WEIGHTED PROJECTIVE VARIETIES

by

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Contents

0. Introduction.

In this paper I discuss the technique of weighted homogeneous coordinates which has appeared in works of various geometers a few years ago and it seems has been appreciated and armed by many people. In many cases this technique allows one to present a nonsingular algebraic variety as a hypersurface in a certain space (a weighted projective space) and deal with it as it would be a nonsingular hypersurface in the projective space. A generalization of this approach is the technique of polyhedral projective spaces for which we refer to [5, 6, 15].

Section 1 deals with weighted projective spaces, the spectrums of graded polynomial rings. Most of the results from this section can be found in [7].

In section 2 we generalize the Bott theorem on the cohomology of twisted sheaves of differentials to the case of weighted projective spaces. Another proof of the same result can be found in [23] and a similar result for torical spaces is discussed in [5].

In section 3 we introduce the notion of a quasismooth subvariety of a weighted projective space. For this we define the affine quasicone over a subvariety and require that this quasicone is smooth outside its vertex. We show that quasismooth weighted complete intersections have many properties of ordinary smooth complete intersections in a projective space. The work of Mori [19] contains a similar result but under more restrictive conditions. Rather surprisingly not everything goes the same as for smooth complete intersections. For example, recent examples of Catanese and Todorov show that the local Torelli theorem fails for some quasismooth weighted complete intersections (see [4, 24]).

In section 4 we generalize to the weighted case the results concerning the Hodge structure of a smooth projective hypersurfaces. Our proof is an algebraic version of one of Steenbrink [23] and can be applied to the calculation of the De Rham cohomology of any such hypersurface over a field of characteristic zero. The present paper is partially based on my talks at a seminar on the Hodge-Deligne theory at Moscow State University in 1975/76. It is a pleasure to thank all of its participants for their attention and criticism.

1. Weighted projective space.

1.1. Notations

$Q = \{q_0, q_1, \ldots, q_r\}$, - a finite set of positive integers;

$$|Q| = q_0 + \ldots + q_r ;$$

$S(Q)$ - the polynomial algebra $k[T_0, \ldots, T_r]$ over a field k, graded by the condition $\deg(T_i) = q_i$, $i = 0, \ldots, r$; $\mathbb{P}(Q) = \mathrm{Proj}(S(Q))$ - weighted projective space of type Q. $U = \mathbb{A}^{r+1} - \{0\} = \mathrm{Spec}(S(Q)) - \{(T_0, \ldots, T_r)\}$; $m = (T_0, \ldots, T_r)$.

Abbreviations:

$\mathbb{P}^r = \mathbb{P}(1, \ldots, 1)$, $S = S(Q)$, $\mathbb{P} = \mathbb{P}(Q)$.

We suppose in the sequel that the characteristic p of k is prime to all q_i, though many results are valid without this assumption. We also assume that $(q_0, \ldots, q_r) = 1$.

The last assumption is not essential in virtue of the following:

Lemma. Let $Q' = \{aq_0, \ldots, aq_r\}$. Then $\mathbb{P}(Q) \simeq \mathbb{P}(Q')$.

Really, $S(Q')_m = S(Q)_{am}$ and hence in the standard notations of [12] we have $S(Q') = S(Q)^{(a)}$. Applying ([12], 2.4.7) we obtain a canonical isomorphism

$$\mathbb{P}(Q) = \mathrm{Proj}(S(Q)) \simeq \mathrm{Proj}(S(Q)^{(a)}) = \mathbb{P}(Q') .$$

We refer to 1.3 for more general results.

For any graded module M over a graded commutative ring A we denote by $M(n)$ the graded A-module obtained by shifting the graduation $M(n)_k = M_{n+k}$.

By $\tilde{M}$ we denote the $\mathcal{O}_{\mathrm{Proj}(A)}$-Module, associated with M. Recall ([12]; 2.5.2) that for any $f \in A_d$

$$\Gamma(D_+(f),\tilde{M}) = M_{(f)} = \{\frac{m}{f^k} : m \in M_{kd}\} ,$$

where open sets $D_+(f) = \mathrm{Spec}(A_{(f)})$ form a base of open sets in $\mathrm{Proj}(A)$.

1.2. Interpretations.

1.2.1. It is well known that a $\mathbb{Z}$-graduation of a commutative ring is equivalent to an action of a 1-dimensional algebraic torus G_m on its spectrum. In our case G_m acts on $\mathbb{A}^{r+1} = \mathrm{Spec}(S(Q))$ as follows

$$S \longrightarrow S \otimes k[X,X^{-1}]$$
$$T_i \longrightarrow T_i \otimes X^{q_i}, \quad i = 0,\dots,r$$

where $G_m = \mathrm{Spec}(k[X,X^{-1}])$.

The corresponding action on points with the value in a field $k' \supset k$ is given by the formulas

$$k'^* \times k'^{r+1} \to k'^{r+1}$$
$$(t,(a_0,\dots,a_r)) \to (a_0 t^{q_0},\dots,a_r t^{q_r})$$

The open set $U = \mathbb{A}^{r+1} - \{0\}$ is invariant with respect to this action and the universal geometric quotient U/G_m exists and coincides with $\mathbb{P}(Q)$.

If $k = \mathbb{C}$ is the field of complex numbers then the analytic space $\mathbb{P}^{an}$ associated to $\mathbb{P}(Q)$ is a complex analytic quotient space $\mathbb{C}^{r+1} - \{0\}/\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}^{r+1}$ by the formulas

$$(t,(z_0,\dots,z_r)) \to (z_0 t^{q_0},\dots,z_r t^{q_r}) .$$

In view of this interpretation the space $\mathbb{P}(Q')$ from the lemma in 1.1 corresponds to a noneffective action of G_m.

1.2.2. For any positive integer q we denote by μ_q the finite group scheme of q-roots of unity. This is a closed subgroup of G_m with the coordinate ring $k[X]/(X^q - 1)$.

Consider the action of the group scheme $\mu_Q = \mu_{q_0} \times \dots \times \mu_{q_r}$ on $\mathbb{P}^r$ which is induced by the action μ_Q on S

$$T_i \to T_i \otimes \bar{X}_i ,$$

where $\bar{X}_i \equiv X \bmod (X^{q_i} - 1)$ in the coordinate ring of μ_{q_i}.

The homomorphism of rings $S(Q) \to S$, $T_i \to T_i^{q_i}$ yields the isomorphism

$S(Q) \simeq S^{\mu_Q}$. It is easy to see that the corresponding morphism of projective spectrums is well defined and gives an isomorphism

$$\mathbb{P}(Q) \simeq \mathrm{Proj}(S^{\mu_Q}) \simeq \mathbb{P}^r/\mu_Q .$$

In case $k = \mathbb{C}$

$$\mathbb{P}(Q)^{an} = \mathbb{P}^r(\mathbb{C})/\mu_Q(\mathbb{C})$$

where $\mu_Q(\mathbb{C})$ acts by the formulas

$$(g,(z_0,\dots,z_r)) \to (z_0g_0,\dots,z_rg_r)$$

$$g = (g_0,\dots,g_r), \quad g_i = \exp(2\pi i b_i/q_i), \quad 0 \le b_i < q_i .$$

1.2.3. The previous interpretation easily gives, for instance, that for $Q = \{1,1,\dots,1,n\}$ the weighted projective space $\mathbb{P}(Q)$ equals the projective cone over the Veronese variety $v_n(\mathbb{P}^{r-1})$.

For example, $\mathbb{P}(1,1,n)$, $n \neq 1$ is obtained by the blowing down the exceptional section of the ruled surface $\mathbb{F}_n$ (when $n=2$ it is an ordinary quadratic cone).

1.2.4. For $Q = \{1,q_1,\dots,q_r\}$ the spaces $\mathbb{P}(Q)$ are compactifications of the affine space $\mathbb{A}^r$. Indeed, the open set $D_+(T_0)$ is isomorphic to the spectrum of the polynomial ring $k\left[\frac{T_1}{T_0},\dots,\frac{T_r}{T_0}\right]$. Its complement coincides with the weighted projective space $\mathbb{P}(q_1,\dots,q_r)$.

1.2.5. Weighted projective spaces are complete toric spaces. More precisely, $\mathbb{P}(q_0,\dots,q_r)$ is isomorphic to the polyhedral space $\mathbb{P}_\Delta$ of [6], where $\Delta = \{(x_0,\dots,x_r) \in \mathbb{R}^{r+1} : \Sigma q_i x_i = q_0 \cdots q_r\}$.

1.3. The first properties

1.3.1. For different Q and Q' the corresponding spaces $\mathbb{P}(Q)$ and $\mathbb{P}(Q')$ can be isomorphic.

Let

$$d_i = (q_0,\dots,q_{i-1},q_{i+1},\dots,q_r)$$

$$a_i = \ell.c.m.(d_0,\dots,d_{i-1},d_{i+1},\dots,d_r)$$

$$a = \ell.c.m.(d_0,\dots,d_r).$$

Note that $a_i | q_i$, $(a_i,d_i) = 1$ and $a_i d_i = a$.

Proposition. (Delorme [7].) Let $Q' = \{q_0/a,\dots,q_r/a_r\}$. Then there exists a natural isomorphism $\mathbb{P}(Q) \simeq \mathbb{P}(Q')$.

For the proof we consider the graded subring $S' = \bigoplus_{n=0}^{\infty} S(Q)_{an}$ of $S(Q)$ and note that $S' \simeq k[X_0,\dots,X_r]$, where $X_i = T_i^{d_i}$ is of degree aq_i/a_i. But then

$S(Q') \simeq S'^{(a)}$ and hence $\mathrm{Proj}(S(Q')) \simeq \mathrm{Proj}(S')$ ([12],2.4.7). Now there exists also an isomorphism $\mathrm{Proj}(S') \simeq \mathrm{Proj}(S(Q)^{(a)}) \simeq \mathrm{Proj}(S(Q))$.

Corollary. Each $\mathbb{P}(Q) \simeq \mathbb{P}(Q')$, where $(q'_0,\dots,q'_{i-1},q'_{i+1},\dots,q'_r) = 1$ for $i = 0,\dots,r$.

Corollary. Assume that $q_i = a_i$ for $i = 0,\dots,r$. Then $\mathbb{P}(Q) \simeq \mathbb{P}^r$.

For example, it is so if all numbers $\ell.c.m.(q_0,\dots,q_r)/q_i$ are coprime. In this case the previous fact was independently discovered by M. Reid.

Note that in case $r = 1$ we can use the previous corollary and obtain that $\mathbb{P}(q_0,q_1) \simeq \mathbb{P}^1$ for any q_0,q_1 . This fact however follows also from interpretation 1.2.2.

