}_x$ in terms of this homomorphism. Let $\mathscr{M}$ be the centralizer of $\mathscr{S}$ in $\mathscr{G}_x$. Then $\mathscr{M}$ is a smooth $\mathfrak{o}$-group scheme, see [SGA3$_{\rm II}$, Exp. XI, Cor. 5.3] or [CGP, Prop. A.8.10(2)]. Applying Proposition 2.1(i) to $\mathscr{M}$ in place of $\mathscr{G}$, we conclude that $\mathscr{M}$ contains a closed $\mathfrak{o}$-torus $\mathscr{T}$ whose special fiber $\overline{\mathscr{T}}$ is a maximal κ-torus of the special fiber $\overline{\mathscr{M}}$ of $\mathscr{M}$. Therefore, $\mathscr{T}$ contains the central $\mathfrak{o}$-torus $\mathscr{S}$ and its generic fiber T is a k-torus of G containing the maximal k-split torus S. As $\overline{\mathscr{T}}$ is a maximal κ-torus of $\overline{\mathscr{M}}$ (and so of $\overline{\mathscr{G}}_x$) and it splits over κ_s, T_K is a maximal K-split torus of G_K. Thus T is a special k-torus of G. $\square$

Proposition 2.4. *Every special k-apartment of $\mathcal{B}(G'/K)$ contains a k-chamber. If κ is perfect and of dimension $\leqslant 1$, then every k-chamber is a chamber of $\mathcal{B}(G'/K)$.*

Proof. Let A be a special k-apartment and T be the corresponding special k-torus. Then T contains a maximal k-split torus S of G and T_K is a maximal K-split torus of G_K. As A is stable under the action of Γ, by the Bruhat-Tits fixed point theorem, it contains a point x which is fixed under Γ. Let F be the facet lying on A which contains x. Then, by definition, F is a k-facet. Let $\mathscr{G}_F^{\circ}$ be the smooth affine $\mathfrak{o}$-group scheme, with connected fibers, associated to $\Omega=F$ in 1.14 and $\overline{\mathscr{G}}_F^{\circ}$ be the special fiber of $\mathscr{G}_F^{\circ}$. Let $\mathscr{T}$ be the closed $\mathfrak{o}$-torus of $\mathscr{G}_F^{\circ}$ with generic fiber T, and let $\mathscr{S}$ be the maximal $\mathfrak{o}$-split subtorus of $\mathscr{T}$ (cf. 1.14). Then the generic fiber of $\mathscr{S}$ is S. Let $\overline{\mathscr{S}}$ and $\overline{\mathscr{T}}$ be the special fibers of $\mathscr{S}$ and $\mathscr{T}$ respectively. We fix a minimal pseudo-parabolic κ-subgroup $\overline{\mathscr{P}}$ of $\overline{\mathscr{G}}_F^{\circ}$ containing $\overline{\mathscr{S}}$, then $\overline{\mathscr{P}}$ contains the centralizer of $\overline{\mathscr{S}}$ [CGP, Prop. C.2.4], and so it contains $\overline{\mathscr{T}}$. Let $\mathcal{P}$ be the inverse image of $\overline{\mathscr{P}}(\kappa_s)$ in $\mathscr{G}_F^{\circ}(\mathcal{O})\,(\subset G(K))$ under the natural homomorphism $\mathscr{G}_F^{\circ}(\mathcal{O})\to\overline{\mathscr{G}}_F^{\circ}(\kappa_s)$. Then $\mathcal{P}$ is a parahoric subgroup of $G(K)$ contained in the parahoric subgroup $\mathscr{G}_F^{\circ}(\mathcal{O})$ (1.10 (5)); $\mathcal{P}$ contains $\mathscr{T}(\mathcal{O})$ and is clearly stable under the action of Γ on $G(K)$. Let C be the facet of the Bruhat-Tits building $\mathcal{B}(G'/K)$ fixed by $\mathcal{P}$. Then C contains F in its closure and is stable under Γ, i.e., it is a k-facet; it is a k-chamber since $\overline{\mathscr{P}}$ is a minimal pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}_F^{\circ}$ (1.18). Moreover, as $\mathcal{P}$ contains the maximal bounded subgroup $\mathscr{T}(\mathcal{O})$ of $T(K)$, C lies in the apartment A (Proposition 2.2(iii)).

If κ is perfect and of dimension $\leqslant 1$, $\overline{\mathscr{G}}_F^\circ$ contains a Borel subgroup defined over κ (1.7), hence the minimal pseudo-parabolic subgroup $\overline{\mathscr{P}}$ is a Borel subgroup of $\overline{\mathscr{G}}_F^\circ$. So, in this case, C is a chamber of the building $\mathcal{B}(G'/K)$. $\square$

Remark 2.5. Let A be a special k-apartment of $\mathcal{B}(G'/K)$. According to Proposition 2.4, there is a k-chamber contained in A, so among the facets of A that meet $\mathcal{B}$, the maximal ones are k-chambers (1.17).

Proposition 2.6. *Given a k-chamber C of the building $\mathcal{B}(G'/K)$ that lies in a special k-apartment A, and a point $x \in \mathcal{B} = \mathcal{B}(G'/K)^\Gamma$, there is a special k-apartment that contains C and x. Therefore, in particular, every point of $\mathcal{B}$ lies in a special k-apartment.*

Proof. Let T be the special k-torus corresponding to the apartment A. Then T contains a maximal k-split torus S of G. We fix a point y of $C \cap \mathcal{B}$, then $\mathscr{G}_y^\circ = \mathscr{G}_C^\circ$. Let $\mathscr{S}$ be the closed $\mathfrak{o}$-split torus in $\mathscr{G}_C^\circ$ with generic fiber S. Let $\overline{\mathscr{S}}$ be the special fiber of $\mathscr{S}$ and $\overline{S}$ be the image of $\overline{\mathscr{S}}$ in $\overline{G}_C^{\,\mathrm{pred}}$. As C is a k-chamber, $\overline{S}$ is central and so every maximal torus of $\overline{G}_y^{\,\mathrm{pred}} = \overline{G}_C^{\,\mathrm{pred}}$ contains it (1.18). By the uniqueness of the geodesic $[xy]$, every point on it is fixed under Γ, i.e., $[xy] \subset \mathcal{B}$. The composite κ-homomorphism

$$\overline{\pi} : \overline{\mathscr{G}}_{[xy]}^\circ \to \overline{\mathscr{G}}_y^\circ \to \overline{G}_y^{\,\mathrm{pred}} \, (= \overline{G}_C^{\,\mathrm{pred}}),$$

where the first homomorphism is the κ-homomorphism $\overline{\rho}_{\Omega',\Omega}$ of 1.18 for $\Omega = [xy]$ and $\Omega' = \{y\}$, and the second homomorphism is the natural projection, restricted to any maximal κ-torus of $\overline{\mathscr{G}}_{[xy]}^\circ$ is an isomorphism onto a maximal κ-torus of $\overline{G}_C^{\,\mathrm{pred}}$. Let $\overline{\mathscr{T}}_{[xy]}$ be a maximal κ-torus of $\overline{\mathscr{G}}_{[xy]}^\circ$. Then the maximal κ-split subtorus of $\overline{\mathscr{T}}_{[xy]}$ is isomorphic to $\overline{S}$ since the isomorphic image $\overline{\pi}(\overline{\mathscr{T}}_{[xy]})$ of $\overline{\mathscr{T}}_{[xy]}$ is a maximal κ-torus of $\overline{G}_C^{\,\mathrm{pred}}$ and so it contains $\overline{S}$.

According to Proposition 2.1(i), $\mathscr{G}_{[xy]}^\circ$ contains a closed $\mathfrak{o}$-torus $\mathscr{T}_{[xy]}$ whose special fiber (as a κ-subgroup of $\overline{\mathscr{G}}_{[xy]}^\circ$) is $\overline{\mathscr{T}}_{[xy]}$. The generic fiber $T_{[xy]}$ of $\mathscr{T}_{[xy]}$ is then a k-torus of G that splits over K and contains a maximal k-split torus of G, so it is a special k-torus. The special k-apartment of $\mathcal{B}(G'/K)$ determined by $T_{[xy]}$ contains $[xy]$ (Proposition 2.2(ii)) and hence it contains C and x. $\square$

Proposition 2.7. *Given points x, y of $\mathcal{B}$, there is a special k-apartment in $\mathcal{B}(G'/K)$ that contains both x and y. Therefore, given any two k-facets of $\mathcal{B}(G'/K)$ (which may not be different), there is a special k-apartment containing them.*

Proof. Let F be the k-facet of $\mathcal{B}(G'/K)$ that contains the point y. Let C be a k-facet that contains F in its closure and is maximal among such facets. Then C is a k-chamber. Let $z \in C \cap \mathcal{B}$. Then according to the previous proposition there is a special k-apartment which contains z, and hence also C. Now the same proposition implies that there is a special k-apartment which contains C and x. This apartment then contains $\overline{C}$, and hence also F, and so it contains y. $\square$

Proposition 2.8. *If G is anisotropic over k, then $\mathcal{B} = \mathcal{B}(G'/K)^{\Gamma}$ $(= \mathcal{B}(G/K)^{\Gamma})$ consists of a single point.*

Proof. To prove the proposition we will use Proposition 2.7. If $\mathcal{B}$ contains points $x \neq y$, then according to that proposition there is a special k-apartment A of $\mathcal{B}(G'/K)$ which contains both x and y. Let T be the special k-torus corresponding to A. Then A is the image of $V(T_K)+x\,(\subset \mathcal{B}(G/K))$ in $\mathcal{B}(G'/K)$, where $V(T_K) = \mathbb{R}\otimes_{\mathbb{Z}}\mathrm{X}_*(T_K)$ with $\mathrm{X}_*(T_K) = \mathrm{Hom}_K(\mathrm{GL}_1, T_K)$. As G is anisotropic over k, T is k-anisotropic, so $\mathrm{X}_*(T_K)^{\Gamma} = \mathrm{Hom}_k(\mathrm{GL}_1, T)$ is trivial. Hence, A^{Γ} consists of a single point x. A contradiction! □

Proposition 2.9. *Let S be a k-split torus of G and x, y be two points of $\mathcal{B}$ lying in apartments corresponding to maximal K-split tori of G_K containing S_K. Then there is a special k-torus T of G containing S such that both x and y lie in the apartment of $\mathcal{B}(G'/K)$ corresponding to T_K.*

Proof. Let $\mathscr{S}$ be the split $\mathfrak{o}$-torus with generic fiber S. Then $\mathscr{S}(\mathcal{O})$ is the maximal bounded subgroup of $S(K)$ and it fixes the geodesic $[xy]$ pointwise. Hence, $\mathscr{S}(\mathcal{O}) \hookrightarrow G(K)^{[xy]}$; this inclusion induces a $\mathfrak{o}$-group scheme homomorphism $\varphi : \mathscr{S} \to \mathscr{G}_{[xy]}$ that is the natural inclusion $S \hookrightarrow G$ on the generic fibers. We identify $\mathscr{S}$ with its image in $\mathscr{G}_{[xy]}$.

We consider the centralizer $\mathscr{M}$ of $\mathscr{S}$ in $\mathscr{G}_{[xy]}$; $\mathscr{M}$ is a smooth $\mathfrak{o}$-group scheme, see [SGA3$_{\mathrm{II}}$, Exp. XI, Cor. 5.3] or [CGP, Prop. A.8.10(2)]. As $[xy]$ lies in a special k-apartment (Proposition 2.7), we see that the dimension of every maximal κ-split torus of the special fiber $\overline{\mathscr{G}}_{[xy]}$ of $\mathscr{G}_{[xy]}$, and so also of the special fiber $\overline{\mathscr{M}}$ of $\mathscr{M}$, is equal to k-rank of G, and the dimension of every maximal torus of $\overline{\mathscr{G}}_{[xy]}$, as well as that of $\overline{\mathscr{M}}$, is equal to K-rank of G. Let $\overline{\mathscr{T}}$ be a maximal κ-torus of $\overline{\mathscr{M}}$ containing a maximal κ-split torus of the latter. Then, in view of Proposition 2.1(i), $\mathscr{M}$ contains a closed $\mathfrak{o}$-torus $\mathscr{T}$ whose special fiber is $\overline{\mathscr{T}}$. It is clear that $\mathscr{T}$ contains $\mathscr{S}$, and hence the generic fiber T of $\mathscr{T}$ contains S, and moreover, T is a special k-torus. According to Proposition 2.2(ii), the geodesic $[xy]$ lies in the apartment corresponding to T. □

Let S be a maximal k-split torus of G. Let $N(S)$ be the normalizer of S in G, and $Z(S)$ (resp. $Z'(S)$) be the centralizers of S in G (resp. G'). The polysimplicial subcomplex of $\mathcal{B}(G'/K)$ consisting of the union of apartments corresponding to maximal K-split tori of G_K that contain S_K will be denoted by $\mathcal{B}(Z'(S)/K)$ (see 3.11). It is obvious that $\mathcal{B}(Z'(S)/K)$ is stable under the action of $N(S)(K)\,(\subset G(K))$ on $\mathcal{B}(G'/K)$.