1.3.2. Remarks. 1. There is a certain difference between the identifications of the proposition and of the lemma in 1.1. In terms of 3.5, the spaces $\mathbb{P}(Q)$ and $\mathbb{P}(Q')$ from the proposition are not projectively isomorphic.

2. It can be shown that the isomorphism $\mathbb{P}(Q) \simeq \mathbb{P}(Q')$ of 1.3.1 induces an isomorphism of sheaves $\mathcal{O}_{\mathbb{P}}(n) \simeq \mathcal{O}_{\mathbb{P}'}((n - \sum_{i=0}^{r} b_i(n)q_i)/a)$, where $b_i(n)$ are uniquely determined by the property

$$n = b_i(n)q_i + c_i(n)d_i , \quad 0 \le b_i < d_i .$$

1.3.2. Let G be a finite group of linear automorphisms of a finite-dimensional vector space V over a field k . An element $g \in G$ is called a pseudoreflection if there exists an element $e_g \in V$ and $f_g \in V^*$ such that

$$g(x) = x + f_g(x)e_g \quad \text{for every } x \in V .$$

Lemma. ([3], ch.V, §5, th.4.) Let B be the symmetric algebra of V and $A = B^G$, the subalgebra of G-invariant elements. Assume that $\#G$ is invertible in k . Then the following assertions are equivalent:

(i) G is generated by pseudoreflections;

(ii) A is a graded polynomial k-algebra.

Example. μ_Q acts on S as a group generated by pseudoreflections. These pseudoreflections act by the formula

$$T_i \to T_i \otimes \overline{X}_i ,$$
$$T_j \to T_j , \quad j \neq i , \quad i = 0,\dots,n .$$

1.3.3. Proposition

(i) $\mathbb{P}(Q)$ is a normal irreducible projective algebraic variety;

(ii) all singularities of $\mathbb{P}(Q)$ are cyclic quotients singularities (in particular, $\mathbb{P}(Q)$ is a V-variety);

(iii) a nonsingular $\mathbb{P}(Q)$ is isomorphic to $\mathbb{P}^r$.

For the proof of property (i) we remark that this property is preserved under an action of a finite group and use interpretation 1.2.2. To see (ii), we use interpretation 1.2.1. Let $\mathbb{P} = \bigcup_{i=0}^{r} U_i$ be the canonical covering of $\mathbb{P}$, where $U_i = D_+(T_i)$. Consider the closed subvariety $V_i = \mathrm{Spec}(S/(T_i - 1))$ of $\mathbb{A}^{r+1} = \mathrm{Spec}(S)$. The action of G_m on $\mathbb{A}^{r+1}$ induces the action of μ_{q_i} on V_i which, after identifying V_i with $\mathrm{Spec}(k[T_0,\ldots,T_{i-1},T_{i+1},\ldots,T_r])$, can be given by the formulas

$$T_j \to T_j \otimes \overline{X}_i^{q_j}, \quad j = 0,\ldots,i-1,i+1,\ldots,r$$

where notations as in 1.2.2.

It is easy to see that $U_i \simeq V_i/\mu_{q_i}$ and, since $V_i \simeq \mathbb{A}^r$, we have property (ii) of $\mathbb{P}(Q)$.

For the proof of (iii) we use 1.3.1 and the previous construction. By 1.3.1 we may assume that $(q_0,\ldots,q_{i-1},q_{i+1},\ldots,q_r) = 1$. Then it is easy to see that the action of μ_{q_i} on V_i is generated by pseudoreflections only in the case $q_i = 1$. It remains to apply 1.3.2.

1.4. Cohomology of $\mathcal{O}_{\mathbb{P}}(n)$.

1.4.1. Recall that $\mathcal{O}_{\mathbb{P}}(n)$ denotes an $\mathcal{O}_{\mathbb{P}}$-Module associated to the graded $S(Q)$-module $S(Q)(n)$. For any homogeneous $f \in S(Q)$ we have a natural homomorphism $S(Q)_n \to S(Q)(n)_{(f)}$ $(a \to a/1)$ which defines a natural homomorphism $\alpha_n : S(Q)_n \to H^0(\mathbb{P}(Q),\mathcal{O}_{\mathbb{P}}(n))$ (the Serre homomorphism).

Theorem.

(i) $\alpha_n : S_n \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ is bijective for any $n \in \mathbb{Z}$;

(ii) $H^i(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)) = 0$ for $i \neq 0,r$, $n \in \mathbb{Z}$;

(iii) $H^r(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)) \simeq S_{-n-|Q|}$.

Proof. According to general properties of projective spectrums we can identify $U = \mathrm{Spec}(S) - \{\mathfrak{m}\}$ with the affine spectrum of the graded $\mathcal{O}_{\mathbb{P}}$-Algebra $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}}(n)$ ([12], 8.3). The corresponding projection $p: U \to \mathbb{P}$ coincides with the quotient morphism $U \to U/G_m$ from 1.2.1. Since p is an affine morphism we have

$$H^i(U,\mathcal{O}_U) \simeq H^i(\mathbb{P}, p_*(\mathcal{O}_U)) \simeq H^i(\mathbb{P}, \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}}(n)) \simeq \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)).$$

Now we use the local cohomology theory ([13]). We have an exact sequence

$$0 \to H^0_{\{\mathfrak{m}\}}(S) \to S \to H^0(U,\mathcal{O}_U) \to H^1_{\{\mathfrak{m}\}}(S) \to 0$$

and isomorphisms

$$H^i_{\{m\}}(S) \simeq H^{i-1}(U,\mathcal{O}_U)\,, \quad i>1\,.$$

It is easy to see that the homomorphism $S \to H^0(U,\mathcal{O}_U)$ induces on each S_n the Serre homomorphism $\alpha_n : S_n \to H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n))$. Since S is a Cohen-Macaulay ring, we have

$$H^i_{\{m\}}(S) = 0\,, \quad i \neq r+1\,.$$

This proves assertions (i), (ii) of the theorem.

For the proof of (iii) we have to use the explicit calculation of $H^{r+1}_{\{m\}}(S)$. We have

$$H^{r+1}_{\{m\}}(S) = \varinjlim_m \operatorname{Ext}^{r+1}(S/(T_0,\dots,T_r)^m,S) =$$
$$= \varinjlim_m \operatorname{Ext}^{r+1}(S/(\underline{T}^m),S)$$

where $(\underline{T}^m) = (T_0^m,\dots,T_r^m)$.

Let V be a free S-module of rang $r+1$ with the basis $(e_0,\dots,e_r)$. Grade V by the condition $\deg(e_i) = q_i m$ and consider the induced gradation on its exterior powers $\overset{p}{\Lambda}(V)$ (where $\deg(e_{i_1}\wedge\dots\wedge e_{i_p}) = m(q_{i_1}+\dots+q_{i_p})$). The Koszul complex for $(\underline{T}^m)$:

$$S \leftarrow \overset{1}{\Lambda}(V) \leftarrow \overset{2}{\Lambda}(V) \leftarrow \dots\dots \leftarrow \overset{r+1}{\Lambda}(V) \leftarrow 0$$

$$e_{i_1}\wedge\dots\wedge e_{i_p} \to \sum_k (-1)^k T^m_{i_k} e_{i_1}\wedge\dots\wedge \hat{e}_{i_k}\wedge\dots\wedge e_{i_p}$$

defines a resolution of graded S-modules for $S/(\underline{T}^m)$ and hence we have an isomorphism of graded S-modules

$$\operatorname{Ext}^{r+1}(S/(\underline{T}^m),S) \simeq \operatorname{Hom}(\overset{r+1}{\Lambda}(V),S)/\operatorname{Im}(\operatorname{Hom}(\overset{r}{\Lambda}(V),S) \simeq$$
$$\simeq (S/(\underline{T}^m))(-m|Q|)$$

Put

$$I_m = (S/(\underline{T}^m))(-m|Q|)\,,$$

then

$$H^{r+1}_{\{m\}}(S) = \varinjlim_m I_m\,,$$

where the inductive system is described as follows.

Let $t^m_{a_0,\dots,a_r}$ be the image of $T_0^{m-a_0}\dots T_r^{m-a_r}$ in I_m. It is clear that for $0 < a_i \leq m$ $t^m_{a_0,\dots,a_r}$ form a basis of I_m. In this notation the transition map

$$u_{m,m+s} : I_m \to I_{m+s}$$

is multiplication by $T_0^s\dots T_r^s$ and

$$u_{m,m+s}(t^m_{a_0,\dots,a_r}) = t^{m+s}_{a_0+s,\dots,a_r+s}\,.$$

Let $\bar{e}_{a_0,\dots a_r}$ be the image of $e^m_{a_0,\dots,a_r}$ in $\varinjlim_m I_m$. Module $H^{r+1}_{\{m\}}(S)$ is a graded S-module and elements $\bar{e}_{a_0,\dots,a_r}$ form its homogeneous basis. We have

$$\deg(\bar{e}_{a_0,\dots,a_r}) = \deg(e^m_{a_0,\dots,a_r}) = (m-a_0)+\dots+(m-a_r)-m|Q| = \sum_{i=0}^{r} a_i q_i$$

Thus we obtain that $\bar{e}_{a_0,\dots,a_r}$ with

$$n = -\sum_{i=0}^{r} a_i q_i \quad (a_i > 0)$$

generate $H^{r+1}_{\{m\}}(S)_n$ as a k-space. Since

$$\dim_k S_{-n-Q} = \#\{(b_0,\dots,b_r) \in \mathbb{N}^{r+1} : -n-|Q| = \sum_{i=0}^{r} b_i q_i\} =$$

$$= \#\{(a_0,\dots,a_r) \in \mathbb{N}^{r+1}_{+} : -n = \sum_{i=0}^{r} a_i q_i\},$$

we have

$$H^{r+1}_{\{m\}}(S)_n \simeq S_{-n-|Q|}.$$

It remains to notice that

$$H^{r+1}_{\{m\}}(S)_n \simeq H^r(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)).$$

1.4.2. Let integers a_n be determined by the identity

$$P_S(t) = \sum_{n=0}^{\infty} a_n t^n = \prod_{i=0}^{r} (1-t^{q_i})^{-1}.$$

Then as a corollary of the previous theorem we have

$$\dim_k H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = \begin{cases} a_n & i = 0 \\ 0 & 1 \neq 0, r \\ a_{-n-|Q|}, & i = r \end{cases}$$

In fact, $P_S(t)$ is the Poincare series of the graded algebra $S(Q)$ (see 3.4) and $a_n = \dim_k S(Q)_n$.