According to Proposition 2.3 there exist special k-tori of G containing S. The special k-apartments contained in $\mathcal{B}(Z'(S)/K)$ are clearly in bijective correspondence with such special k-tori. Now let T be a special k-torus of G containing S and S' (resp. T') be the maximal k-subtorus of S (resp. T) contained in the derived subgroup G' of G. The apartment A corresponding to T is stable under Γ, so it contains a point, say x, fixed under Γ. Let $V(S') = \mathbb{R}\otimes_{\mathbb{Z}}\mathrm{X}_*(S')$ and $V(T') = \mathbb{R}\otimes_{\mathbb{Z}}\mathrm{X}_*(T'_K)$. As $A = V(T') + x$, and $V(T')^{\Gamma} = V(S')$ since S' is the maximal k-split torus in

T', we infer that $A^\Gamma = V(S') + x$, so A^Γ is an affine space under $V(S')$. Now let y be an arbitrary point of $\mathcal{B}(Z'(S)/K)^\Gamma$ ($\subset \mathcal{B}$). In view of the preceding proposition, there exists a special k-apartment A' ($\subset \mathcal{B}(Z'(S)/K)$) that contains both x and y. Now working with A' in place of A, we see that ${A'}^\Gamma = V(S') + x$, and hence ${A'}^\Gamma = A^\Gamma$, in particular, A^Γ contains y. Therefore, $\mathcal{B}(Z'(S)/K)^\Gamma = A^\Gamma$ for any k-special apartment A contained in $\mathcal{B}(Z'(S)/K)$. We state this as the following proposition.

Proposition 2.10. *For every special k-apartment A contained in $\mathcal{B}(Z'(S)/K)$, we have $\mathcal{B}(Z'(S)/K)^\Gamma = A^\Gamma$.*

(Note that A^Γ is an affine space under $V(S')$.)

2.11. Let $N(S)$ be the normalizer of S in G. Under the natural action of $N(S)(k)$ ($\subset N(S)(K)$) on $\mathcal{B}(Z'(S)/K)$, $\mathcal{B}(Z'(S)/K)^\Gamma$ is stable. For $n \in N(S)(K)$, the action of n carries an apartment $A \subset \mathcal{B}(Z'(S)/K)$ to the apartment $n \cdot A$ by an affine transformation.

Let A be a spacial k-apartment contained in $\mathcal{B}(Z'(S)/K)$ and T ($\supset S$) be the corresponding special k-torus. Let S' be the unique maximal k-split torus of G' contained in S. It follows from the previous proposition that $\mathcal{B}(Z'(S)/K)^\Gamma = A^\Gamma$. So we can view $\mathcal{B}(Z'(S)/K)^\Gamma$ as an affine space under $V(S') = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S')$. We will now show, using the proof of the lemma in 1.6 of [PY1], that $\mathcal{B}(Z'(S)/K)^\Gamma$ has the properties required of an apartment corresponding to the maximal k-split torus S in the Bruhat-Tits building of $G(k)$ if such a building exists. We need to check the following three conditions.

A1: *The action of $N(S)(k)$ on $\mathcal{B}(Z'(S)/K)^\Gamma = A^\Gamma$ is by affine transformations and the maximal bounded subgroup $Z(S)(k)_b$ of $Z(S)(k)$ acts trivially.*

Let $\mathrm{Aff}(A^\Gamma)$ be the group of affine automorphisms of A^Γ and $f : N(S)(k) \to \mathrm{Aff}(A^\Gamma)$ be the action map.

A2: *The group $Z(S)(k)$ acts on $\mathcal{B}(Z'(S)/K)^\Gamma$ by translations, and the action is characterized by the following formula: for $z \in Z(S)(k)$,*

$$\chi(f(z)) = -\omega(\chi(z)) \text{ for all } \chi \in \mathrm{X}_k^*(Z(S)) \,(\hookrightarrow \mathrm{X}_k^*(S)),$$

here we regard the translation $f(z)$ as an element of $V(S')$ ($\hookrightarrow V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$).

A3: *For $g \in \mathrm{Aff}(A^\Gamma)$, denote by $dg \in \mathrm{GL}(V(S'))$ the derivative of g. Then the map $N(S)(k) \to \mathrm{GL}(V(S'))$, $n \mapsto df(n)$, is induced from the action of $N(S)(k)$ on $\mathrm{X}_*(S')$ (i.e., it is the Weyl group action).*

Moreover, as G' is semi-simple, these three conditions determine the affine structure on $\mathcal{B}(Z'(S)/K)^\Gamma$, see [T2, 1.2].

Proposition 2.12. *Conditions* A1, A2 *and* A3 *hold.*

Proof. The action of $n \in N(S)(k)$ on $\mathcal{B}(G'/K)$ carries the special k-apartment $A = A_T$ via an affine isomorphism $\varphi(n) : A \to A_{nTn^{-1}}$ to the special k-apartment $A_{nTn^{-1}}$ corresponding to the special k-torus nTn^{-1} containing S. As $(A_{nTn^{-1}})^\Gamma =$

$\mathcal{B}(Z'(S)/K)^{\Gamma} = A^{\Gamma}$, we see that $\varphi(n)$ keeps A^{Γ} stable and so $f(n) = \varphi(n)|_{A^{\Gamma}}$ is an affine automorphism of A^{Γ}.

Let $V(T_K) = \mathbb{R}\otimes_{\mathbb{Z}} \mathrm{Hom}_K(\mathrm{GL}_1, T_K)$ and $V(nT_Kn^{-1}) = \mathbb{R}\otimes_{\mathbb{Z}} \mathrm{Hom}_K(\mathrm{GL}_1, nT_Kn^{-1})$. The derivative $d\varphi(n) : V(T_K) \to V(nT_Kn^{-1})$ is induced from the map

$$\mathrm{Hom}_K(\mathrm{GL}_1, T_K) = \mathrm{X}_*(T_K) \to \mathrm{X}_*(nT_Kn^{-1}) = \mathrm{Hom}_K(\mathrm{GL}_1, nT_Kn^{-1}),$$

$\lambda \mapsto \mathrm{Int}\, n \cdot \lambda$, where $\mathrm{Int}\, n$ is the inner automorphism of G determined by n. So, the restriction $df(n) : V(S') \to V(S')$ is induced from the homomorphism $\mathrm{X}_*(S') \to \mathrm{X}_*(S')$, $\lambda \mapsto \mathrm{Int}\, n \cdot \lambda$. This proves A3.

Condition A3 implies that df is trivial on $Z(S)(k)$. Therefore, $Z(S)(k)$ acts on $\mathcal{B}(Z'(S)/K)^{\Gamma} = A^{\Gamma}$ by translations. The action of the bounded subgroup $Z(S)(k)_b$ on A^{Γ} admits a fixed point; see Proposition 1.13 and the observation following its statement. Hence, $Z(S)(k)_b$ acts by the trivial translation. This proves A1.

Since the image of $S(k)$ in $Z(S)(k)/Z(S)(k)_b \simeq \mathbb{Z}^r$ is a subgroup of finite index, to prove the formula in A2, it suffices to prove it for $z \in S(k)$. But for $z \in S(k)$, $zTz^{-1} = T$, and $f(z)$ is a translation of the apartment A ($f(z)$ is regarded as an element of $V(T_K)$) which satisfies (see 1.9):

$$\chi(f(z)) = -\omega(\chi(z)) \quad \text{for all } \chi \in \mathrm{X}^*_K(T_K).$$

This implies the formula in A2, since the restriction map $\mathrm{X}^*_K(T_K) \to \mathrm{X}^*_K(S_K)\,(= \mathrm{X}^*_k(S))$ is surjective and the image of the restriction map $\mathrm{X}^*_k(Z(S)) \to \mathrm{X}^*_k(S)$ is of finite index in $\mathrm{X}^*_k(S)$. □

2.13. By definition, the *apartments of* $\mathcal{B}$ are A^{Γ}, for special k-apartments A of $\mathcal{B}(G'/K)$. Let T be a special k-torus of G, S the maximal k-split torus of G contained in T and S' be the maximal k-split torus of G' contained in S. Let A be the apartment of $\mathcal{B}(G'/K)$ corresponding to T. Then (Proposition 2.10) $A^{\Gamma} = \mathcal{B}(Z'(S)/K)^{\Gamma}$, thus the apartment A^{Γ} of $\mathcal{B}$ is uniquely determined by S, and it is an affine space under $V(S') = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S')$. As maximal k-split tori of G are conjugate to each other under $G(k)$, we conclude that $G(k)$ acts transitively on the set of apartments of $\mathcal{B}$.

3. Main results

We will use the notations introduced in §§1, 2. Thus G will denote a connected reductive k-group, G' its derived group. We assume that for G, and hence also for G', Bruhat-Tits theory is available over K. As before, $\mathcal{B}(G'/K)$ will denote the Bruhat-Tits building of $G(K)$; $\mathcal{B}(G'/K)$ is also the Bruhat-Tits building of $G'(K)$. In this section we will prove that $\mathcal{B} = \mathcal{B}(G'/K)^{\Gamma}$ is an affine building. For this purpose, *after replacing G with G', we will assume in 3.1–3.11 that G is semi-simple.*

Theorem 3.1. *Let A_1 and A_2 be special k-apartments of $\mathcal{B}(G/K)$; T_1, T_2 be the corresponding special k-tori. Let S be a k-split torus contained in $T_1 \cap T_2$. Let Ω be a nonempty Γ-stable bounded subset of $A_1 \cap A_2$ and $\mathscr{G}^{\circ}_{\Omega}$ be the smooth affine $\mathfrak{o}$-group* *scheme associated to Ω in 1.14. Then there is an element $g \in \mathscr{G}_\Omega^\circ(\mathfrak{o})\,(\subset G(k))$ that commutes with S and carries ${A_1}^\Gamma$ onto ${A_2}^\Gamma$.*

If the residue field κ of k is perfect and of dimension $\leqslant 1$, then there exists an element $g \in \mathscr{G}_\Omega^\circ(\mathfrak{o})\,(\subset G(k))$ that commutes with S and conjugates T_1 onto T_2, hence it carries the apartment A_1 onto the apartment A_2.

As g belongs to $\mathscr{G}_\Omega^\circ(\mathfrak{o})$, it fixes Ω pointwise.

Proof. Let S_1 and S_2 be the maximal k-split tori of G contained in T_1 and T_2 respectively. Let $\mathscr{G} := \mathscr{G}_\Omega^\circ$ and $\mathscr{T}_1$ and $\mathscr{T}_2$ be the closed $\mathfrak{o}$-tori of $\mathscr{G}$ with generic fibers T_1 and T_2 respectively (see 1.14). Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be the maximal $\mathfrak{o}$-split subtori of $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. Then the generic fibers of $\mathscr{S}_1$ and $\mathscr{S}_2$ are S_1 and S_2 respectively. Let $\mathscr{S}$ be the closed $\mathfrak{o}$-torus contained in $\mathscr{S}_1 \cap \mathscr{S}_2$ whose generic fiber is S. Let $\mathscr{M}$ be the centralizer of $\mathscr{S}$ in $\mathscr{G}$; $\mathscr{M}$ is a smooth affine $\mathfrak{o}$-subgroup scheme ([SGA3$_{\rm II}$, Exp. XI, Cor. 5.3] or [CGP, Prop. A.8.10(2)]) and its fibers are connected since the centralizer of a torus in a connected smooth affine algebraic group is connected [Bo, Cor. 11.12]. Using Proposition 2.1(i) for $\mathscr{M}$ in place of $\mathscr{G}$, and the remark following that proposition, we see that the special fibers $\overline{\mathscr{S}}_1$ and $\overline{\mathscr{S}}_2$ of $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively are maximal κ-split tori in the special fiber $\overline{\mathscr{M}}$ of $\mathscr{M}$. Hence there exists an element $\overline{g} \in \overline{\mathscr{M}}(\kappa)$ that conjugates $\overline{\mathscr{S}}_1$ onto $\overline{\mathscr{S}}_2$ [CGP, Thm. C.2.3]. By Proposition 2.1(ii), there exists an element $g \in \mathscr{M}(\mathfrak{o})\,(\subset G(k))$ lying over $\overline{g}$ that conjugates $\mathscr{S}_1$ onto $\mathscr{S}_2$. As $g\mathscr{S}_1 g^{-1} = \mathscr{S}_2$, we infer that $gS_1g^{-1} = S_2$, so

$$g \cdot {A_1}^\Gamma = g \cdot \mathcal{B}(Z(S_1)/K)^\Gamma = \mathcal{B}(Z(S_2)/K)^\Gamma = {A_2}^\Gamma,$$

and g fixes Ω pointwise.

To prove the second assertion of the theorem, let $\overline{\mathscr{T}}_1$ and $\overline{\mathscr{T}}_2$ be the special fibers of $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. Both of them are maximal κ-tori of $\overline{\mathscr{M}}$. Now let us assume that κ is perfect and of dimension $\leqslant 1$. Then the reductive κ-group $\overline{\mathscr{M}}^{\rm red} := \overline{\mathscr{M}}/\mathscr{R}_{u,\kappa}(\overline{\mathscr{M}})$ is quasi-split (1.7) and hence any maximal κ-split torus of $\overline{\mathscr{M}}^{\rm red}$ is contained in a unique maximal torus. Therefore, as the element $\overline{g} \in \overline{\mathscr{M}}(\kappa)$ chosen in the preceding paragraph conjugates $\overline{\mathscr{S}}_1$ onto $\overline{\mathscr{S}}_2$, it conjugates $\overline{\mathscr{T}}_1$ onto a maximal κ-torus of the solvable κ-subgroup $\overline{\mathscr{H}} := \overline{\mathscr{T}}_2 \cdot \mathscr{R}_{u,\kappa}(\overline{\mathscr{M}})$. Since any two maximal κ-tori of the solvable κ-group $\overline{\mathscr{H}}$ are conjugate to each other under an element of $\overline{\mathscr{H}}(\kappa)$ [Bo, Thm. 19.2], we conclude that $\overline{\mathscr{T}}_2$ is conjugate to $\overline{\mathscr{T}}_1$ under an element of $\overline{\mathscr{M}}(\kappa)$. Now Proposition 2.1(ii) implies that there is an element $g \in \mathscr{M}(\mathfrak{o})\,(\subset (G(k))$ that conjugates $\mathscr{T}_1$ onto $\mathscr{T}_2$, so $gT_1g^{-1} = T_2$, and hence g carries A_1 onto A_2 fixing Ω pointwise. $\square$

3.2. Polysimplicial structure on $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$. The *facets* (resp. *chambers*) of $\mathcal{B}$ are by definition the subsets $\mathcal{F} := F \cap \mathcal{B}$ (resp. $\mathcal{C} := C \cap \mathcal{B}$) for k-facets F (resp. k-chambers C) of $\mathcal{B}(G/K)$. As the subset of points of $\mathcal{B}(G/K)$ fixed under $\mathscr{G}_F^\circ(\mathcal{O}) = \mathscr{G}_{\mathcal{F}}^\circ(\mathcal{O})$ is $\overline{F}$ (1.9), the subset of points of $\mathcal{B}$ fixed under $\mathscr{G}_{\mathcal{F}}^\circ(\mathcal{O})$ is $\overline{F} \cap \mathcal{B} = \overline{\mathcal{F}}$.

Let F be a minimal k-facet in $\mathcal{B}(G/K)$ and A be a special k-apartment containing F (Proposition 2.6). We will presently show that F contains a unique point fixed under Γ (i.e., F meets $\mathcal{B}$ in a single point). Every special k-apartment is stable under the action of the Galois group Γ which acts on it by affine automorphisms. Now if x and y are two distinct points in $F \cap \mathcal{B}$, then the whole straight line in the apartment A passing through x and y is pointwise fixed under Γ. This line must meet the boundary of F, contradicting the minimality of F. By definition, a *vertex* of $\mathcal{B}$ is the unique point of $F \cap \mathcal{B}$ for any minimal k-facet F in $\mathcal{B}(G/K)$.

Let F be a k-facet in $\mathcal{B}(G/K)$ (F is not assumed to be minimal) and $\mathscr{V}_F$ be the set of vertices of $\mathcal{B}$ contained in $\overline{F}$. For $v \in \mathscr{V}_F$, let F_v be the face of F which contains v. Since v is a vertex of $\mathcal{B}$, F_v is a minimal k-facet. Now if x and y are two distinct vertices in $\mathscr{V}_F$, then $\overline{F}_x \cap \overline{F}_y$ is empty. For, this intersection is convex and stable under Γ and hence if it is nonempty, it would contain a Γ-fixed point (i.e., a point of $\mathcal{B}$). This would contradict the minimality of k-facets F_x and F_y. Thus the sets of vertices (we call them K-vertices) of the facets F_x and F_y are disjoint, and each one of these sets is Γ-stable. The union of the sets of K-vertices of F_v, for $v \in \mathscr{V}_F$, is the set of K-vertices of F. To see this, we observe that any K-vertex of F is a K-vertex of a face of F which is a minimal k-facet and so it contains a (unique) point of $\mathscr{V}_F$. Arguing by induction on dimension of F, we easily see that $\overline{F} \cap \mathcal{B}$ is the convex hull of the set $\mathscr{V}_F$ of vertices of $\mathcal{B}$ contained in $\overline{F}$. The points of $\mathscr{V}_F$ are by definition the vertices of the facet $\mathcal{F} := F \cap \mathcal{B}$ of $\mathcal{B}$.

Given a k-facet F of $\mathcal{B}(G/K)$, using the description of pseudo-parabolic κ-subgroups of $\overline{G}_F^{\rm pred}$ up to conjugacy, we see (1.18) that κ-rank of the derived subgroup of $\overline{G}_F^{\rm pred}$ is equal to the codimension of $\mathcal{F} := F \cap \mathcal{B}$ in $\mathcal{B}$.

Let F be a k-facet of $\mathcal{B}(G/K)$, and $\mathcal{F} = F \cap \mathcal{B}$ be the corresponding facet of $\mathcal{B}$. Then, for $g \in G(k)$, $g \cdot F$ is also a k-facet and $g \cdot \mathcal{F} = g \cdot (F \cap \mathcal{B}) = (g \cdot F) \cap \mathcal{B}$ is the facet of $\mathcal{B}$ corresponding to $g \cdot F$. Thus the action of $G(k)$ on $\mathcal{B}$ is by polysimplicial automorphisms.

We assume in this paragraph that G is absolutely almost simple. Then the Bruhat-Tits building $\mathcal{B}(G/K)$ is a simplicial complex, and in this case $\mathcal{B}$ is also a simplicial complex with simplices $\mathcal{F} := F \cap \mathcal{B}$, for k-facets F of $\mathcal{B}(G/K)$ (F is a simplex!). To see this, note that given a nonempty subset $\mathscr{V}'$ of $\mathscr{V}_F$, the k-facet F' whose set of K-vertices is the union of the set of K-vertices of F_v, for $v \in \mathscr{V}'$, is a face of F, so $\mathcal{F}' := F' \cap \mathcal{B}$ is a face of $\mathcal{F}$ and its set of vertices is $\mathscr{V}'$.

3.3. If G is simply connected and quasi-split over K, then for any k-facet F, the stabilizer of the facet $\mathcal{F} = F \cap \mathcal{B}$ of $\mathcal{B}$ in $G(k)$ (resp. $G(K)$) is $\mathscr{G}_F^\circ(\mathfrak{o})$ (resp. $\mathscr{G}_F^\circ(\mathcal{O})$), hence the stabilizer of $\mathcal{F}$ fixes both F and $\mathcal{F}$ pointwise. This follows from the fact that the stabilizer of $\mathcal{F}$ also stabilizes F since F is the unique facet of $\mathcal{B}(G/K)$ containing $\mathcal{F}$. But, in case G is simply connected and quasi-split over K, the stabilizer of F in $G(K)$ is the subgroup $\mathscr{G}_F^\circ(\mathcal{O})$ $(= \mathscr{G}_{\mathcal{F}}^\circ(\mathcal{O}) \subset G(K))$ (1.9) and this subgroup fixes F pointwise.

Proposition 3.4. *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$. Then there is a unique maximal k-split torus S of G such that $\mathcal{A} = \mathcal{B}(Z(S)/K)^{\Gamma}$. So the stabilizer of $\mathcal{A}$ in $G(k)$ is $N(S)(k)$.*

Proof. We fix a maximal k-split torus S of G such that $\mathcal{A} = \mathcal{B}(Z(S)/K)^{\Gamma}$. We will show that S is uniquely determined by $\mathcal{A}$. For this purpose, we observe that as $N(S)(k)$ acts on $\mathcal{A}$ and the maximal bounded subgroup $Z(S)(k)_b$ of $Z(S)(k)$ acts trivially (Proposition 2.12), the subgroup $\mathcal{Z}$ of $G(k)$ consisting of elements that fix $\mathcal{A}$ pointwise is a bounded subgroup of $G(k)$ that is normalized by $N(S)(k)$ and contains $Z(S)(k)_b$. Using the Bruhat decomposition of $G(k)$ with respect to S, we see that every bounded subgroup of $G(k)$ that is normalized by $N(S)(k)$ is a normal subgroup of the latter. So the identity component of the Zariski-closure of $\mathcal{Z}$ is $Z(S)$. As S is the unique maximal k-split torus of G contained in $Z(S)$, our assertion follows. □

Proposition 3.5. *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$, and $\mathcal{C}$, $\mathcal{C}'$ two chambers in $\mathcal{A}$. Then there is a gallery joining $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{A}$, i.e., there is a finite sequence*

$$\mathcal{C} = \mathcal{C}_0,\ \mathcal{C}_1,\ \dots\ ,\ \mathcal{C}_m = \mathcal{C}'$$

of chambers in $\mathcal{A}$ such that for i with $1 \leqslant i \leqslant m$, $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ share a face of codimension 1.

Proof. Let $\mathcal{A}_2$ be the codimension 2-skelton of $\mathcal{A}$, i.e., the union of all facets in $\mathcal{A}$ of codimension at least 2. Then $\mathcal{A}_2$ is a closed subset of the affine space $\mathcal{A}$ of codimension 2, so $\mathcal{A} - \mathcal{A}_2$ is arcwise connected. This implies that given points $x \in \mathcal{C}$ and $x' \in \mathcal{C}'$, there is a piecewise linear curve in $\mathcal{A} - \mathcal{A}_2$ joining x and x'. Now the chambers in $\mathcal{A}$ that meet this curve make a gallery joining $\mathcal{C}$ to $\mathcal{C}'$. □

The dimension of any apartment, so of any chamber, in $\mathcal{B}$ is equal to the k-rank of $G\,(= (G, G))$. A *panel* in $\mathcal{B}$ is by definition a facet of codimension 1.