1.5. Pathologies

If $\mathbb{P} = \mathbb{P}^r$ then the following properties are well known.

(i) for any $n \in \mathbb{Z}$ $\mathcal{O}_{\mathbb{P}}(n)$ is an invertible sheaf;

(ii) an invertible sheaf $\mathcal{O}_{\mathbb{P}}(n)$ is ample;

(iii) the homorphism of multiplication $S(n) \otimes S(m) \to S(n+m)$ induces the isomorphism $\mathcal{O}_{\mathbb{P}}(n) \otimes \mathcal{O}_{\mathbb{P}}(m) \simeq \mathcal{O}_{\mathbb{P}}(n+m)$;

(iv) for any graded S-module M and $n \in \mathbb{Z}$

$$\underline{M(n)} \simeq \underline{M} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n)$$

None of these properties is valid for general $\mathbb{P}(Q)$.

1.5.1. Let $Q = \{1,1,2\}$. The restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to $D_+(T_2)$ is given by the

$S_{(T_2)}$-module

$$S(1)_{(T_2)} = \{\frac{a}{T_2^k}:\ a \in S_{2k-1}\} .$$

It is clear that $S(1)_{(T_2)} = S_{(T_2)}\frac{T_0}{T_2} + S_{(T_2)}\frac{T_1}{T_2}$ is not a free $S_{(T_2)}$-module of rang one.

This is a counterexample to property (i).

1.5.2. On a weighted projective line $\mathbb{P}(q_0,q_1)$ all sheaves $\mathcal{O}_{\mathbb{P}}(n)$ are invertible. In fact, $\mathcal{O}_{\mathbb{P}}(n)|D_+(T_i)$ is associated to the $S_{(T_i)}$-module $S(n)_{(T_i)}$, freely generated by T_j^p/T_i^k, where $n = kq_i - pq_j$ and k/n, p/n are integers coprime with q_j and q_i respectively.

Since $\mathbb{P}(q_0,q_1) \simeq \mathbb{P}^1$ (1.3.1), an invertible sheaf $\mathcal{O}_{\mathbb{P}}(n)$ is equal to some $\mathcal{O}_{\mathbb{P}^1}(b_n)$. Moreover, if $\Gamma(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)) \neq 0$, then

$$b_n = \dim_k \Gamma(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)) - 1 .$$

Thus, $\mathcal{O}_{\mathbb{P}}(n)$ is ample if $\dim_k \Gamma(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)) \geq 2$. But, if $n < \min\{q_0,q_1\}$ and $n > 0$, $\Gamma(\mathbb{P},\mathcal{O}_{\mathbb{P}}(n)) = 0$ (1.4.1).

This is a counterexample to property (ii).

1.5.3. In notations of 1.5.2 assume that $q_1 = q_0 + 1$, $q_0 > 1$. Then $b_{q_0} = b_{q_0+q_1+1} = 0$, $b_{q_1+1} < 0$. But

$$\mathcal{O}_{\mathbb{P}}(q_0) \otimes \mathcal{O}_{\mathbb{P}}(q_1+1) = \mathcal{O}_{\mathbb{P}^1}(b_{q_0}) \otimes \mathcal{O}_{\mathbb{P}^1}(b_{q_1+1}) \simeq \mathcal{O}_{\mathbb{P}^1}(b_{q_0}+b_{q_1+1})$$

$$\mathcal{O}_{\mathbb{P}}(q_0+q_1+1) \simeq \mathcal{O}_{\mathbb{P}^1}(b_{q_0+q_1+1}) .$$

This is a counterexample to property (iii).

1.5.4. To obtain a counterexample to property (iv) we can take $M = S(m)$, note that $S(m)(n) = S(m+n)$ and use the counterexample from 1.5.3.

1.5.5. We refer to the paper of Delorme ([7]) for more details concerning properties of the sheaves $\mathcal{O}_{\mathbb{P}}(n)$. For example, one can find there a generalization of the duality theorem for $\mathbb{P}(Q)$, the particular case of which we have proved in 1.4.1.

We remark also that according to Mori ([19]) everything is well in the open set $V = \bigcap_{k>1} D_k$, where $D_k = \bigcup_{k \nmid q_i} D_+(T_i)$. Namely, V is the maximal open subscheme such that $\mathcal{O}_{\mathbb{P}}(1)|V$ is invertible and $(\mathcal{O}_{\mathbb{P}}(1)|V)^{\otimes m} \simeq \mathcal{O}_{\mathbb{P}}(m)|V$, $\forall m \in \mathbb{Z}$.

2. Bott's theorem.

2.1. Sheaves $\bar{\Omega}^i_{\mathbb{P}}$.

2.1.1. Let Ω_S^1 be the S-module of k-differentials of S. This is a free module with a basis $dT_0,\dots,dT_r$. Denote by Ω_S^i its exterior i^{th} power $\overset{i}{\Lambda}(\Omega_S^1)$ (as usual we put $\Omega_S^0 = S$). This is a free S-module with the basis $\{dT_{s_1}\wedge\dots\wedge dT_{s_i} : 0 \le s_1 < \dots < s_i \le r\}$. Grade Ω_S^i by the condition

$$\deg(dT_{s_1}\wedge\dots\wedge dT_{s_i}) = q_{s_1}+\dots+q_{s_i} .$$

We have an isomorphism of graded S(Q)-modules

$$\Omega_S^i \simeq \bigoplus_{0\le s_1<\dots<s_i\le r} S(-q_{s_1} - \cdots -q_{s_i}) .$$

For $i = r$ we obtain

$$\Omega_S^{r+1} \simeq S(-|Q|)$$

Let $d:S \to \Omega_S^1$ be the canonical universal differentiation. By definition of the partial derivatives we have

$$da = \sum_{j=0}^{r} \frac{\partial a}{\partial T_j} dT_j , \quad a \in S .$$

The k-linear map d extends to the exterior differentiation

$$d : \Omega_S^i \to \Omega_S^{i+1}$$

uniquely determined by the conditions

$$d(w\wedge w') = dw\wedge w' + (-1)^i w\wedge dw' , \quad w \in \Omega_S^i , \quad w' \in \Omega_S^j$$

$$d(d(w)) = 0 , \quad \forall w \in \Omega_S^i .$$

2.1.2. Recall the <u>Euler formula</u>:

$$na = \sum_{j=0}^{r} \frac{\partial a}{\partial T_j} q_j T_j , \quad \forall a \in S(Q)_n .$$

Using the linearity of both sides of this identity we may verify this formula only in the case when a is a monomial $T_0^{s_0}\dots T_r^{s_r}$. But in this case it can be done without any difficulties.

2.1.3. Define the homomorphism of grades S-modules

$$\Delta : \Omega_S^i \to \Omega_S^{i-1} , \quad i \ge 1$$

by the formula

$$\Delta(dT_{s_1}\wedge\dots\wedge dT_{s_i}) = \sum_{k=1}^{i} (-1)^{k+1} q_{s_k} T_{s_k} dT_{s_1}\wedge\dots\wedge \widehat{dT_{s_k}}\wedge\dots\wedge dT_{s_i}$$

<u>Lemma</u>.

(i) $\Delta(w\wedge w') = \Delta(w)\wedge w' + (-1)^i w\wedge\Delta(w')$, $w \in \Omega_S^i$, $w' \in \Omega_S^j$;

(ii) $\Delta(da) = na$, $a \in S_n$;

(iii) $\Delta(dw) + d(\Delta(w)) = nw$, $w \in (\Omega_S^i)_n$.

Using the linearity of Δ we may verify (i) only in the case $w = dT_{s_1} \wedge \dots \wedge dT_{s_i}$, $w' = dT_{s'_1} \wedge \dots \wedge dT_{s'_j}$. But this is easy.

Property (ii) is a corollary of the Euler formula. To verify property (iii) it suffices to consider the case $w = adT_{s_1} \wedge \dots \wedge dT_{s_i}$, $a \in S_k$. We have

$$\Delta(dw) = \Delta(da \wedge dT_{s_1} \wedge \dots \wedge dT_{s_i}) = \Delta(da) \wedge dT_{s_1} \wedge \dots \wedge dT_{s_i} - da \wedge \Delta(dT_{s_1} \wedge \dots \wedge dT_{s_i}) =$$
$$= kadT_{s_1} \wedge \dots \wedge dT_{s_i} - da \wedge \Delta(dT_{s_1} \wedge \dots \wedge dT_{s_i})$$

$$d(\Delta(w)) = d(a\Delta(dT_{s_1} \wedge \dots \wedge dT_{s_i}) = da \wedge \Delta(dT_{s_1} \wedge \dots \wedge dT_{s_i}) + ad(\Delta(dT_{s_1} \wedge \dots \wedge dT_{s_i}) =$$
$$= da \wedge \Delta(dT_{s_1} \wedge \dots \wedge dT_{s_i}) + ad(\sum_{\ell=1}^{i} (-1)^{\ell+1} q_{s_\ell} T_{s_\ell} dT_{s_1} \wedge \dots \wedge \widehat{dT_{s_\ell}} \wedge \dots \wedge dT_{s_i}) =$$
$$= da \wedge \Delta(dT_{s_1} \dots dT_{s_i}) + (\sum_{\ell=1}^{i} q_{s_\ell}) adT_{s_1} \wedge \dots \wedge dT_{s_i} \quad .$$

Adding we get

$$\Delta(dw) + d(\Delta(w)) = (k + \sum_{\ell=1}^{i} q_{s_\ell}) w = nw .$$

2.1.4. It is easy to identify the sequence

$$0 \to \Omega_S^{r+1} \xrightarrow{\Delta} \Omega_S^r \to \dots \to \Omega_S^1 \to S$$

with the Koszul complex for the regular sequence $(q_0T_0, \dots, q_rT_r)$. Thus, we obtain that it is an exact sequence.

Now put

$$\overline{\Omega}_S^i = \operatorname{Ker}(\Omega_S^i \xrightarrow{\Delta} \Omega_S^{i-1}) = \operatorname{Im}(\Omega_S^{i+1} \xrightarrow{\Delta} \Omega_S^i)$$

with the induced grading.

So, we have the exact sequences of graded S-modules:

$$0 \to \overline{\Omega}_S^i(n) \to \Omega_S^i(n) \to \overline{\Omega}_S^{i-1}(n) \to 0 , \quad i \geq 1 , \quad n \in \mathbb{Z}$$

2.1.5. Define the sheaf $\overline{\Omega}_{\mathbb{P}}^i$ on $\mathbb{P}(Q)$ by

$\overline{\Omega}_{\mathbb{P}}^i = \underline{\overline{\Omega}_S^i}$, where $\underline{M}$ denotes the sheaf associated to a graded S-module M.