Proposition 3.6. *Let $\mathcal{A}$ be an apartment of $\mathcal{B}$ and S be the maximal k-split torus of G corresponding to this apartment.* (*Then $\mathcal{A} = (\mathcal{B}(Z(S)/K)^{\Gamma}$.*) *The group $N(S)(k)$ acts transitively on the set of chambers of $\mathcal{A}$.*

Proof. According to the previous proposition, given any two chambers in $\mathcal{A}$, there exists a minimal gallery in $\mathcal{A}$ joining these two chambers. So to prove the proposition by induction on the length of a minimal gallery joining two chambers, it suffices to prove that given two different chambers $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{A}$ which share a panel $\mathcal{F}$, there is an element $n \in N(S)(k)$ such that $n \cdot \mathcal{C} = \mathcal{C}'$. Let $\mathscr{G} := \mathscr{G}_{\mathcal{F}}^{\circ}$ be the smooth $\mathfrak{o}$-group scheme associated with the panel $\mathcal{F}$ and $\mathscr{S} \subset \mathscr{G}$ be the closed $\mathfrak{o}$-split torus with generic fiber S. Let $\overline{\mathscr{G}}$ be the special fiber of $\mathscr{G}$, $\overline{\mathscr{S}}$ the special fiber of $\mathscr{S}$. Then $\overline{\mathscr{S}}$ is a maximal κ-split torus of $\overline{\mathscr{G}}$. The chambers $\mathcal{C}$ and $\mathcal{C}'$ correspond to minimal pseudo-parabolic κ-subgroups $\overline{\mathscr{P}}$ and $\overline{\mathscr{P}}'$ of $\overline{\mathscr{G}}$, see 1.18. Both of these minimal pseudo-parabolic κ-subgroups contain $\overline{\mathscr{S}}$ since the chambers $\mathcal{C}$ and $\mathcal{C}'$ lie on $\mathcal{A}$. But then by Theorems C.2.5 and C.2.3 of [CGP], there is an element $\overline{n} \in \overline{\mathscr{G}}(\kappa)$ which normalizes $\overline{\mathscr{S}}$ and conjugates $\overline{\mathscr{P}}$ onto $\overline{\mathscr{P}}'$. Now from Proposition 2.1(iii) we conclude that there is an element $n \in N_{\mathscr{G}}(\mathscr{S})(\mathfrak{o})$ lying over $\overline{n}$. It is clear that n normalizes S and hence it lies in $N(S)(k)$; it fixes $\mathcal{F}$ pointwise and $n \cdot \mathcal{C} = \mathcal{C}'$. □

Proposition 3.7. *$\mathcal{B}$ is thick, that is any panel is a face of at least three chambers, and every apartment of $\mathcal{B}$ is thin, that is any panel lying in an apartment is a face of exactly two chambers of the apartment.*

Proof. Let F be a k-facet of $\mathcal{B}(G/K)$ that is not a k-chamber, and C be a k-chamber of which F is a face. Then there is an $\mathfrak{o}$-group scheme homomorphism $\mathscr{G}_C^\circ \to \mathscr{G}_F^\circ$. The image of $\overline{\mathscr{G}}_C^\circ$ in $\overline{\mathscr{G}}_F^\circ$, under the induced homomorphism of special fibers, is a minimal pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}_F^\circ$, and conversely, any minimal pseudo-parabolic κ-subgroup of the latter determines a k-chamber with F as a face. Now if κ is infinite, $\overline{\mathscr{G}}_F^\circ$ clearly contains infinitely many minimal pseudo-parabolic κ-subgroups. On the other hand, if κ is a finite field, then pseudo-parabolic κ-subgroups of $\overline{\mathscr{G}}_F^\circ$ are parabolic and as any nontrivial irreducible projective κ-variety has at least three κ-rational points, we see that F is a face of at least three distinct k-chambers.

To prove the second assertion, let $\mathcal{F} := F^\Gamma$ be a panel in an apartment $\mathcal{A}$ of $\mathcal{B}$, where F is a k-facet in $\mathcal{B}(G/K)$. Let S be the maximal k-split torus of G corresponding to $\mathcal{A}$. Let $\mathscr{G}_F^\circ$ be the smooth affine $\mathfrak{o}$-group scheme associated with F in 1.14 and $\overline{G}_F^{\mathrm{pred}}$ be the maximal pseudo-reductive quotient of the special fiber of this group scheme. Let $\mathscr{S}$ be the closed $\mathfrak{o}$-split torus of $\mathscr{G}_F^\circ$ with generic fiber S. Then the chambers of $\mathcal{B}$ lying in $\mathcal{A}$, with F as a face, are in bijective correspondence with minimal pseudo-parabolic κ-subgroups of $\overline{G}_F^{\mathrm{pred}}$ that contain the image $\overline{S}$ of the special fiber of $\mathscr{S}$ (1.18). The κ-rank of the derived subgroup of $\overline{G}_F^{\mathrm{pred}}$ is 1 since $\mathcal{F}$ is of codimension 1 in $\mathcal{B}$ (3.2). This implies that $\overline{G}_F^{\mathrm{pred}}$ has exactly two minimal pseudo-parabolic κ-subgroups containing $\overline{S}$.

The second assertion also follows at once from the following well-known result in algebraic topology: In any simplicial complex whose geometric realization is a topological manifold without boundary (such as an apartment $\mathcal{A}$ in $\mathcal{B}$), any simplex of codimension 1 is a face of exactly two chambers (i.e., maximal dimensional simplices). □

In 3.2 we saw that $\mathcal{B}$ carries a natural polysimplicial structure with facets $\mathcal{F} := F \cap \mathcal{B}$, for k-facets F of $\mathcal{B}(G/K)$. Propositions 2.7, 3.5, 3.7 and Theorem 3.1 show that the conditions, recalled in 1.8, in the definition of buildings hold for $\mathcal{B}$ if $\mathcal{B}(Z(S)/K)^\Gamma$ $(= \mathcal{B}(Z(S)/K) \cap \mathcal{B})$, for maximal k-split tori S of G, are taken to be its apartments. Thus we obtain the following:

Theorem 3.8. *$\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ is a building: Its apartments are the affine spaces $\mathcal{B}(Z(S)/K)^\Gamma$ under $V(S) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$, for maximal k-split tori S of G. Its chambers are $\mathcal{C} := C \cap \mathcal{B}$ for k-chambers C of $\mathcal{B}(G/K)$, and its facets are $\mathcal{F} := F \cap \mathcal{B}$ for k-facets F of $\mathcal{B}(G/K)$. The group $G(k)$ acts on $\mathcal{B}$ by polysimplicial isometries.*

Definition 3.9. $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ is called the *Bruhat-Tits building of $G(k)$*.

Since $G(k)$ acts transitively on the set of maximal k-split tori of G, it acts transitively on the set of apartments of $\mathcal{B}$ (cf. 2.13). Now Proposition 3.6 implies the following:

Proposition 3.10. *$G(k)$ acts transitively on the set of ordered pairs $(\mathcal{A}, \mathcal{C})$ consisting of an apartment $\mathcal{A}$ of $\mathcal{B}$ and a chamber $\mathcal{C}$ lying in the apartment $\mathcal{A}$.*

3.11. Bruhat-Tits theory for Levi subgroups of G. Let $S\,(\subset G)$ be a k-torus that splits over K and $M := Z(S)$ be the centralizer of S in G. Then M is a connected reductive k-subgroup and Bruhat-Tits theory is available for M, as well as for its derived subgroup $\mathscr{D}(M) = (M, M)$, over K. In fact, the enlarged Bruhat-Tits building $\mathcal{B}(M/K)$ of $M(K)$ is the union of apartments of $\mathcal{B}(G/K)$ corresponding to maximal K-split tori of G_K containing S_K. Proposition 2.9 and Theorem 3.1 (with K in place of k) verify conditions (B2) and (B3) of 1.8 for $\mathcal{B}(M/K)$.

Let $\mathcal{S}$ be the maximal k-torus contained in the center of M that splits over K, and let $V(\mathcal{S}_K) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathcal{S}_K)$. Then $V(\mathcal{S}_K)$ operates on each apartment of the enlarged building $\mathcal{B}(M/K)$ by translations. The quotient of $\mathcal{B}(M/K)$ by $V(\mathcal{S}_K)$ is the Bruhat-Tits building of $M(K)$, as well as that of $\mathscr{D}(M)(K)$. Its apartments are the quotients of the apartments of $\mathcal{B}(M/K)$ by $V(\mathcal{S}_K)$.

Let T be a maximal K-split torus of G_K containing S_K, and Ω be a nonempty bounded subset of the apartment of $\mathcal{B}(G/K)$ corresponding to T. Let $\mathscr{G}_\Omega^\circ$ be the Bruhat-Tits group scheme with generic fiber G, and connected special fiber, associated to Ω in 1.9. Let $\mathscr{S}$ be the schematic closure of $\mathcal{S}_K$ in $\mathscr{G}_\Omega^\circ$. Then $\mathscr{S}$ is a $\mathcal{O}$-torus of $\mathscr{G}_\Omega^\circ$ with generic fiber $\mathcal{S}_K$. The Bruhat-Tits smooth affine $\mathcal{O}$-group scheme, with generic fiber M_K, associated to Ω, is the centralizer $\mathscr{M}_\Omega^\circ$ of $\mathscr{S}$ in $\mathscr{G}_\Omega^\circ$ (smoothness of such centralizers is known; see [SGA3$_{\mathrm{II}}$, Exp. XI, Cor. 5.3] or [CGP, Prop. A.8.10(2)]). The special fiber of $\mathscr{M}_\Omega^\circ$ is connected since the centralizer of a torus in a smooth connected affine algebraic group is connected [Bo, Cor. 11.12].

Now we see from our earlier results that Bruhat-Tits theory is available for M over k, and the enlarged Bruhat-Tits building of $M(k)$ is $\mathcal{B}(M/K)^\Gamma$.

3.12. Parahoric subgroups of $G(k)$. For $x \in \mathcal{B}$, the $\mathfrak{o}$-group scheme $\mathscr{G}_x^\circ$ with connected fibers, described in 1.14, is by definition the Bruhat-Tits parahoric $\mathfrak{o}$-group scheme, and $\mathscr{G}_x^\circ(\mathfrak{o})$ is the parahoric subgroup of $G(k)$ associated to the point x. If $\mathcal{F}$ is the facet of $\mathcal{B}$ containing x and F is the k-facet of $\mathcal{B}(G'/K)$ containing $\mathcal{F}$, then $\mathscr{G}_x^\circ = \mathscr{G}_{\mathcal{F}}^\circ = \mathscr{G}_F^\circ$. The generic fiber of $\mathscr{G}_x^\circ$ is G, and the subgroup $\mathscr{G}_x^\circ(\mathfrak{o}) = \mathscr{G}_F^\circ(\mathcal{O})^\Gamma$ $(= \mathscr{G}_F^\circ(\mathcal{O}) \cap G(k))$ of $G(k)$ fixes F pointwise. Since F is the unique facet of $\mathcal{B}(G'/K)$ containing $\mathcal{F}$, the stabilizer of $\mathcal{F}$ also stabilizes F. But $\mathscr{G}_F^\circ(\mathcal{O})$ is of finite index in the stabilizer of F in $G(K)$. Therefore, $\mathscr{G}_x^\circ(\mathfrak{o}) = \mathscr{G}_{\mathcal{F}}^\circ(\mathfrak{o})$ is of finite index in the stabilizer of $\mathcal{F}$ in $G(k)$. For a k-chamber C of $\mathcal{B}(G'/K)$, let $\mathcal{C} = C \cap \mathcal{B}$ denote the corresponding chamber of $\mathcal{B}$. The subgroup $\mathscr{G}_{\mathcal{C}}^\circ(\mathfrak{o})$ is then a minimal parahoric subgroup of $G(k)$, and all minimal parahoric subgroups of $G(k)$ arise this way.

Let P be a parahoric subgroup of $G(K)$ which is stable under the action of Γ on $G(K)$, then the facet F of $\mathcal{B}(G'/K)$ corresponding to P is Γ-stable, i.e., it is a

k-facet. Let $\mathcal{F} = F \cap \mathcal{B}$ be the corresponding facet of $\mathcal{B}$, and $\mathscr{G}_{\mathcal{F}}^{\circ}$ be the associated $\mathfrak{o}$-group scheme with generic fiber G and with connected special fiber. Then $\mathscr{G}_{\mathcal{F}}^{\circ}(\mathfrak{o}) = \mathscr{G}_{F}^{\circ}(\mathcal{O})^{\Gamma} = P^{\Gamma}$ is a parahoric subgroup of $G(k)$. Thus the parahoric subgroups of $G(k)$ are the subgroups of the form P^{Γ}, for Γ-stable parahoric subgroups P of $G(K)$.

Proposition 3.13. *The minimal parahoric subgroups of $G(k)$ are conjugate to each other under $G'(k)$.*

Proof. The minimal parahoric subgroups of $G(k)$ are the subgroups $\mathscr{G}_{\mathcal{C}}^{\circ}(\mathfrak{o})$ for chambers $\mathcal{C}$ in the building $\mathcal{B}$. Proposition 3.10 implies that $G'(k)$ acts transitively on the set of chambers of $\mathcal{B}$. □

3.14. We say that G is *residually quasi-split* if every k-chamber in $\mathcal{B}(G'/K)$ is actually a chamber, or, equivalently, if for any k-chamber C, the special fiber of the $\mathfrak{o}$-group scheme $\mathscr{G}_{C}^{\circ}$ is solvable. If the residue field κ of k is perfect and of dimension $\leqslant 1$, then every semi-simple k-group is quasi-split over K (1.7) and by Proposition 2.4, it is residually quasi-split. For residually quasi-split G, the minimal parahoric subgroups of $G(k)$ are called the *Iwahori subgroups* of $G(k)$. They are of the form I^{Γ} for Γ-stable Iwahori subgroups I of $G(K)$.