Also we put

$$\overline{\Omega}_{\mathbb{P}}^i(n) = \underline{\overline{\Omega}_S^i(n)} , \quad n \in \mathbb{Z} .$$

Since $M \to \underline{M}$ is an exact functor we have exact sequences of sheaves on $\mathbb{P}(Q)$:

$$0 \to \overline{\Omega}_{\mathbb{P}}^i(n) \to \underline{\Omega_S^i(n)} \to \overline{\Omega}_{\mathbb{P}}^{i-1}(n) \to 0 , \quad i \geq 1 .$$

It is clear that for $i = r+1$, $\overline{\Omega}_{\mathbb{P}}^i(n) = 0$, thus for $i = r$

$$\overline{\Omega}_{\mathbb{P}}^r(n) = \underline{\Omega_S^{r+1}(n)} = \underline{S(n - |Q|)} = \mathcal{O}_{\mathbb{P}}(n - |Q|) .$$

2.1.6. Note that in the case $\mathrm{char}(k) = 0$ property (iii) in lemma 2.1.3 gives an algebraic proof of the acyclicity of the De Rham complex

$$0 \to k \to S \xrightarrow{d} \Omega_S \xrightarrow{d} \cdots \to \Omega_S^{r+1} \to 0 .$$

2.2. Justifications.

In this section we try to convince the reader that the sheaves $\overline{\Omega}^i_{\mathbb{P}}$ introduced in the previous section are good substitutes for the sheaves of germs of differentials $\Omega^i_{\mathbb{P}^r}$ on the usual projective space $\mathbb{P}^r$.

2.2.1. Let $\mathbb{P} = \mathbb{P}^r$. Let us show that

$$\overline{\Omega}^i_{\mathbb{P}}(n) = \Omega^i_{\mathbb{P}^r}(n) .$$

In this case

$$U = V(\mathcal{O}_{\mathbb{P}^r}(-1))^* = \mathrm{Spec}\big(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^r}(n)\big)$$

is the complement to the zero section of the tautological line bundle $V(\mathcal{O}_{\mathbb{P}^r}(-1))$ on $\mathbb{P}^r$ and the canonical morphism $p : U \to \mathbb{P}^r$ is smooth.

The standard exact sequence

$$0 \to p^*\Omega^1_{\mathbb{P}} \to \Omega^1_U \to \Omega^1_{U/\mathbb{P}} \to 0$$

induces the exact sequences

$$0 \to p^*\Omega^i_{\mathbb{P}} \to \Omega^i_U \to \Omega^1_{U/\mathbb{P}} \otimes p^*\Omega^{i-1}_{\mathbb{P}} \to 0 .$$

The homomorphism

$$\Delta : \Omega^1_S \to S \qquad (\sum_i a_i dT_i \to \sum_i a_i q_i T_i)$$

induces after restriction to U a surjective homomorphism of sheaves

$$\Delta : \Omega^1_U \to \mathcal{O}_U$$

(here we use that $(q_i,\mathrm{char}(k)) = 1!$). It is easy to verify that $\Delta(p^*\Omega^1_{\mathbb{P}}) = 0$ and hence Δ defines a surjective homomorphism

$$\tilde{\Delta} : \Omega^1_{U/\mathbb{P}} \to \mathcal{O}_U .$$

Since $\Omega^1_{U/\mathbb{P}}$ is invertible we obtain that $\tilde{\Delta}$ is in fact an isomorphism.

Thus we have exact sequences

$$0 \to p^*\Omega^i_{\mathbb{P}} \to \Omega^i_U \to p^*\Omega^{i-1}_{\mathbb{P}} \to 0$$

and applying p_* we obtain exact sequences

$$0 \to \bigoplus_{n\in\mathbb{Z}} \Omega^i_{\mathbb{P}}(n) \to \bigoplus_{n\in\mathbb{Z}} \underline{\Omega}^i_S(n) \to \bigoplus_{n\in\mathbb{Z}} \Omega^{i-1}_{\mathbb{P}}(n) \to 0 .$$

It is easy to see that in this way we obtain exact sequence 2.1.5 of the definition of $\overline{\Omega}^i_{\mathbb{P}}(n)$.

2.2.2. Lemma. In the notation of 1.3.2, let us assume that G is generated by pseudoreflections and its order is invertible in k. Then the canonical homomorphism

$$\Omega^i_{A/k} \to (\Omega^i_{B/k})^G$$

is an isomorphism of A-modules.

Proof. Since B (resp. A) is a polynomial algebra, the B-module $\Omega^i_{B/k}$ (resp. $\Omega^i_{A/k}$) is a free B-module (resp. A-module). Since B is a free A-module ([3],ch.5, 5, th.5), $\Omega^i_{B/k}$ is a free A-module. Let $\Omega^i_{A/k} \to \Omega^i_{B/k}$ be the canonical homomorphism of A-modules (the inverse image of a differential form). It is injective (because $\Omega^i_{A/k}$ is free and it is injective over a dense open subset of Spec A). Let T be its cokernel and

$$0 \to \Omega^i_{A/k} \to \Omega^i_{B/k} \to T \to 0$$

be the corresponding exact sequence.

Now, for every G-B-module M, the homomorphism $m \to \frac{1}{\#G} \sum_g g(m)$ is a projector onto a direct summand (here we use the assumption that $\#G$ is invertible in k), thus the functor $(\ )^G$ is exact. Applying this functor to the above exact sequence, we get an exact sequence

$$0 \to \Omega^i_{A/k} \to (\Omega^i_{B/k})^G \to T^G \to 0$$

where $(\Omega^i_{B/k})^G$, being a direct summand of a free A-module, is a projective A-module. This shows that dim. proj. $(T^G) \leq 1$ and, hence, depth $(T^G) \geq \dim B - 1$. This implies that $T^G = 0$ if its localization $(T^G)_P = 0$ for any prime P of A of height 1. Let Q be a prime ideal of B such that $Q \cap A = P$ and $G_Q = \{g \in G : g(Q) = Q\}$ be the decomposition group of Q. Then $(T^G)_P = (T_Q)^{G_Q} = \mathrm{Coker}(\Omega^i_{A_P/k} \to (\Omega^i_{B_Q/k})^{G_Q})$. Let G'_Q be the inertia group of Q, the subgroup of G_Q of elements which act trivially in the residue field K of B_Q. Then $B_Q \supset B'_Q = (B_Q)^{G'_Q} \supset (B_Q)^{G_Q} = A_P$, the extension $B'_Q \supset A_P$ is etale, the group G'_Q is a cyclic group of order e equal to the ramification index of the extension $B_Q \supset B'_Q$ ([3],ch.V, 5, n°5). This shows that $\Omega^i_{A_P/k} \otimes_{A_P} A'_Q \simeq \Omega^i_{B'_Q}$ and, hence, $\Omega^i_{A_P} = (\Omega^i_{B'_Q})^{G'_Q/G'_Q}$. Thus, it suffices to show that

$$\Omega^i_{B'_Q} = (\Omega^i_{B_Q})^{G'_Q} .$$

Passing to the completions, we may assume that $B_Q = K[[T]]$, $B'_Q = K[[T^e]]$ and a generator g of G'_Q acts on B_Q by multiplying T by a primitive e-th root of unity ζ. Let $t_1,\ldots,t_{n-1}$ be a separable transcendence basis of K over k. Then

$$\Omega^i_{B'_Q} = \Sigma B'_Q \, dt_{j_1} \wedge \ldots \wedge dt_{j_{i-1}} \, dT^e$$

$$\Omega^i_{B_Q} = \Sigma B_Q \, dt_{j_1} \wedge \ldots \wedge dt_{j_{i-1}} \wedge dT$$

A direct computation shows that $(\Omega^i_{B_Q/k})^{G'_Q} = \Omega^i_{B'_Q/k}$.

2.2.3. Let $a : \mathbb{P}^r \to \mathbb{P}$ be the natural projection $\mathbb{P}^r \to \mathbb{P}^r/\mu_Q = \mathbb{P}(Q)$ from 1.2.2. Let us show that

$$\bar{\Omega}^i_{\mathbb{P}} \simeq a^G_*(\Omega^i_{\mathbb{P}}) ,$$

where $G = \mu_Q$ and a^G_* is the functor of invariant direct image ([11],5.1).

The action of G on $\mathbb{P}^r$ is induced by one on S . Since S^G is a polynomial algebra, this latter action is generated by pseudoreflections and hence, by lemma 2.2.2, we have an isomorphism of $S(Q)$-modules

$$\Omega^i_{S(Q)} \simeq (\Omega^i_S)^G$$

and, hence, an isomorphism of sheaves

$$\underline{\Omega}^i_{S(Q)} \simeq a^G_*(\underline{\Omega}^i_S) .$$

Applying a^G_* to the exact sequence (see 2.1.5)

$$0 \to \Omega^i_{\mathbb{P}^r} \to \underline{\Omega}^i_S \to \Omega^{i-1}_{\mathbb{P}^r} \to 0$$

and using the exactness of a^G_* (p is an affine morphism and $(\ )^G$ is an exact functor) we obtain an exact sequence:

$$0 \to a^G_*(\Omega^i_{\mathbb{P}^r}) \to \Omega^i_{S(Q)} \to a^G_*(\Omega^{i-1}_{\mathbb{P}^r}) \to 0 .$$

Since $a^G_*(\mathcal{O}_{\mathbb{P}^r}) = \mathcal{O}_{\mathbb{P}}$ we obtain by induction that $a^G_*(\Omega^i_{\mathbb{P}^r}) \simeq \bar{\Omega}^i_{\mathbb{P}}$.

2.2.4. Let us show that $\bar{\Omega}^i_{\mathbb{P}}$ coincides with the sheaf $\tilde{\Omega}^i_{\mathbb{P}}$ introduced for any V-variety in [23].