3.15. Assume that G is semi-simple simply connected and quasi-split over K. Let $\mathcal{F}$ be a facet of $\mathcal{B}$ and F the k-facet of $\mathcal{B}(G/K)$ containing $\mathcal{F}$. Then the stabilizer of $\mathcal{F}$ in $G(K)$ is $\mathscr{G}_{\mathcal{F}}^{\circ}(\mathcal{O}) = \mathscr{G}_{F}^{\circ}(\mathcal{O})$, so the stabilizer of $\mathcal{F}$ in $G(k)$ is $\mathscr{G}_{\mathcal{F}}^{\circ}(\mathfrak{o})$ and $\mathscr{G}_{\mathcal{F}}^{\circ} = \mathscr{G}_{\mathcal{F}} = \mathscr{G}_{F}$, hence the stabilizer of $\mathcal{F}$ in $G(k)$ fixes $\mathcal{F}$ and F pointwise (cf. 1.9). The normalizer of a parahoric subgroup P of $G(k)$ is P itself, for if P is the stabilizer of the facet $\mathcal{F}$ of $\mathcal{B}$, then the normalizer of P also stabilizes $\mathcal{F}$, and hence it coincides with P.

3.16. Tits systems in suitable subgroups of $G(k)$ provided by the building. We assume in this paragraph that G is semi-simple. Let $\mathcal{G}$ be a subgroup of $G(k)$ that acts on $\mathcal{B}$ by type-preserving automorphisms and acts transitively on the set of ordered pairs consisting of an apartment of $\mathcal{B}$ and a chamber lying in the apartment. We fix an apartment $\mathcal{A}$ of $\mathcal{B}$ and a chamber $\mathcal{C}$ lying in $\mathcal{A}$. Let S be the maximal k-split torus of G corresponding to $\mathcal{A}$ and $N(S)$ be the normalizer of S in G. Let B and N respectively be the stabilizer of $\mathcal{C}$ and $\mathcal{A}$ in $\mathcal{G}$. Then according to [T1, Prop. 3.11], (B, N) is a saturated Tits system in $\mathcal{G}$, and $\mathcal{B}$ is the Tits building corresponding to this Tits system. Note that $\mathcal{G} \cap \mathscr{G}_{\mathcal{C}}^{\circ}(\mathfrak{o})$ is a subgroup of B of finite index, and $N = \mathcal{G} \cap N(S)(k)$ since the stabilizer of $\mathcal{A}$ in $G(k)$ is $N(S)(k)$ by Proposition 3.4.

For example, if G is quasi-split over K, and $\pi : \widehat{G} \to G$ is the simply connected central cover of G, then $\widehat{G}(k)$ acts on $\mathcal{B}$ by type-preserving automorphisms (1.9), and according to Proposition 3.10, this group acts transitively on the set of ordered pairs consisting of an apartment of $\mathcal{B}$ and a chamber lying in the apartment. Therefore, $\mathcal{G} := \pi(\widehat{G}(k))$ has the properties required in the preceding paragraph. Let $\mathcal{A}$ and $\mathcal{C}$ be as in the preceding paragraph. Then the stabilizer of $\mathcal{C}$ in $\mathcal{G}$ is $\mathcal{G} \cap \mathscr{G}_{\mathcal{C}}^{\circ}(\mathfrak{o})$ (cf. 1.9). Hence there is a Tits system (B, N) in $\mathcal{G}$ with $B = \mathcal{G} \cap \mathscr{G}_{\mathcal{C}}^{\circ}(\mathfrak{o})$, and $N = \mathcal{G} \cap N(S)(k)$, and $\mathcal{B}$ is the building given by this Tits system. For a facet $\mathcal{F}$ of $\mathcal{B}$, as $\mathcal{G} \cap \mathscr{G}_{\mathcal{F}}^{\circ}(\mathfrak{o})$ is the stabilizer of $\mathcal{F}$ in $\mathcal{G}$, $\mathcal{F} \mapsto \mathcal{G} \cap \mathscr{G}_{\mathcal{F}}^{\circ}(\mathfrak{o})$ is an order-reversing injective map. This implies that $\mathcal{F} \mapsto \mathscr{G}_{\mathcal{F}}^{\circ}(\mathfrak{o})$ is an order-reversing bijective map from the set of facets of $\mathcal{B}$ to the set of parahoric subgroups of $G(k)$. So, for a facet $\mathcal{F}$ of $\mathcal{B}$, the subset of points of $\mathcal{B}$ fixed under $\mathscr{G}_{\mathcal{F}}^{\circ}(\mathfrak{o})$ is precisely the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$.

4. Hyperspecial points of $\mathcal{B}$ and hyperspecial parahoric subgroups of $G(k)$

We will continue to use the notation introduced earlier.

4.1. A point x of $\mathcal{B}$ is said to be a *hyperspecial point* if the $\mathfrak{o}$-group scheme $\mathscr{G}_x^{\circ}$ is reductive. As the generic fiber G of $\mathscr{G}_x^{\circ}$ is reductive, the latter is reductive if and only if its special fiber $\overline{\mathscr{G}}_x^{\circ}$ is reductive. From the definition it is clear that every hyperspecial point of $\mathcal{B}$ is also a hyperspecial point of the building $\mathcal{B}(G/K)$ of $G(K)$.

In case G is semi-simple, every hyperspecial point $x \in \mathcal{B}$ is a vertex. In fact, if $\mathcal{F}$ is the facet of $\mathcal{B}$ containing x, and y is a vertex of the compact polyhedron $\overline{\mathcal{F}}$, then unless $x = y$, the image of the homomorphism $\overline{\rho}_{\{y\},\mathcal{F}} : \overline{\mathscr{G}}_{\mathcal{F}}^{\circ} = \overline{\mathscr{G}}_x^{\circ} \to \overline{\mathscr{G}}_y^{\circ}$, induced by the inclusion of $\{y\}$ in $\overline{\mathcal{F}}$, is a proper pseudo-parabolic κ-subgroup of $\overline{\mathscr{G}}_y^{\circ}$. However, as $\overline{\mathscr{G}}_x^{\circ}$ is reductive, its image $\overline{\rho}_{\{y\},\mathcal{F}}(\overline{\mathscr{G}}_x^{\circ})$ in $\overline{\mathscr{G}}_y^{\circ}$ is a reductive group. But a proper pseudo-parabolic subgroup cannot be reductive. We conclude that $x = y$, i.e., x is a vertex. Moreover, since x is a hyperspecial point of $\mathcal{B}(G/K)$, it is also a vertex of this building.

A *hyperspecial parahoric* subgroup of $G(k)$ is by definition the parahoric subgroup $\mathscr{G}_x^{\circ}(\mathfrak{o})$ for a hyperspecial point x of $\mathcal{B}$. Let $x \in \mathcal{B}$ be a hyperspecial point and A be a special k-apartment of $\mathcal{B}(G/K)$ containing x (Proposition 2.6). Let T be the special k-torus corresponding to A, and S be the maximal k-split torus of G contained in T. Let $\mathscr{G}_x^{\circ}$ be the reductive $\mathfrak{o}$-group scheme corresponding to x and $\mathscr{S} \subset \mathscr{T}$ be the closed $\mathfrak{o}$-tori in $\mathscr{G}_x^{\circ}$ with generic fibers $S \subset T$. Let $Z_{\mathscr{G}_x^{\circ}}(\mathscr{S})$ and $Z_{\mathscr{G}_x^{\circ}}(\mathscr{T})$ respectively be the centralizers of $\mathscr{S}$ and $\mathscr{T}$ in $\mathscr{G}_x^{\circ}$. Both these subgroup schemes are smooth (see, for example, [SGA3$_{\rm II}$, Exp. XI, Cor. 5.3] or [CGP, Prop. A.8.10(2)]), and hence their generic and special fibers are of equal dimension.

The special fibers $\overline{\mathscr{S}}$ and $\overline{\mathscr{T}}$ of $\mathscr{S}$ and $\mathscr{T}$ are respectively a maximal κ-split torus and a maximal κ-torus (containing $\overline{\mathscr{S}}$) of the special fiber $\overline{\mathscr{G}}_x^{\circ}$ of $\mathscr{G}_x^{\circ}$ (1.14). As the residue field κ_s of K is separably closed, $\overline{\mathscr{T}}$ splits over κ_s and hence the torus T splits over K. Also, since $\overline{\mathscr{G}}_x^{\circ}$ is reductive, the centralizer of the maximal torus $\overline{\mathscr{T}}$ in $\overline{\mathscr{G}}_x^{\circ}$ is itself, so the special fiber of $Z_{\mathscr{G}_x^{\circ}}(\mathscr{T})$ is $\overline{\mathscr{T}}$. By dimension consideration, this implies that $Z_{\mathscr{G}_x^{\circ}}(\mathscr{T}) = \mathscr{T}$, so the centralizer of T in G equals T. Hence T is a maximal torus of G (and this maximal torus splits over K). Thus, if $\mathcal{B}$ contains a hyperspecial point, then G splits over K.

Now let us assume that the residue field κ of k is of dimension $\leqslant 1$. Then the reductive κ-group $\overline{\mathscr{G}}_x^{\circ}$ is quasi-split, i.e., it contains a Borel subgroup defined over κ (see 1.7), or, equivalently, the centralizer in $\overline{\mathscr{G}}_x^{\circ}$ of the maximal κ-split torus $\overline{\mathscr{S}}$ is a torus, and hence this centralizer is $\overline{\mathscr{T}}$. Thus the special fiber of the group scheme $(\mathscr{T} \subset)\, Z_{\mathscr{G}_x^{\circ}}(\mathscr{S})$ is $\overline{\mathscr{T}}$. From this we conclude that $Z_{\mathscr{G}_x^{\circ}}(\mathscr{S}) = \mathscr{T}$, so the centralizer of the maximal k-split torus S of G in the latter is the torus T. Therefore, $\mathscr{G}_x^\circ$ and G are quasi-split.

Thus we have proved the following:

Proposition 4.2. *If the Bruhat-Tits building $\mathcal{B}$ of $G(k)$ contains a hyperspecial point, then G splits over the maximal unramified extension K of k. Moreover, if the residue field of k is of dimension $\leqslant 1$, then the Bruhat-Tits parahoric $\mathfrak{o}$-group scheme associated to any hyperspecial parahoric subgroup is quasi-split and hence G is quasi-split over k.*

Now we will establish the following partial converse of this proposition (cf. [BrT2, Prop. 4.6.31]).

Proposition 4.3. *The Bruhat-Tits building of a quasi-split reductive k-group that splits over the maximal unramified extension K of k contains hyperspecial points.*

Proof. We begin by recalling a construction that produces all quasi-split reductive k-groups that split over K. Let $\mathscr{G}$ be a Chevalley $\mathfrak{o}$-group scheme (i.e., a split reductive $\mathfrak{o}$-group scheme). Let $\mathscr{B}$ be a Borel $\mathfrak{o}$-subgroup scheme of the adjoint group of $\mathscr{G}$. Let $\mathscr{T}$ be a maximal $\mathfrak{o}$-torus of $\mathscr{B}$; this torus splits over $\mathfrak{o}$. Let G, B and T be the generic fibers of $\mathscr{G}$, $\mathscr{B}$ and $\mathscr{T}$ respectively. Then G is a k-split reductive group, B is a Borel subgroup of the adjoint group of G and T is a k-split maximal torus of B. We may (and we will) identify the outer automorphism group $\mathrm{Out}(G/k)$ of G/k with the subgroup of the automorphism group of $\mathscr{G}$ that keeps $\mathscr{B}$, $\mathscr{T}$ and a pinning stable. Let c be 1-cocycle on $\mathrm{Gal}(K/k)$ with values in $\mathrm{Out}(G/k)$. The Galois twist ${}_cG$ is a quasi-split k-group that splits over K, and all such groups arise in this way from suitable $\mathscr{G}$ and c. Now we identify $\mathrm{Gal}(K/k)$ with the automorphism group of $\mathcal{O}/\mathfrak{o}$ and consider c to be a 1-cocycle on this automorphism group to obtain the twist ${}_c\mathscr{G}$ of $\mathscr{G}$. The generic fiber of the reductive $\mathfrak{o}$-group scheme ${}_c\mathscr{G}$ is clearly ${}_cG$.

Thus, we conclude from the above that given a quasi-split reductive k-group that splits over K, there is reductive $\mathfrak{o}$-group scheme whose generic fiber is the given (quasi-split reductive) k-group. For simplicity, we now change notation. Let G be a quasi-split reductive k-group that splits over K and $\mathscr{G}$ be a reductive $\mathfrak{o}$-group scheme with generic fiber G. Let $\mathcal{B}(G/K)$ and $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ be the Bruhat-Tits buildings of $G(K)$ and $G(k)$ respectively. Let x be a point of $\mathcal{B}(G/K)$ fixed by $\Gamma \ltimes \mathscr{G}(\mathcal{O})$. The point x lies in $\mathcal{B}$. We will presently show that x is a hyperspecial point of $\mathcal{B}$.

As the smooth $\mathfrak{o}$-group scheme $\mathscr{G}$ is "étoffé", the inclusion $\mathscr{G}(\mathcal{O}) \hookrightarrow \mathscr{G}_x(\mathcal{O})$ induces a $\mathfrak{o}$-group scheme homomorphism $\mathscr{G} \to \mathscr{G}_x$ that is the identity homomorphism on the generic fiber G. Since the fibers of $\mathscr{G}$ are connected, this homomorphism factors through a homomorphism $f : \mathscr{G} \to \mathscr{G}_x^\circ$. The kernel of the induced homomorphism $\overline{f} : \overline{\mathscr{G}} \to \overline{\mathscr{G}}_x^\circ$ between the special fibers is a unipotent normal subgroup scheme of the (connected) reductive group $\overline{\mathscr{G}}$, and so this kernel is zero-dimensional and it follows by dimension consideration that $\overline{f}$ is surjective. Hence the special fiber $\overline{\mathscr{G}}_x^\circ$ of $\mathscr{G}_x^\circ$ is reductive. This proves that $\mathscr{G}_x^\circ$ is reductive. So, by definition, x is a hyperspecial point of $\mathcal{B}$. $\square$

4.4. Let x be a hyperspecial point of $\mathcal{B}$. We will now show that the special fiber $\overline{\mathscr{G}}_x$ of $\mathscr{G}_x$ is connected and hence $\mathscr{G}_x = \mathscr{G}_x^\circ$. We may (and do) replace k with K and assume that $\mathfrak{o} = \mathcal{O}$ and the residue field κ is separably closed. Let $\mathscr{T}$ be a maximal $\mathfrak{o}$-torus of $\mathscr{G}_x^\circ$. Let T be the generic fiber of $\mathscr{T}$ and $\overline{\mathscr{T}}$ be its special fiber. Then T is a maximal k-torus of G and this torus is split (4.1). The centralizer $Z_{\mathscr{G}_x}(\mathscr{T})$ of $\mathscr{T}$ in $\mathscr{G}_x$ is a smooth subgroup scheme ([SGA3$_{\rm II}$, Exp. XI, Cor. 5.3] or [CGP, Prop. A.8.10(2)]) containing $\mathscr{T}$ as a closed subgroup scheme, and its generic fiber is $Z_G(T) = T$. We see that the closed immersion $\mathscr{T} \hookrightarrow Z_{\mathscr{G}_x}(\mathscr{T})$ between smooth (and hence flat) $\mathfrak{o}$-schemes is an equality on generic fibers and hence is an equality. Therefore, the inclusion $N_{\mathscr{G}_x}(\mathscr{T})(\mathfrak{o}) \hookrightarrow N_G(T)(k)$ gives an embedding $N_{\mathscr{G}_x}(\mathscr{T})(\mathfrak{o})/\mathscr{T}(\mathfrak{o}) \hookrightarrow N_G(T)(k)/T(k)$, and $Z_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}}) = \overline{\mathscr{T}}$.

As $\overline{\mathscr{T}}$ is a maximal κ-torus of $\overline{\mathscr{G}}_x$, by the conjugacy of maximal κ-tori in $\overline{\mathscr{G}}_x^\circ$ under $\overline{\mathscr{G}}_x^\circ(\kappa)$ (κ is separably closed so every κ-torus is split!), we see that $\overline{\mathscr{G}}_x(\kappa) = N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa) \cdot \overline{\mathscr{G}}_x^\circ(\kappa)$. We know from Proposition 2.1(iii) that $N_{\mathscr{G}_x}(\mathscr{T})(\mathfrak{o}) \to N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa)$ is surjective, and hence $N_{\mathscr{G}_x}(\mathscr{T})(\mathfrak{o})/\mathscr{T}(\mathfrak{o}) \to N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa)/\overline{\mathscr{T}}(\kappa)$ is surjective too. So the order of $N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa)/\overline{\mathscr{T}}(\kappa)$ is less than or equal to that of $N_{\mathscr{G}_x}(\mathscr{T})(\mathfrak{o})/\mathscr{T}(\mathfrak{o})$ ($\hookrightarrow N_G(T)(k)/T(k)$). On the other hand, $N_G(T)(k)/T(k)$ is the Weyl group of the root system of (G, T) and $N_{\overline{\mathscr{G}}_x^\circ}(\overline{\mathscr{T}})(\kappa)/\overline{\mathscr{T}}(\kappa)$ is the Weyl group of the root system of $(\overline{\mathscr{G}}_x^\circ, \overline{\mathscr{T}})$, but these root systems are isomorphic ([SGA3$_{\rm III}$, Exp. XXII, Prop. 2.8]), hence their Weyl groups are isomorphic. We conclude from these observations that the inclusion $N_{\overline{\mathscr{G}}_x^\circ}(\overline{\mathscr{T}})(\kappa)/\overline{\mathscr{T}}(\kappa) \hookrightarrow N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa)/\overline{\mathscr{T}}(\kappa)$ is an isomorphism. So $N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa) = N_{\overline{\mathscr{G}}_x^\circ}(\overline{\mathscr{T}})(\kappa)$, and therefore,

$$\overline{\mathscr{G}}_x(\kappa) = N_{\overline{\mathscr{G}}_x}(\overline{\mathscr{T}})(\kappa) \cdot \overline{\mathscr{G}}_x^\circ(\kappa) = \overline{\mathscr{G}}_x^\circ(\kappa).$$

This implies that $\mathscr{G}_x = \mathscr{G}_x^\circ$.

Using [SGA3$_{\rm III}$, Exp. XXIV, 1.5, 3.9.1 and 3.10] we obtain the following.

Proposition 4.5. *Assume that the residue field of k is of dimension $\leqslant 1$. Given two hyperspecial parahoric subgroups P, P' of $G(k)$, there is a unique inner k-automorphism of G that maps P onto P'.*

5. Filtration of the root groups and valuation of root datum

5.1. We fix a maximal k-split torus S of G, and let $\Phi := \Phi(G, S)$ be the root system of G with respect to S. Let $\mathcal{B}$ be the Bruhat-Tits building of $G(k)$ and $\mathcal{A}$ be the apartment corresponding to S. For a nondivisible root a, let U_a be the root group corresponding to a. If $2a$ is also a root, the root group U_{2a} is a subgroup of U_a. Let S_a be the identity component of the kernel of a. Let M_a be the centralizer of S_a and G_a be the derived subgroup of M_a. Then M_a is a Levi-subgroup of G and G_a is a semi-simple subgroup of k-rank 1. Let C_a be the central torus of M_a. Then S_a

is the maximal k-split subtorus of C_a. The root groups of G_a and M_a with respect to S are $U_{\pm a}$, and also $U_{\pm 2a}$ in case $\pm 2a$ are roots too.

There is a $G_a(K)$-equivariant embedding of the Bruhat-Tits building $\mathcal{B}(G_a/K)$ of $G_a(K)$ into the Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$, cf. 3.11 above or [BrT1, §7.6]; such an embedding is unique up to translation by an element of $V((C_a)_K) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*((C_a)_K)$. Thus the set of $G_a(K)$-equivariant embeddings of $\mathcal{B}(G_a/K)$ into $\mathcal{B}(G/K)$ is an affine space under $V((C_a)_K)$ on which the Galois group Γ of K/k acts through a finite quotient. Therefore, there is a $\Gamma \ltimes G_a(K)$-equivariant embedding of $\mathcal{B}(G_a/K)$ into $\mathcal{B}(G/K)$. This implies that there is a $G_a(k)$-equivariant embedding ι of the Bruhat-Tits building $\mathcal{B}(G_a/K)^\Gamma$ of $G_a(k)$ into the Bruhat-Tits building $\mathcal{B}(G/K)^\Gamma$ of $G(k)$. (In fact, such embeddings form an affine space under $V(S_a) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S_a) = V((C_a)_K)^\Gamma$.) We shall consider the Bruhat-Tits building of $G_a(k)$, which is a Bruhat-Tits tree since G_a is of k-rank 1, embedded in the Bruhat-Tits building of $G(k)$ in terms of ι.

5.2. Filtration of the root groups. Given a real valued affine function ψ on $\mathcal{A}$ with gradient a, let z be the point on the apartment $\mathcal{A}_a$ ($\subset \mathcal{A}$), corresponding to the maximal k-split torus of G_a contained in S, in the Bruhat-Tits tree of $G_a(k)$, such that $\psi(z) = 0$. Let $\mathscr{G}$ be the smooth affine $\mathfrak{o}$-group scheme with generic fiber G_a such that $\mathscr{G}(\mathcal{O})$ is the subgroup of $G_a(K)$ consisting of elements that fix z (1.14). Let $\mathscr{G}^\circ$ be the neutral component of $\mathscr{G}$. Let $\mathscr{S}$ be the closed 1-dimensional $\mathfrak{o}$-split torus of $\mathscr{G}^\circ$ whose generic fiber is the maximal k-split torus of G_a contained in S and let $\lambda : \mathrm{GL}_1 \to \mathscr{S}\ (\hookrightarrow \mathscr{G})$ be the $\mathfrak{o}$-isomorphism such that $\langle a, \lambda\rangle > 0$. Let $\mathscr{U}$ ($:= U_{\mathscr{G}}(\lambda)$, see [CGP, Lemma 2.1.5]) be the $\mathfrak{o}$-subgroup scheme of $\mathscr{G}$ representing the functor

$$R \rightsquigarrow \{g \in \mathscr{G}(R) \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\}.$$

Using the last assertion of 2.1.8(3), and the first assertion of 2.1.8(4), of [CGP] (with k, which is an an arbitrary commutative ring in these assertions, replaced by $\mathfrak{o}$, and G replaced by $\mathscr{G}$), we see that $\mathscr{U}$ is a closed smooth unipotent $\mathfrak{o}$-subgroup scheme with connected fibers; its generic fiber is clearly U_a and $\mathscr{U}(\mathcal{O}) = \mathscr{G}(\mathcal{O}) \cap U_a(K)$. Since the fibers of $\mathscr{U}$ are connected, it is actually contained in $\mathscr{G}^\circ$. Denote by U_ψ the subgroup $\mathscr{U}(\mathfrak{o}) = \mathscr{G}(\mathfrak{o}) \cap U_a(k)\ (= \mathscr{G}^\circ(\mathfrak{o}) \cap U_a(k))$ of $U_a(k)$. This subgroup consists precisely of elements of $U_a(k)$ that fix every point of $\mathcal{A}_a$ where ψ takes non-negative values. For $\psi' \leqslant \psi$, $U_\psi \subseteq U_{\psi'}$ and the union of the U_ψ's is $U_a(k)$.

5.3. We will now work with a given $u \in U_a(k)$–$\{1\}$. Let ψ_u be the largest real valued affine function on $\mathcal{A}$ with gradient a such that u lies in U_{ψ_u} and let $z = z(u)$ be the unique point on the apartment $\mathcal{A}_a$ where ψ_u vanishes. We observe that z is a vertex in $\mathcal{A}_a$. For otherwise, it would be a point of a chamber $\mathcal{C}$ (i.e., a 1-dimensional simplex) of $\mathcal{A}_a$ and then since u fixes z it would fix the chamber $\mathcal{C}$ pointwise, and hence it would fix both the vertices of $\mathcal{C}$. Now let $\psi > \psi_u$ be the affine function with gradient a which vanishes at the vertex of $\mathcal{C}$ where ψ_u takes a negative value. Then u belongs to U_ψ, contradicting the choice of ψ_u to be the largest of such affine functions. As in the previous paragraph, let $\mathscr{G}^\circ$ be the Bruhat-Tits group scheme, corresponding to the vertex $z = z(u)$, with generic fiber G_a and connected special fiber. Then u lies in $\mathscr{G}^\circ(\mathfrak{o})$. Let $\overline{\mathscr{G}}^\circ$ be the special fiber of $\mathscr{G}^\circ$. We assert that the image $\overline{u}$ of u in $\overline{\mathscr{G}}^\circ(\kappa)$ does not lie in $\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ)(\kappa)$, for if it did, then u would fix the unique chamber of $\mathcal{A}_a$ which has z as a vertex and on which ψ_u takes negative values. Then, as above, we would be able to find an affine function $\psi > \psi_u$ with gradient a such that $u \in U_\psi$, contradicting the choice of ψ_u.