Recall that

$$\tilde{\Omega}^i_{\mathbb{P}} = j_*(\Omega^i_W)$$

where $j : W \to \mathbb{P}$ is the open immersion of the smooth locus of $\mathbb{P}$. In notations of 2.2.3, let us consider a commutative diagram

$$\begin{array}{ccc} a^{-1}(W) & \xrightarrow{\ j'\ } & \mathbb{P}^r \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\ W & \xrightarrow{\ j\ } & \mathbb{P} \end{array}$$

Here $a' = a|a^{-1}(W)$ and j' is the natural immersion. Since W is smooth, the action of μ_Q on $a^{-1}(W)$ is generated locally by pseudoreflections. Then, by lemma 2.2.2, we get

$$\Omega^i_W = a'^G_*(\Omega^i_{a^{-1}(W)}) .$$

Since $\operatorname{codim}(\mathbb{P} - W, \mathbb{P}) \geq 2$ ($\mathbb{P}$ is a normal scheme) and $\mathbb{P}^r$ is smooth,

$$j'_*(\Omega^i_{a^{-1}(W)}) \simeq \Omega^i_{\mathbb{P}^r} .$$

Thus

$$\tilde{\Omega}^i_{\mathbb{P}} = j_*(\Omega^i_W) = j_*(a'^G_*(\Omega^i_{a^{-1}(W)}) = a^G_*(j'_*(\Omega^i_{a^{-1}(W)}) = a^G_*(\Omega^i_{\mathbb{P}^r})$$

2.3. Cohomology of $\overline{\Omega}^i_{\mathbb{P}}(n)$.

2.3.1. Let us consider the graded $S(Q)$-modules $\overline{\Omega}^i_S$, introduced in 2.1.4 and let $H^i_{\{m\}}$ denote the local cohomology group for a S-module M (cf. 1.4).

Proposition.

$$H^j_{\{m\}}(\overline{\Omega}^i_S) = \begin{cases} 0, & j \neq i+1, r+1 \\ k, & j = i+1 \neq r+1 . \end{cases}$$

Proof. We have exact sequences (2.1.4)

$$0 \to \overline{\Omega}^i_S \to \Omega^i_S \to \overline{\Omega}^{i-1}_S \to 0, \qquad i > 1$$

which, after applying the functor $H^i_{\{m\}}$, yield the exact sequences of local cohomology

$$\cdots \to H^{j-1}_{\{m\}}(\Omega^i_S) \to H^{j-1}_{\{m\}}(\overline{\Omega}^{i-1}_S) \to H^j_{\{m\}}(\overline{\Omega}^i_S) \to H^j_{\{m\}}(\Omega^i_S) \to \cdots .$$

Since $\Omega^i_S = S(-n)$ for some $n \in \mathbb{Z}$ and S is a Cohen-Macaulay ring, $H^j_{\{m\}}(\Omega^i_S) = 0$ if $j \neq r+1$. Thus, we have an isomorphism

$$H^j_{\{m\}}(\overline{\Omega}^i_S) \simeq H^{j-1}_{\{m\}}(\overline{\Omega}^{i-1}_S) \quad \text{for } j \neq r+1 .$$

By induction, we obtain

$$H^j_{\{m\}}(\overline{\Omega}^i_S) \simeq H^{j-i+1}_{\{m\}}(\overline{\Omega}_S) .$$

Now, first terms of the Koszul complex from (2.1.4) give an exact sequence

$$0 \to \overline{\Omega}^1_S \to \Omega^1_S \to m \to 0 ,$$

which easily implies that

$$H^1_{\{m\}}(\overline{\Omega}^1_S) = \begin{cases} 0, & 1 \neq 2, \ r+1 \\ k, & 1 = 2 \neq r+1 . \end{cases}$$

This proves the proposition.

Corollary. $\overline{\Omega}^i_S$ is a Cohen-Macaulay S-module if and only if $i = r$.

2.3.2. For any subset $J \subset [0,r] = \{0,\dots,r\}$ denote by $|Q_J|$ the sum $\sum_{j \in J} q_j$. Notice that $|Q_{[0,r]}| = |Q|$ in our old notations. Put $a_n = \dim_k S(Q)_n$.

Theorem. Let $h(j,i;n) = \dim_k H^j(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n))$. Then

$$h(0,i;n) = \sum_{\#J=i} a_{n-|Q_J|} - h(0,i-1;n), \quad i \geq 1, \quad n \in \mathbb{Z}$$

$$h(j,i;n) = 0, \quad \text{if } j \neq 0,i,r, \quad n \in \mathbb{Z}$$

$$h(i,i;0) = 1, \quad i = 0,\dots,r$$

$$h(i,i;n) = 0, \quad n \neq 0, \quad i \neq r,0$$

$$h(r,i;n) = \sum_{\#J=r+1-i} a_{-n-|Q_J|} - h(r,i-1;n), \quad i \geq 0, \quad n \in \mathbb{Z}$$

Proof. Using the same arguments as in the proof of theorem 1.4.1 we obtain the exact sequence

$$0 \to H^0_{\{m\}}(\overline{\Omega}^i_S) \to \overline{\Omega}^i_S \to \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n)) \to H_{\{m\}}(\overline{\Omega}^i_S) \to 0$$

and an isomorphism

$$H^j_{\{m\}}(\overline{\Omega}^i_S) \to \bigoplus_{n \in \mathbb{Z}} H^{j-1}(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n)).$$

Applying 2.3.1 we get that

$$H^0(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n)) = (\overline{\Omega}^i_S)_n = \mathrm{Ker}(\Omega^i_S \xrightarrow{\Delta} \overline{\Omega}^{i-1}_S)_n$$

$$H^j(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n)) = 0, \quad j \neq 0,i,r, \quad n \in \mathbb{Z}$$

$$H^i(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n)) = k, \quad n = 0, \quad i \neq r$$

$$H^i(\mathbb{P}, \overline{\Omega}^i_{\mathbb{P}}(n)) = 0, \quad n \neq 0, \quad i \neq 0,r.$$

Now $\Omega^i_S = \bigoplus_{\#J=i} S(-|Q_J|)$ and Δ is surjective (2.1). So, we get all the assertions except the last one.

Consider exact sequence 2.1.5

$$0 \to \overline{\Omega}^i_{\mathbb{P}}(n) \to \underline{\Omega}^i_S(n) \to \Omega^{i-1}_{\mathbb{P}}(n) \to 0$$

and the corresponding cohomology sequence

$$H^{r-1}(\underline{\Omega}^i_S(n)) \to H^{r-1}(\overline{\Omega}^{i-1}_{\mathbb{P}}(n)) \to H^r(\overline{\Omega}^i_{\mathbb{P}}(n)) \to H^r(\underline{\Omega}^i_S(n)) \to H^r(\Omega^{i-1}_{\mathbb{P}}(n)) \to 0.$$

Since $\underline{\Omega}^i_S(n) \simeq \bigoplus_{\#J=i} \mathcal{O}_{\mathbb{P}}(n-|Q_J|)$ we can apply theorem 1.4.1 and obtain that

$$\dim_k H^r(\underline{\Omega}^i_S(n)) = \sum_{\#J=r+1-i} a_{-n-|Q_J|}.$$

Using this sequence and preceeding results we obtain the last equality.

2.3.3. Corollary.

$$H^0(\overline{\mathbb{P}},\overline{\Omega}^i_{\mathbb{P}}(n)) = 0\,, \qquad \text{if} \quad n < \min\{|Q_J| : \#J = i\}$$

$$H^r(\mathbb{P},\overline{\Omega}^i_{\mathbb{P}}(n)) = 0\,, \qquad \text{if} \quad n > -\min\{|Q_J| : \#J = r+1-i\}\,.$$

2.3.4. Corollary (Bott-Steenbrink). If $n > 0$ then

$$H^j(\mathbb{P},\overline{\Omega}^i_{\mathbb{P}}(n)) \neq 0$$

only when $j = 0$ and $n > \min\{|Q_J| : \#J = i\}$.

2.3.5. Corollary.

$$h(0,i;n) = \sum_{\ell=0}^{i} (-1)^{\ell+i} \sum_{\#J=\ell} a_{n-|Q_J|}$$

$$h(r,i;n) = h(0,r-i;-n)\ .$$

Here the first assertion immediately follows from 2.3.2 and to verify the second one we have to consider the identity

$$h(r,i;n) - h(0,r-i;-n) = h(0,r+1;-n) = \dim_k H^0(\mathbb{P},\overline{\Omega}^{r+1}_{\mathbb{P}}(-n)) = 0\ .$$

2.3.6. Corollary. If $k = \mathbb{C}$, then

$$H^i(\mathbb{P},\mathbb{C}) = \begin{cases} \mathbb{C}\,, & i \text{ even} \\ 0\,, & i \text{ odd} \end{cases}$$

$$h^{p,q}(\mathbb{P}) = \begin{cases} 1\,, & p = q \\ 0\,, & p \neq q \end{cases}\ .$$

This follows from the degeneracy of the spectral sequence $E_1^{p,q} = H^q(\mathbb{P},\overline{\Omega}^p_{\mathbb{P}}) \Rightarrow H^{p+q}(X,\mathbb{C})$ proven by Steenbrink [23].

3. Weighted complete intersections.

3.1. Quasicones.

3.1.1. Let X be a closed subscheme of a weighted projective space $\mathbb{P}(Q)$ and $p : U \to \mathbb{P}(Q)$ be the canonical projection.

The scheme closure of $p^{-1}(X)$ in A^{r+1} is called the affine quasicone over X. The point $0 \in C_X$ is called the vertex of C_X.

Let $\mathcal{J}$ be the Ideal of X in $\mathbb{P}$ then the ideal I of C_X in S is equal to $H^0(U,p^*(\mathcal{J})\otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_U) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P},\mathcal{J}_{\mathcal{O}_{\mathbb{P}}}\otimes\mathcal{O}_{\mathbb{P}}(n))$.

3.1.2. Proposition.

(i) I is a homogeneous ideal of $S(Q)$;

(ii) the maximal ideal m_0 of the vertex of C_X coincides with the irrelevant ideal of the graded ring $S(Q)/I$ and has no immersed components (i.e. $\operatorname{depth}_m(S/I) \geq 1$);

(iii) The closed embedding $\operatorname{Proj}(S/I) \to \operatorname{Proj}(S) = \mathbb{P}$ corresponding to the natural projection $S \to S/I$ determines an isomorphism $\operatorname{Proj}(S/I) \simeq X$;

(iv) I is uniquely determined by the properties above.

This is an easy exercise in the theory of projective schemes, which we omit (it will not be used in the sequel).

3.1.3. An affine variety V is called quasiconic (or quasicone) if there is an effective action of G_m on V such that the intersection of the closures of all orbits is a closed point. This point is called the vertex of a quasicone.

3.1.4. Proposition. Let V be an affine algebraic variety over a field k. The following properties are equivalent:

(i) V is a quasicone;

(ii) $k[V] = \Gamma(V, O_V)$ has a nonnegative grading with $k[V]_0 \simeq k$;

(iii) there is a closed embedding $j : V \to \mathbb{A}^{r+1}$ such that $j(V)$ is invariant with respect to the action of G_m on $\mathbb{A}^{r+1}$ defined as in 1.2.1;

(iv) there is a closed embedding $j : V \to \mathbb{A}^{r+1}$ such that the ideal of $j(V)$ is generated by weighted-homogeneous polynomials with integer positive weights (i.e. homogeneous elements of some $S(Q)$).

The proof consists of standard arguments of the algebraic group theory (cf. [8], [20]).

Corollary. Any affine quasicone is a quasicone. Conversely any quasicone without immersed components in its vertex is an affine quasicone for some $X \subset \mathbb{P}(Q)$.

3.1.5. A closed subscheme $X \subset \mathbb{P}(Q)$ is called quasismooth (with respect to the embedding $X \to \mathbb{P}(Q)$) if its affine quasicone is smooth outside its vertex.

3.1.6. Theorem. A quasismooth closed subscheme $X \subset \mathbb{P}(Q)$ is a V-variety.

Proof. Let C_X be the affine quasicone over X and $x \in X$ be a closed point. In notations of the proof of 1.3.2 let $W_i = V_i \cap C_X$. Let us show that for any $y \in W_i$ over $x \notin W_i$ is nonsingular in y. We have to show that the tangent space $T_{C_X}(y)$ is not contained in the tangent space $T_{V_i}(y)$. Let $p' : C_X^* \to X$ be the restriction of p to $C_X^* = C_X - \{0\}$ and $F = p'^{-1}(x)_{red}$. The fibre F is an orbit of the point y with respect to the action of G_m restricted to C_X^*. If $(\bar{y}_0,\ldots,\bar{y}_{i-1},1,\bar{y}_{i+1},\ldots,\bar{y}_r)$ denotes the coordinates of y, then F coincides with the image of the map $G_m = \mathrm{Spec}(k[t,t^{-1}]) \to \mathbb{A}^{r+1} = \mathrm{Spec}(S)$ which is given by the formula:

$$(T_0,\ldots,T_{i-1},T_i,T_{i+1},\ldots,T_r) \to (\bar{y}_0 t^{q_0},\ldots,\bar{y}_{i-1}t^{q_{i-1}},t^{q_i},\bar{y}_{i+1}t^{q_{i+1}},\ldots,\bar{y}_r t^{q_2}).$$

The tangent line to the curve F is the image of the corresponding tangent map and defined by the equation

$$T_0 - \bar{y}_0 = q_0\bar{y}_0,\ldots,T_{i-1} - \bar{y}_{i-1} = q_{i-1}\bar{y}_{i-1},T_i - 1 = q_i,\ldots,T_r - \bar{y}_r = q_r\bar{y}_r .$$

It is clear that $T_F(y) \not\subset T_{V_i}(y) = V_i$ and, since $T_F(y) \subset T_{C_X}(y)$, we obtain that y is a nonsingular point of W_i.

The end of the proof is the same as in the proof of 1.3.2: we obtain that $U_k \subset X$ is locally isomorphic to the quotient of the nonsingular variety W_i by the isotropy group $G_y \subset G_m$ of the point y_i.

3.2. Weighted complete intersections.

3.2.1. Assume that the ideal $I \subset S$ of the affine quasicone C_X of $X \subset \mathbb{P}$ is generated by a regular sequence of homogeneous elements of the ring $S(Q)$. If $d_1,\ldots,d_k$ are the degrees of these elements then we say that X is a weighted complete intersection of multidegree $\underline{d} = (d_1,\ldots,d_k)$ and denote X by $V_{\underline{d}}(Q)$.

In case I is a principal ideal (F) and $F \in S(Q)_d$ we say that X is a weighted hypersurface of degree d and denote X by $V_d(Q)$.

3.2.2. In the sequel, C_X^* will denote the punctured affine quasicone $C_X - \{0\}$. Let $p : C_X^* \to X$ be the corresponding projection.

Lemma. Assume that $X = V_{\underline{d}}(Q)$ is quasismooth. Then

(i) $\mathrm{Pic}(C_X^*) = 0$ if $\dim X \geq 3$;

(ii) any G_m-equivariant etale covering of C_X^* is trivial if $\dim X \geq 2$;

(ii)' $\pi_1(C_X^*) = 0$ if $k = \mathbb{C}$ and $\dim X \geq 2$;

(iii) $H^i(C_X^*,\mathcal{O}_{C_X^*}) = 0$, $0 < i < \dim X$.

Proof. (i) Since the local ring $\mathcal{O}_{C_X,0}$ is a complete intersection ring of dimension 4, regular outside its maximal ideal, it is a factorial ring ([13], exp.XI). This shows that $\mathrm{Pic}(C_X^*) = \mathrm{Pic}(C_X)$. The latter group, being isomorphic to the group of classes of invertible divisorial ideals of a graded commutative ring, is trivial ([10]). This proves (i).

(ii) A similar reference ([13], exp.X) shows that $\mathcal{O}_{C_X 0}$ is pure. Hence, every etale covering of C_X^* is a restriction of an etale covering of C_X. Moreover, the same is true for G_m-equivariant coverings. Let $f: Y \to C_X$ be an irreducible G_m-equivariant etale covering of C_X. Then $Y = \mathrm{Spec}\, B$, where $B = \bigoplus_{n\in\mathbb{Z}} B_n$ is an integral $\mathbb{Z}$-graded k-algebra, and f is defined by an inclusion of graded rings $k[C_X] \subset B$. Let $m = k[C]_{>0} = \bigoplus_{n>0} k[C_X]_n$ be the maximal ideal of the vertex $o \in C_X$. Then $mB \subset B_{>0}$ and B/mB is a finite separable k-algebra. Since B is integral, this easily implies that $B_m = 0$ for $m < 0$ and B_0 is a finite algebra over $k[C_X]/m = k$. Since B_0 is a subalgebra of an integral algebra B, this implies that B_0 is a field. Thus, we obtain that Y is a quasicone and its vertex is the only point lying over the vertex of C_X. Because f is etale and k is algebraically closed, this implies that f is an isomorphism. This proves (ii).

(ii)' Let C_X be G_m-equivariantly embedded into $\mathbb{C}^n$. The subgroup $\mathbb{R}_+$ of positive real numbers of the group $\mathbb{C}^*$ acts freely on C_X^*. Intersecting every $\mathbb{R}_+$-orbit with a sphere S_ε^{2n-1} of small radius ε with the center at the origin, we get a map $C_X^* \to \mathbb{R}_+ \times K_\varepsilon$, where $K_\varepsilon = S^{2n-1} \cap C_X$. It is easily verified that this map is a diffeomorphism of C_X^* onto $\mathbb{R}_+ \times K_\varepsilon$. Now, since the vertex of C_X is a complete intersection isolated singularity, the space K_ε is (d-2)-connected ($d = \dim C_X = \dim X + 1$) (see [14,18]). Thus, $\pi_1(C_X^*) \simeq \pi_1(K_\varepsilon) = 0$ if $\dim X \geq 2$.

To verify (iii) we again use the local cohomology theory. Since $C_X = \mathrm{Spec}(S/I)$ is affine,

$$H^i(C_X^*, \mathcal{O}_{C_X^*}) = H^{i+1}_{\{0\}}(C_X, \mathcal{O}_{C_X}) = H^{i+1}_{\{m_0\}}(S/I).$$

Since S/I, being a quotient of a regular ring by a regular sequence, is a Cohen-Macaulay ring, $H^{i+1}_{\{m_0\}}(S/I) = 0$, if $i+1 \neq \dim(S/I) = \dim X + 1$.

3.2.3. Remark. If $\mathrm{char}\, k > 0$, then $\pi_1^{alg}(C_X^*)$ may be not trivial. For example, $\pi_1^{alg}(\mathbb{A}^n - \{0\}) \neq 0$, because A^n has nontrivial etale coverings.

3.2.4. Theorem. Under the conditions of the lemma

(i) $\mathrm{Pic}(X) \simeq \mathbb{Z}$, if $\dim X \geq 3$;

(ii) $\pi_1^{alg}(X) = 0$, if $\dim X \geq 2$;

(ii)' $\pi_1(X) = 0$, if $k = \mathbb{C}$ and $\dim X \geq 2$;

(iii) $H^i(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, $0 < i < \dim X$.

Proof. Let L be an invertible sheaf on X. Since $\mathrm{Pic}(C_X^*) = 0$, $p^*(L) = \mathcal{O}_{C_X^*}$ and is determined as a G_m-sheaf by some character $\chi_L \in H^1(G_m, \underline{\mathrm{Aut}}(\mathcal{O}_{C_X^*})) = H^1(G_m, G_m) \simeq \simeq \mathbb{Z}$. In this way we obtain a homomorphism $f: \mathrm{Pic}(X) \to \mathbb{Z}$. If $p^*(L) = p^*(L')$ as G_m-sheaves, then $L = p_*^{G_m}(p^*(L)) = L' = p_*^G(p^*L'))$ and, hence, f is injective. This proves (i).

Let X' be an etale finite covering of X and A be a corresponding $\mathcal{O}_X$-Algebra (i.e. $X' = \mathrm{Spec}(A)$). Since the covering $\overline{X}' = X' \times_X C_X^*$ of C_X^* is a

G_m-equivariant covering, by lemma 3.2.2 (ii) we get that $\overline{X}' = \mathrm{Spec}(p^*(A))$ is trivial.

Hence the G_m-$\mathcal{O}_{C_X^*}$-Algebra $\mathcal{B} = p^*(A) = A_{\mathcal{O}_X} \otimes \mathcal{O}_{C_X^*}$ splits, i.e. $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_i$ are nontrivial. Since G_m is connected, the Subalgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are invariant Subalgebras and we have a splitting of G_m-Algebras $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. Applying $p_*^{G_m}$, we obtain that $A = p_*^{G_m}(\mathcal{B}) = p_*^{G_m}(\mathcal{B}_1) \times p_*^{G_m}(\mathcal{B}_2)$ splits. This shows that the covering X' splits and proves (ii).

To prove (ii)' we apply Lemma 3.2.2 (ii) and notice that the canonical homomorphism $\pi_1(C_X^*) \to \pi_1(X)$ is surjective because the fibres of $C_X^* \to X$ are pathwise connected.

To prove (iii) we note that

$$H^i(C_X^*, \mathcal{O}_{C_X^*}) = H^i(C_X^*, \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_U) = H^i(X, \mathcal{O}_X \otimes p_* \mathcal{O}_U) = \bigoplus_{n \in \mathbb{Z}} H^i(X, \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n)) .$$

But $\mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n) \simeq \mathcal{O}_X(n)$ and we can apply 3.2.2 and obtain (iii).