5.4. Let $\mathscr{S}$, and $\lambda : \mathrm{GL}_1 \to \mathscr{S}$ be as in 5.2 for $z = z(u)$. Let $\overline{\mathscr{S}}$ be the special fiber of $\mathscr{S}$. Since $\overline{\mathscr{G}}^\circ(\kappa)$ contains an element which normalizes $\overline{\mathscr{S}}$ and whose conjugation action on $\overline{\mathscr{S}}$ is by inversion, as in the proof of Proposition 3.6, by considering the smooth normalizer subgroup scheme $N_{\mathscr{G}^\circ}(\mathscr{S})$, we conclude that $\mathscr{G}^\circ(\mathfrak{o})$ contains an element n which normalizes $\mathscr{S}$ and whose conjugation action on this torus is by inversion.

We shall now use the notation introduced in §2.1 of [CGP]. According to Remark 2.1.11 and the last assertion of Proposition 2.1.8(3) of [CGP] (with k, which is an arbitrary commutative ring in that assertion, replaced by $\mathfrak{o}$, and G replaced by $\mathscr{G}^\circ$), the multiplication map

$$U_{\mathscr{G}^\circ}(-\lambda) \times Z_{\mathscr{G}^\circ}(\lambda) \times U_{\mathscr{G}^\circ}(\lambda) \to \mathscr{G}^\circ$$

is an open immersion of $\mathfrak{o}$-schemes. We shall denote $U_{\mathscr{G}^\circ}(-\lambda)$, $Z_{\mathscr{G}^\circ}(\lambda)(= Z_{\mathscr{G}^\circ}(\mathscr{S}))$ and $U_{\mathscr{G}^\circ}(\lambda)$ by $\mathscr{U}_{-a}$, $\mathscr{Z}$ and $\mathscr{U}_a$ respectively, and the special fibers of these $\mathfrak{o}$-subgroup schemes by $\overline{\mathscr{U}}_{-a}$, $\overline{\mathscr{Z}}$ and $\overline{\mathscr{U}}_a$ respectively. Note that $\mathscr{U}_{\pm a}$ are the $\pm a$-root groups of $\mathscr{G}^\circ$ with respect to $\mathscr{S}$, and $\overline{\mathscr{U}}_{\pm a}$ are the $\pm a$-root groups of $\overline{\mathscr{G}}^\circ$ with respect to $\overline{\mathscr{S}}$. Now since $n\mathscr{U}_{-a}n^{-1} = \mathscr{U}_a$, we see that $\Omega := \mathscr{U}_{-a}\mathscr{Z}n\mathscr{U}_{-a}$ is an open subscheme of $\mathscr{G}^\circ$. Let $\overline{\Omega} = \overline{\mathscr{U}}_{-a}\overline{\mathscr{Z}}n\overline{\mathscr{U}}_{-a}(\subset \overline{\mathscr{G}}^\circ)$ be the special fiber of Ω.

Let $\pi : \overline{\mathscr{G}}^\circ \to \overline{\mathscr{G}}^\circ/\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ)$ be the maximal pseudo-reductive quotient of $\overline{\mathscr{G}}^\circ$. As $\overline{u} \notin \mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ)(\kappa)$, $\pi(\overline{u})$ is a nontrivial element of $\pi(\overline{\mathscr{U}}_a)(\kappa)$. Note that $\pi(\overline{\Omega}) = \pi(\overline{\mathscr{U}}_{-a})\pi(\overline{\mathscr{Z}})\pi(n)\pi(\overline{\mathscr{U}}_{-a})$, and $\pi(\overline{\mathscr{U}}_{\pm a})$ are the $\pm a$-root groups of the pseudo-reductive κ-group $\overline{\mathscr{G}}^\circ/\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ)$ with respect to the maximal κ-split torus $\pi(\overline{\mathscr{S}})$ [CGP, Cor. 2.1.9]. Now using Proposition C.2.24(i) of [CGP] we infer that $\pi(\overline{u})$ lies in $\pi(\overline{\Omega})(\kappa)$. We claim that $\overline{u} \in \overline{\Omega}(\kappa)$. To establish this claim, it would suffice to prove that $\overline{\Omega} \cdot \mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) = \overline{\Omega}$.

According to [CGP, Prop. 2.1.12(1)], the open immersion

$$(\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) \cap \overline{\mathscr{U}}_a) \times (\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) \cap \overline{\mathscr{Z}}) \times (\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) \cap \overline{\mathscr{U}}_{-a}) \to \mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ),$$

defined by multiplication, is an isomorphism of schemes. Using this, and the normality of $\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ)$ in $\overline{\mathscr{G}}^\circ$, we see that

$$\overline{\Omega} \cdot \mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) = \overline{\mathscr{U}}_{-a}\overline{\mathscr{Z}}n\overline{\mathscr{U}}_{-a} \cdot \mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) = \overline{\mathscr{U}}_{-a}\overline{\mathscr{Z}}n \cdot \mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ)\overline{\mathscr{U}}_{-a}$$

$$= \overline{\mathscr{U}}_{-a}\overline{\mathscr{Z}}n(\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) \cap \overline{\mathscr{U}}_a)(\mathscr{R}_{u,\kappa}(\overline{\mathscr{G}}^\circ) \cap \overline{\mathscr{Z}})\overline{\mathscr{U}}_{-a} = \overline{\mathscr{U}}_{-a}\overline{\mathscr{Z}}n\overline{\mathscr{U}}_{-a} = \overline{\Omega}.$$

Now the following well-known lemma implies at once that u is contained in $\Omega(\mathfrak{o})$. Therefore, there exist $u', u'' \in \mathscr{U}_{-a}(\mathfrak{o})$, such that $m(u) := u'uu'' \in \mathscr{Z}(\mathfrak{o})n\,(\subset N_{\mathscr{G}^\circ}(\mathscr{S})(\mathfrak{o}) \subset \mathscr{G}^\circ(\mathfrak{o}))$.

Lemma 5.5. *Let X be a scheme, and $\Omega \subset X$ an open subscheme. If for a local ring R, $f : \mathrm{Spec}(R) \to X$ is a map carrying the closed point into Ω, then f factors through Ω.*

Proof. Since Ω is an open subscheme of X, the property of f factoring through Ω is purely topological; i.e., it is equivalent to show that the open subset $f^{-1}(\Omega) \subset \mathrm{Spec}(R)$ is the entire space. Our hypothesis says that this latter open subset contains the closed point, so our task reduces to showing that the only open subset of a local scheme that contains the unique closed point is the entire space. Said equivalently in terms of its closed complement, we want to show that the only closed subset Z of $\mathrm{Spec}(R)$ not containing the closed point is the empty set. For an ideal $J \subset R$ defining Z, this is the obvious assertion that if J is not contained in the unique maximal ideal of R then $J = (1)$. □

5.6. We recall that there exist unique $u', u'' \in U_{-a}(k)$ such that $u'uu''$ normalizes S [CGP, Prop. C.2.24(i)]. Thus the above $m(u)$ is uniquely determined by u. It acts on the apartment $\mathcal{A}$ by an affine reflection $r(u)$ whose derivative (or, vector part) is the reflection associated with a. As $m(u) \in \mathscr{G}^\circ(\mathfrak{o})$, $r(u)$ fixes the point $z = z(u)$ defined above. Hence, the fixed point set of the affine reflection $r(u)$ is the hyperplane spanned by $S_a(k) \cdot z$ in $\mathcal{A}$. As $\psi_u(z) = 0$, this hyperplane is the vanishing hyperplane of the affine function ψ_u. This observation implies at once that the filtration subgroups of $U_a(k)$ as defined in [T2, §1.4] are same as the subgroups U_ψ described above. We also note that the largest half-apartment in $\mathcal{A}$ that is fixed pointwise by the element u is $\psi_u^{-1}([0, \infty))$.

5.7. As above, let $u', u'' \in U_{-a}(k)$ be such that $m(u) = u'uu''$ normalizes S. Then $m(u) = (m(u)u''m(u)^{-1})u'u = uu''(m(u)^{-1}u'm(u))$. Since $m(u)^{-1}u'm(u)$ and $m(u)u''m(u)^{-1}$ belong to $U_a(k)$, we conclude that $m(u') = m(u) = m(u'')$. Hence, $\psi_{u'} = -\psi_u = \psi_{u''}$. Also, $m(u^{-1}) = m(u)^{-1}$, and hence $r(u) = r(u^{-1})$, and so $\psi_u = \psi_{u^{-1}}$.

5.8. Now assume that $2a$ is also a root of G with respect to S, and $u \in U_{2a}(k)$–$\{1\}(\subset U_a(k)$–$\{1\})$. Let u', u'' be as in 5.6. Considering the semi-simple subgroup generated by the root groups $U_{\pm 2a}$, we see that $u', u'' \in U_{-2a}(k)$. Let ψ_u be the affine function as in 5.3. Then $2\psi_u$ is the affine function with gradient $2a$ whose vanishing hyperplane is the fixed point set of the reflection $r(u)$. Thus if we consider u to be an element of $U_{2a}(k)$–$\{1\}$, then the associated affine function with gradient $2a$ is $2\psi_u$.

5.9. Valuation of root datum. Given a point $s \in \mathcal{A}$, the corresponding valuation φ_a of the root group $U_a(k)$ is defined as follows: For $u \in U_a(k)$–$\{1\}$, let $\varphi_a(u) = \psi_u(s)$. According to a result of Tits (Theorem 10.11 of [R]), $(\varphi_a)_{a\in\Phi}$ is a valuation of the root groups $(U_a(k))_{a\in\Phi}$. From the results in 5.7, 5.8 we see that for $u \in$

$U_a(k)$–$\{1\}$, if $m(u) = u'uu''$ is as above, then $\varphi_{-a}(u') = -\varphi_a(u) = \varphi_{-a}(u'')$, and $\varphi_a(u) = \varphi_a(u^{-1})$. Moreover, if $2a$ is also a root, then $\varphi_{2a} = 2\varphi_a$ on $U_{2a}(k)$–$\{1\}$.

6. Residue field κ perfect and of dimension $\leqslant 1$

We will assume throughout this section that the residue field κ of k is perfect and is of dimension $\leqslant 1$. According to Proposition 2.4, then every k-chamber is a chamber of $\mathcal{B}(G'/K)$; in other words, every reductive k-group is residually quasi-split.

Theorem 6.1. (i) *Any two special k-tori of G are conjugate to each other under an element of $G'(k)$.*

(ii) *Let S be a maximal k-split torus of G, then any two special k-tori contained in $Z(S)$ are conjugate to each other under an element of the bounded subgroup $Z(S)'(k)$ of $Z(S)(k)$, where $Z(S)' = (Z(S), Z(S))$ is the derived subgroup of $Z(S)$.*

Proof. For $i = 1, 2$, let T_i be a special k-torus of G and A_i the corresponding special k-apartment of $\mathcal{B}(G'/K)$. If $A_1 \cap A_2$ is nonempty, the first assertion follows immediately from the second assertion of Theorem 3.1. So let us assume that $A_1 \cap A_2$ is empty. We fix a k-chamber C_i in A_i, for $i = 1, 2$ (Proposition 2.4). According to Proposition 2.7, there is a special k-apartment A containing C_1 and C_2. Let T be the special k-torus of G corresponding to this apartment. Then using the second assertion of Theorem 3.1 twice, first for the pair $\{A, A_1\}$, and then for the pair $\{A, A_2\}$ we see that T is conjugate to both T_1 and T_2 under $G'(k)$. So T_1 and T_2 are conjugate to each other under an element of $G'(k)$.