3.2.5. Remark. The proof of (i) easily gives that $\mathrm{Pic}(X)$ is generated by some $\mathcal{O}_X(n)$, where, in general, $n \neq 1$. For example, $\mathrm{Pic}(\mathbb{P}(1,\ldots,1,n))$ is generated by $\mathcal{O}_{\mathbb{P}}(2)$.

3.2.6. One can also prove 3.2.4 (and its generalizations to torical spaces) using the methods of [13] (cf. [9]).

3.3. The dualizing sheaf.

3.3.1. Recall that according to Grothendieck for any normal integral projective Cohen-Macaulay variety X there is a sheaf ω_X (the dualizing sheaf) such that

$$H^i(X,F) = (\mathrm{Ext}^{n-i}(X;F,\omega_X))^* \qquad (n = \dim X)$$

for any coherent $\mathcal{O}_X$-Module F. The sheaf ω_X can be determined as the sheaf of germs of differential forms which are regular at nonsingular points of X (see, for example [16]).

In other words,

$$\omega_X = j_*(\Omega_Z^n)$$

where $j : Z \to X$ is the open immersion of the nonsingular locus of X.

In this section we shall compute ω_X for a quasismooth weighted complete intersection.

3.3.2. Lemma. Let X be a closed quasismooth subscheme of $\mathbb{P}$, C_X be its projecting quasicone, Z be the nonsingular locus of X, then

$$p_*^{G_m}(\Omega^1_{C_X^*/X})\,|\,Z \simeq \mathcal{O}_Z .$$

Proof. The embedding of smooth schemes $C_X^* \to U$ defines an exact sequence

$$0 \to J/J^2 \to \Omega^1_{U/\mathbb{P}} \otimes_{\mathcal{O}_U} \mathcal{O}_{C_X^*} \to \Omega^1_{C_X^*/X} \to 0$$

where J is the ideal sheaf of C_X^* in U. Consider the surjective homomorphism $\tilde{\Delta} : \Omega^1_{U/\mathbb{P}} \to \mathcal{O}_U$ from 2.2.1 (its construction does not use the assumption that $\mathbb{P} = \mathbb{P}^r$, the latter is used only to check that $\tilde{\Delta}$ is an isomorphism). It is easy to see that the induced map $\bar{\Delta} : \Omega^1_{C_X^*/X} \to \mathcal{O}_{C_X^*}$ is well defined and is surjective.

Since $p_*^{G_m}$ is exact, the map

$$p_*^{G_m}(\bar{\Delta}) : p_*^{G_m}(\Omega^1_{C_X^*/X}) \to \mathcal{O}_X$$

is surjective. Thus, it is sufficient to show that the restriction of the left hand side sheaf to Z is an invertible sheaf.

This verification is local. Let $x \in Z$ and Z_x be its neighbourhood of the form W/G where W is a nonsingular subvariety of C_X^* of codimension 1 and G is a finite subgroup of G_m constructed in the proof of theorem 3.1.6.

Since W is regularly embedded in C_X^*, we have an exact sequence:

$$0 \to N_{W/C_X} \to \Omega^1_{C_X^*/X} \otimes_{\mathcal{O}_{C_X^*}} \mathcal{O}_W \to \Omega^1_{W/Z_x} \to 0 .$$

Since N_{W/C_X} is locally free of rank 1 we may assume (replacing W by smaller one) that $N_{W/C_X} \simeq \mathcal{O}_W$.

It is clear that

$$p_*^{G_m}(\Omega^1_{C_X^*/X} \otimes_{\mathcal{O}_{C_X}} \mathcal{O}_W) = p_*^{G}(\Omega^1_{C_X^*/X} \otimes_{\mathcal{O}_{C_X^*}} \mathcal{O}_W) .$$

Since x is nonsingular, G acts by pseudoreflections and hence $p_*^G(\Omega^1_{W/Z_X}) = 0$ (see the proof of 2.2.2). Applying $p_*^{G_m}$ to the above sequence we obtain

$$p_*^{G_m}(\Omega^1_{C_X^*/X})|Z_x = p_*^G(\mathcal{O}_W) = \mathcal{O}_{Z_x} .$$

This proves the lemma.

3.3.3. Proposition. In conditions of 3.3.2

$$\omega_X = p_*^{G_m}(\Omega^{n+1}_{C_X^*}) \qquad (n = \dim X) .$$

Proof. Since X is a normal Cohen-Macaulay variety (it follows easily from 3.1.6), by 3.3.1 it is sufficient to show that

$$p_*^{G_m}(\Omega^{n+1}_{C_X^*})|Z \simeq \Omega^n_Z .$$

Consider the exact sequence

$$0 \to p^*\Omega^1_Z \to \Omega^1_{C_X^*}|p^{-1}(Z) \to \Omega^1_{C_X^*}|p^{-1}(Z) \to 0 .$$

Applying $p_*^{G_m}$ and using 3.3.2 we obtain the exact sequence

$$0 \to \Omega_Z^1 \to p_*^{G_m}(\Omega^1_{C_X^*})|Z \to \mathcal{O}_Z \to 0 .$$

Taking the exterior power we get

$$\Omega_Z^n \simeq \Omega_Z^n \otimes_{\mathcal{O}_Z} \mathcal{O}_Z \simeq p_*^{G_m}(\Omega^{n+1}_{C_X^*})|Z$$

q.e.d.

3.3.4. Theorem. Let $X = V_{\underline{d}}(Q)$ be a quasismooth weighted complete intersection of multidegree $\underline{d} = (d_1,\dots,d_s)$. Then

$$\omega_X \simeq \mathcal{O}_X(|\underline{d}| - |Q|)$$

where $|\underline{d}| = d_1+\dots+d_s$.

Proof. Let I be the ideal of the projecting quasicone over X and $B = S/I$. There is an isomorphism of graded A-modules

$$I/I^2 = B(-d_1) + \cdots + B(-d_s) .$$

The exact sequence

$$0 \to I/I^2 \to \Omega_S^1 \otimes_S B \to \Omega_B^1 \to 0$$

gives the homomorphism

$$f : \Omega_B^{r+1-s}(-|\underline{d}|) = \overset{s}{\Lambda}(I/I^2) \otimes_B \overset{r+1-s}{\Lambda}(\Omega_B^1) \to \overset{2+1}{\Lambda}(\Omega_S^1) \otimes_S B = B(-|Q|) .$$

Since C_X^* is smooth, the restriction of f to C_X^* is an isomorphism. Hence

$$\Omega^{r+1-s}_{C_X^*} = B(|\underline{d}| - |Q|) .$$

It remains to use the above proposition.

3.4. The Poincare series.

3.4.1. Let $A = \bigoplus_{n\geq 0} A_n$ be a graded k-algebra of finite type. Then its Poincare series is defined by

$$P_A(t) = \sum_{n=0}^{\infty} (\dim_k A_n) t^n .$$

If $x_0,\dots,x_r$ are homogeneous generators of A and $q_0,\dots,q_r$ are its degrees, then $P_A(t)$ is a rational function of the form

$$P_A(t) = F(t)/\prod_{i=0}^{r} (1 - t^{q_i})$$

where $F(t)$ is a polynomial ([2], 11.1).

3.4.2. Assume that $A = S(Q)$ is a graded polynomial k-algebra. Then ([3], ch.V, §5, n°1)

$$P_{S(Q)}(t) = 1/\prod_{i=0}^{r}(1-t^{q_i}) .$$

Let $f_1,\ldots,f_s$ be a regular sequence of homogeneous elements of the ring $S(Q)$ and $d_1,\ldots,d_r$ be its degrees, let $A = S(Q)/(f_1,\ldots,f_s)$. Then

$$P_A(t) = \prod_{i=1}^{s}(1-t^{d_i})/\prod_{i=0}^{r}(1-t^{q_i}) .$$

This formula follows from the above formula. Put $A^0 = S(Q)$, $A^i = S(Q)/(f_1,\ldots,f_i)$. Then $A^i = A^{i-1}/(f_i)$ and obviously

$$t^{d_i}P_{A^{i-1}}(t) + P_{A^i}(t) = P_{A^{i-1}}(t) .$$

Thus

$$P_{A^i}(t) = (1-t^{d_i})P_{A^{i-1}}(t) , \qquad i = 1,\ldots,s$$

and we obtain our formula.

3.4.3. For $X = \mathrm{Proj}(A)$ we put

$$P_X(t) = \sum_{n=0}^{\infty}(\dim_k H^0(X,\mathcal{O}_X(n)))t^n .$$

Lemma. Let $m = \bigoplus_{n>0} A_n$ be the irrelevant ideal of A. Assume that $\mathrm{depth}_m(A) \geq 2$ (for example, A is normal). Then

$$P_A(t) = P_X(t) .$$

The same argument as in the proof of 3.2.4 (iii) and 1.4.2(i) shows that the Serre homomorphism of graded algebras

$$A \to \bigoplus_{n\in\mathbb{Z}} H^0(X,\mathcal{O}_X(n))$$

is bijective.

3.4.4. Theorem. Let $X = V_{\underline{d}}(Q)$ be a quasismooth weighted complete intersection, $P_X(t) = \sum_{n=0}^{\infty} a_n t^n$ be the power series defined above. Then

$$P_X(t) = \prod_{i=1}^{s}(1-t^{d_i})/\prod_{i=0}^{r}(1-t^{q_i}) .$$

Corollary. Define $p_g(X) = \dim_k H^{\dim X}(X,\mathcal{O}_X)$, then in notations of the theorem

$$p_g(V_{\underline{d}}(Q)) = a_{|\underline{d}| - |Q|} .$$

Indeed, since $\omega_X = \mathcal{O}_X(|\underline{d}| - |Q|)$ is the dualizing sheaf (3.3.4) we have that $\dim_k H^{\dim X}(X,\mathcal{O}_X) = \dim_k \mathrm{Hom}(\mathcal{O}_X,\omega_X) = \dim_k H^0(X,\omega_X) = a_{|\underline{d}| - |Q|}$.

3.5. Examples.

3.5.1. We shall say that two closed subvarieties $X \subset \mathbb{P}$ and $X' \subset \mathbb{P}'$ are affine isomorphic if their affine quasicones are isomorphic and projectively isomorphic if their quasicones are G_m-isomorphic. It is clear that in general there are only two implications

projectively isomorphic $\Rightarrow$ affine isomorphic

projectively isomorphic $\Rightarrow$ isomorphic

between these three notions.