The second assertion follows from the first applied to $Z(S)$ in place of G. □

Theorem 6.2. *Let T be a special k-torus of G and S be the maximal k-split torus of G contained in T. Then $N(T)(k) \subset N(S)(k) = Z(S)'(k) \cdot N(T)(k)$. Therefore, the natural homomorphism $N(T)(k) \to N(S)(k)/Z(S)(k)_b$, induced by the inclusion of $N(T)(k)$ in $N(S)(k)$, is surjective.*

Proof. Any k-automorphism of T carries the unique maximal k-split subtorus S to itself. So $N(T)(k) \subset N(S)(k)$. Now let $n \in N(S)(k)$, then nTn^{-1} is a special k-torus that contains S. So T and nTn^{-1} are special k-tori contained in $Z(S)$. Now Theorem 6.1(ii) implies that there is a $g \in Z(S)'(k)$ such that $gTg^{-1} = nTn^{-1}$. Hence, $g^{-1}n$ belongs to $N(T)(k)$, and $n = g \cdot g^{-1}n$. □

The following result is in [BrT3, 4.4-4.5] for complete k.

Theorem 6.3. *Assume that G is absolutely almost simple and anisotropic over k. Then it splits over the maximal unramified extension K of k and is of type A_n for some n.*

Proof. We know from Proposition 2.8 that $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ consists of a single point, say x. Let A be a special k-apartment of $\mathcal{B}(G/K)$, and C be a k-chamber in A (Proposition 2.4). Then $C^\Gamma = C \cap \mathcal{B}$ is nonempty, and hence it equals $\{x\}$. Let

I be the Iwahori subgroup of $G(K)$ determined by the chamber C and T be the k-torus of G corresponding to the apartment A. Then I is stable under Γ, and T_K is a maximal K-split torus of G_K. We consider the affine root system of G_K with respect to T_K and let Δ denote its basis determined by the Iwahori subgroup I. Then Δ is stable under the natural action of Γ on the affine root system and there is a natural Γ-equivariant bijective correspondence between the set of vertices of C and Δ. As $\mathcal{B}$ does not contain any facets of positive dimension, we see from the discussion in 3.2 that Γ acts transitively on the set of vertices of C, and hence it acts transitively on Δ. Now from the description of irreducible affine root systems, we see that G_K is K-split and its root system with respect to the split maximal K-torus T_K is of type A_n for some n, for otherwise, the action of the automorphism group of the Dynkin diagram of Δ cannot be transitive on Δ. □

Remark 6.4. If k is a locally compact nonarchimedean field (that is, k is complete and its residue field κ is finite), then any absolutely almost simple k-anisotropic group G is of *inner* type A_n for some n. This assertion was proved by Martin Kneser for fields of characteristic zero, and Bruhat and Tits in general. In view of the previous theorem, to prove it, we just need to show that any simply connected absolutely almost simple k-group G of *outer* type A_n for $n \geqslant 2$ is k-isotropic. Since there does not exist a noncommutative finite dimensional division algebra with center a quadratic Galois extension of k which admits an involution of the second kind with fixed field k (see [Sch, Ch. 10, Thm. 2.2(ii)]), if G is of outer type, then there is a quadratic Galois extension ℓ of k and a nondegenerate hermitian form h on ℓ^{n+1} such that $G = \mathrm{SU}(h)$. But any hermitian form in at least 3 variables over a nonarchimedean locally compact field represents zero nontrivially, and hence $\mathrm{SU}(h)$ is isotropic for $n \geqslant 2$.

The following example of an absolutely almost simple k-anisotropic group of *outer* type A_{r-1} (over a discretely valued complete field k with residue field of dimension $\leqslant 1$) was communicated to me by Philippe Gille. As usual, $\mathbb{C}$ will denote the field of complex numbers; for a positive integer r, let μ_r denote the group of r-th roots of unity; $F = \mathbb{C}(x)$ and $F' = \mathbb{C}(x')$ with $x' = \sqrt{x}$. We take $k = F((t))$ and $k' = F'((t))$. Since the Brauer groups of F and F' are trivial, the residue maps induce isomorphisms:

$$\ker({}_r\mathrm{Br}(k') \xrightarrow{N_{k'/k}} {}_r\mathrm{Br}(k)) \xrightarrow{\simeq} \ker(\mathrm{H}^1(F', \mu_r) \xrightarrow{N_{F'/F}} \mathrm{H}^1(F, \mu_r))$$

$$\xrightarrow{\simeq} \ker({F'}^{\times}/{F'}^{\times r} \xrightarrow{N_{F'/F}} F^{\times}/{F^{\times}}^{r});$$

see [S, §2 of the Appendix after Ch. II]. The element $u := \frac{1+x'}{1-x'} \in {F'}^{\times}$ has trivial norm over F, and has a pole of order 1 at $x' = 1$, so it cannot be an r-th power. It defines a central simple k'-algebra $\mathscr{D}$ which is division and cyclic of degree r. By Albert's theorem, $\mathscr{D}$ carries a k'/k-involution τ of the second kind. The k-group $\mathrm{SU}(\mathscr{D}, \tau)$ is of outer type A_{r-1} and is anisotropic over k.

In the following theorem, and in its proof, we will use the notation introduced earlier in the paper, and assume that k is a discretely valued field with Henselian valuation ring and perfect residue field κ of dimension $\leqslant 1$.

Theorem 6.5. *Let G be a simply connected semi-simple k-group. Then the Galois cohomology set $\mathrm{H}^1(k, G)$ is trivial.*

Proof. By Steinberg's theorem (1.7), $\mathrm{H}^1(K, G)$ is trivial, so $\mathrm{H}^1(k, G) \simeq \mathrm{H}^1(K/k, G(K))$. Let $c : \gamma \mapsto c(\gamma)$ be a 1-cocycle on the Galois group Γ of K/k with values in $G(K)$ and ${}_cG$ be the Galois-twist of G with the cocycle c. The k-groups G and ${}_cG$ are isomorphic over K and we will identify ${}_cG(K)$ with $G(K)$. (Recall that with identification of ${}_cG(K)$ with $G(K)$ as an abstract group, the "twisted" action of Γ on ${}_cG(K)$ is described as follows: For $x \in {}_cG(K)$, and $\gamma \in \Gamma$, $\gamma \circ x = c(\gamma)\gamma(x)c(\gamma)^{-1}$, where $\gamma(x)$ denotes the γ-transform of x considered as a K-rational element of the given k-group G.)

Now let I be a Γ-stable Iwahori subgroup of $G(K)$, say $I = \mathscr{G}_C^\circ(\mathcal{O})$ for a k-chamber C of $\mathcal{B}(G/K)$. The subgroup I is also an Iwahori subgroup of ${}_cG(K)$ (as ${}_cG(K)$ has been identified with $G(K)$ in terms of a K-isomorphism $({}_cG)_K \to G_K$). However, under the twisted action of Γ on ${}_cG(K)$, I may not be Γ-stable. But as ${}_cG$ is a residually quasi-split semi-simple k-group, ${}_cG(K)$ certainly contains an Iwahori subgroup which is stable under the twisted action of Γ. Since any two Iwahori subgroups of ${}_cG(K)$ are conjugate under ${}_cG(K) = G(K)$ (Proposition 3.10 for K in place of k), there exists a $g \in G(K)$ such that gIg^{-1} is stable under the twisted action of Γ. Then $c(\gamma)\gamma(g)I\gamma(g)^{-1}c(\gamma)^{-1} = gIg^{-1}$ for all $\gamma \in \Gamma$. Hence, for $\gamma \in \Gamma$, $c'(\gamma) := g^{-1}c(\gamma)\gamma(g) \in G(K)$ normalizes the Iwahori subgroup I. As the normalizer of I is I itself (3.15 for K in place of k), we conclude that c', which is a 1-cocycle on Γ cohomologous to c, takes values in $I = \mathscr{G}_C^\circ(\mathcal{O})$. So to prove the theorem, it suffices to prove the triviality of $\mathrm{H}^1(\Gamma, \mathscr{G}_C^\circ(\mathcal{O}))$.

By unramified Galois descent over discrete valuation rings [BLR, §6.2, Ex. B], this cohomology set classifies $\mathscr{G}_C^\circ$-torsors $\mathscr{X}$ over $\mathfrak{o}$ which admit an $\mathcal{O}$-point. (As $\mathscr{X}$ inherits $\mathfrak{o}$-smoothness from $\mathscr{G}_C^\circ$, and $\mathcal{O}$ is Henselian with algebraically closed residue field $\overline{\kappa}$, $\mathscr{X}$ does automatically admit $\mathcal{O}$-points.) Thus, it suffices to prove that every such torsor admits an $\mathfrak{o}$-point. By $\mathfrak{o}$-smoothness of $\mathscr{X}$, and the henselian property of $\mathfrak{o}$, it suffices to prove that the special fiber of $\mathscr{X}$ admits a rational point. But the isomorphism class of the special fiber as a torsor is determined by an element of the set $\mathrm{H}^1(\Gamma, \mathscr{G}_C^\circ(\overline{\kappa}))$, and this cohomology set is trivial by Steinberg's Theorem (1.7) since κ has been assumed to be perfect and of dimension $\leqslant 1$. □

Remark 6.6. The above theorem was first proved by a case-by-case analysis by Martin Kneser for k a nonarchimedean local field of characteristic zero with finite residue field. It was proved for all discretely valued *complete* fields k with perfect residue field of dimension $\leqslant 1$ by Bruhat and Tits [BrT3, Thm. in §4.7]. For the completion $\widehat{k}$ of k, the natural map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(\widehat{k}, G)$ is bijective [GGM, Prop. 3.5.3(ii)]. So the vanishing theorem of Bruhat and Tits over the completion $\widehat{k}$ also implies the above theorem.

References

[Be] V. G. Berkovich, *Étale cohomology for non-archimedean analytic spaces.* Publ. Math. IHES **78** (1993), 5-161.

[Bo] A. Borel, *Linear algebraic groups* (second edition). Springer-Verlag, New York (1991).

[BoT] A. Borel and J. Tits, *Groupes réductifs.* Publ. Math. IHES **27** (1965), 55-150.

[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models.* Springer-Verlag, Heidelberg (1990).

[BrT1] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local.* Publ. Math. IHES **41**(1972).

[BrT2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local, II.* Publ. Math. IHES **60**(1984).

[BrT3] F. Bruhat and J. Tits, *Groupes algébriques sur un corps local, III.* J. Fac. Sc. U. Tokyo **34**(1987), 671-698.

[C] B. Conrad, *Reductive group schemes.* In "Group schemes: A celebration of SGA3", Volume I, Panoramas et Synthèses #**42-43**(2014), Soc. Math. France.

[CP] B. Conrad and G. Prasad, *Structure and classification of pseudo-reductive groups.* Proc. Symp. Pure Math. #**94**, 127-276, American Math. Soc. (2017).

[CGP] B. Conrad, O. Gabber and G. Prasad, *Pseudo-reductive groups* (second edition). Cambridge U. Press, New York (2015).

[EGA IV$_3$] A. Grothendieck, *Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES **28**(1966), 5-255.

[EGA IV$_4$] A. Grothendieck, *Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES **32**(1967), 5-361.

[SGA3$_{\mathrm{II}}$] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome II.* Lecture Notes in Math. #**152**, Springer-Verlag, Heidelberg (1970).

[SGA3$_{\mathrm{III}}$] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome III* (revised). Documents Mathématiques, Société Math. France (2011).

[GGM] O. Gabber, P. Gille and L. Moret-Bailly, *Fibrés principaux sur les corps valués henséliens.* Algebr. Geom. **1**(2014), 573-612.

[P] G. Prasad, *Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits.* Bull. Soc. Math. France **110**(1982), 197-202.

[PY1] G. Prasad and J.-K. Yu, *On finite group actions on reductive groups and buildings.* Invent. Math. **147**(2002), 545–560.

[PY2] G. Prasad and J.-K. Yu, *On quasi-reductive group schemes.* J. Alg. Geom. **15** (2006), 507-549.

[R] M. Ronan, *Lectures on buildings.* University of Chicago Press, Chicago (2009).

[Sch] W. Scharlau, *Quadratic and hermitian forms.* Springer-Verlag, Heidelberg (1985).

[S] J-P. Serre, *Galois cohomology.* Springer-Verlag, New York (1997).

[T1] J. Tits, *Buildings of spheirical type and finite* BN-*pairs.* Lecture Notes in Math. #**386**, Springer-Verlag, Berlin (1974).

[T2] J. Tits, *Reductive groups over local fields.* Proc. Symp. Pure Math. #**33**, Part I, 29–69, American Math. Soc. (1979).

[Y] J.-K. Yu, *Smooth models associated to concave functions in Bruhat-Tits theory.* In "Group schemes: A celebration of SGA3", Volume III, Panoramas et Synthèses #**47**(2016), Soc. Math. France.

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