3.5.2. Weighted plane curves. A quasismooth hypersurface $X = V_d(Q)$ in a weighted projective plane $\mathbb{P}(q_0,q_1,q_2)$ is a smooth projective curve. Its dualizing sheaf coincides with the canonical sheaf Ω^1_X and we have (3.3.4):

$$\Omega^1_X = \mathcal{O}_X(d-q_0-q_1-q_2) .$$

Its genus is calculated by the formula

$$g = \text{coefficient at } t^{d-|Q|} \text{ in the formal series}$$
$$(1-t^d)/\prod_{i=0}^{2}(1-t^{q_i}) .$$

The affine quasicone of such a curve is given by a weighted-homogeneous equation $f(x_0,x_1,x_2) = 0$ with an isolated singularity at the origin. Such singularities were studied by many authors ([1,8,18,20]).

Let

$$m = d - q_0 - q_1 - q_2 .$$

Each weighted plane curve with $m < 0$ is affine isomorphic to one of the following curves

$\mathbb{P}(q_0,q_1,q_2)$	d	Equation	Name
$\mathbb{P}(1,1,1)$	1	$x_0 = 0$	
$\mathbb{P}(1,k,k)$	2k	$x_0^{2k}+x_1^2+x_2^2 = 0$	A_{2k-1}, $k \geq 1$
$\mathbb{P}(2,2k+1,2k+1)$	4k+2	$x_0^{2k+1}+x_1^2+x_2^2 = 0$	A_{2k}, $k \geq 1$
$\mathbb{P}(2,k-2,k-1)$	2k-2	$x_0^{k-1}+x_1^2x_0+x_2^2 = 0$	D_k, $k \geq 4$
$\mathbb{P}(3,4,6)$	12	$x_0^4+x_1^3+x_2^2 = 0$	E_6
$\mathbb{P}(4,6,9)$	18	$x_0^3x_1+x_1^3+x_2^2 = 0$	E_7
$\mathbb{P}(6,10,15)$	30	$x_0^5+x_1^3+x_2^2 = 0$	E_8

The equations of corresponding projecting quasicones are well known two-dimensional singularities, which are called the platonic singularities, Du Val singularities, Klein singularities, ADE singularities, double rational singularities, simple singularities, 0-modal singularities.

Note that any curve which is affine isomorphic to a curve of type D_k or E is projectively isomorphic to this curve.

It is clear that all such curves with $m < 0$ are isomorphic to $\mathbb{P}^1$.

When $m = 0$ each weighted plane curve is projectively isomorphic to one of the following curves:

$\mathbb{P}(q_0,q_1,q_2)$	d	Equation	Name
$\mathbb{P}(1,1,1)$	3	$x_0^3 + x_1^3 + x_2^3 + ax_0x_1x_2 = 0, \quad a^3 + 27 \neq 0$	$\tilde{E}_6$ or P_8
$\mathbb{P}(1,1,2)$	4	$x_0^4 + x_1^4 + x_2^2 + ax_0^2x_1^2 = 0, \quad a^2 - 4 \neq 0$	$\tilde{E}_7$ or X_9
$\mathbb{P}(1,2,3)$	6	$x_0^6 + x_1^3 + x_2^2 + ax_1^2x_2^2 = 0, \quad 4a^3 + 27 \neq 0$	$\tilde{E}_8$ or J_{10}

It can be shown (V. I. Arnold) that for any fixed m there is only a finite number of collections $(q_0,q_1,q_2;d)$ for which there is a smooth weighted plane curve $V_d(q_0,q_1,q_2)$.

For $m = 1$ there are exactly 31 collections. The corresponding affine quasicones have a canonical quasihomogeneous singularity embeddable in $\mathbb{A}^3$. There is a natural correspondence between the 31 collections and the 31 possible signatures of the Fuchsian groups of the first kind with compact quotient for which the algebra of automorphic forms is generated by three elements ([8,25]).

Of course, a general smooth projective curve is not isomorphic to any weighted plane curve.

3.5.3. Surfaces. There are no classification results in this case, there are only some interesting examples.

Let $f(x_0,x_1,x_2) = 0$ be an equation of a smooth weighted plane curve $V_d(Q)$. Then the equation

$$f(x_0,x_1,x_2) + x_3^d = 0$$

defines a quasismooth hypersurface $V_d(q_0,q_1,q_2,1)$.

For curves with $m = 0$ we obtain in this way dell Pezzo surfaces [15] of degree 3, 2 and 1 respectively (M. Reid).

For curves with $m = 1$ we obtain simply-connected projective surfaces with the dualizing sheaf $\omega_X \approx \mathcal{O}_X$. Resolving its singularities (which are double rational points) we get minimal models of nonsingular K3-surfaces. One example of such a surface is the following Klein surface:

$$V_{42}(6,14,21,1) : x_0^7 + x_1^3 + x_2^2 + x_3^{42} = 0 .$$

This surface has 3 singular points

$(1,-1,0,0)$ of type A_1

$(0,-1,1,0)$ of type A_6

$(-1,0,1,0)$ of type A_2 .

For any such surface the complement to the curve $x_3 = 0$ is isomorphic to the affine surface with an equation

$$f(x_0,x_1,x_2) + 1 = 0$$

which is diffeomorphic to the Milnor space F_θ for the singularity $f(x_0,x_1,x_2) = 0$ ([18]). This fact can be used for the explanation of some observations in the singularity theory by means of the theory of algebraic surfaces (see [21]).

3.5.4. Multiple spaces. Let $X \to \mathbb{P}^{r-1}$ be a finite Galois covering with a cyclic automorphism group of order m branched along a smooth surface $W \subset \mathbb{P}^{r-1}$ of degree d. Let $f(x_0,\dots,x_{r-1}) = 0$ be the equation of W. Assume that $(d,\mathrm{char}(k)) = 1$. Then X is isomorphic to a weighted quasismooth hypersurface

$$V_d(Q) : f(x_0,\dots,x_{r-1}) + x_r^m = 0$$

where $Q = \{1,\dots,1,d/m\}$.

It is easy to see that such X is smooth. From 3.2.4 we obtain that all such varieties are simply-connected if $r \geq 3$ (i.e. $\pi_1^{alg}(X) = 0$ or $\pi_1(X) = 0$ if $k = \mathbb{C}$) (cf. [22]). Moreover, $\mathrm{Pic}(X) \simeq \mathbb{Z}$ if $r \geq 4$.

The Poincare series $P_X(t)$ has the form (3.4.4):

$$P_X(t) = (1-t^d)/(1-t)^r(1-t^{d/m}) = (1+t^{d/m}+\dots+t^{d(m-1)/m})/(1-t)^r .$$

In particular,

$$p_g(X) = \text{the coefficient at } t^{d-r-d/m} = \sum_{s=0}^{m-1} \begin{pmatrix} \frac{d(m-1-s)}{m} - 1 \\ r-1 \end{pmatrix} .$$

For example

$m = 2$, $r = 2$ (hyperelliptic curve) $p_g = d/2 - 1$

$m = 2$, $r = 3$, $d = 6$ (K3-surface) $p_g = 1$.

It is very useful for the construction problems in algebraic geometry to consider also weighted multiple planes, cyclic coverings of weighted projective spaces. For example, the Klein surface from 3.5.3 is such a multiple plane.

4. The Hodge structure on the cohomology of weighted hypersurfaces.

4.1. A resolution of $\tilde{\Omega}_X^i$.

Let $X = V_N(Q)$ be a quasismooth weighted hypersurface, C_X its affine quasicone, $I \subset S(Q)$ the ideal of C_X, $f \in S(Q)_N$ its generator, $A = S(Q)/I$ the coordinate ring of C_X, m_0 the maximal ideal of the vertex of C_X, $C_X^* = C_X - \{0\}$.

Since X is a V-variety (3.1.6) its cohomology has (in case $k = \mathbb{C}$) a pure Hodge structure and the corresponding Hodge numbers are calculated by the formula (see [23])

$$h^{p,q}(X) = \dim_k(H^q(X,\tilde{\Omega}_X^p)) .$$

In this section we shall construct a suitable resolution for the sheaf $\tilde{\Omega}^p_X$.

4.1.1. Define a k-linear map

$$d_f : \Omega^i_S \to \Omega^{i+1}_S , \qquad i \geq 0$$

setting for homogeneous elements of the S-module Ω^i_S (2.1.1)

$$d_f(w) = fdw + (-1)^{i+1}\frac{|w|}{N} w\wedge df ,$$

where $|w|$ denotes for brevity the degree of w.

Lemma.

(i) $d_f(w\wedge w') = d_f(w)\wedge w' + (-1)^i w\wedge d_f(w')$, $w \in \Omega^i_S$, $w' \in \Omega^j_S$;

(ii) $d_f(d_f(w)) = 0$, $\forall w \in \Omega^i_S$;

(iii) $d(d_f(w)) = (1 + \frac{|w|}{N})df\wedge dw$, $\forall w \in \Omega^i_S$;

(iv) $d_f(dw) = \frac{|w|}{N} df\wedge dw$, $\forall w \in \Omega^i_S$;

(v) $d_f((\Omega^i_S)_n) \subset (\Omega^{i+1}_S)_{n+N}$.

This is directly verified.

Let us show that d_f induces a linear map of S-modules $\overline{\Omega}^i_S = \mathrm{Ker}(\Omega^i_S \xrightarrow{\Delta} \Omega^{i-1}_S)$ (2.1.4).

Lemma (continuation).

(vi) $d_f(\Delta(w)) = -\Delta(d_f(w))$, $w \in \Omega^i_S$;

(vii) $d_f(\overline{\Omega}^i_S) \subset \overline{\Omega}^{i+1}_S$.

It is clear that (vii) follows from (vi). Let us prove (vi). Recalling properties of the map Δ (2.1.3), we obtain

$$\begin{aligned}\Delta(d_f(w)) &= \Delta(fdw + (-1)^{i+1}\frac{|w|}{N} w\wedge df) = f\Delta(dw) + (-1)^{i+1}\frac{|w|}{N}\Delta(w\wedge df) = \\ &= -fd(\Delta(w)) + f|w|w + (-1)^{i+1}\frac{|w|}{N}\Delta(w)\wedge df - |w|fw = \\ &= -fd(\Delta(w)) + (-1)^i\frac{|\Delta(w)|}{N}\Delta(w)\wedge df = -d_f(\Delta(w)) .\end{aligned}$$

4.1.2. Properties (ii) and (v) of the lemma make possible to introduce the following complex $R^{\bullet}_i$ of graded S-modules:

$$R^k_i = \Omega^k_S((k-i)N)$$

$$d_k = (-1)^k d_f : R^k_i \to R^{k+1}_i .$$

Property (vi) implies that the homomorphisms $\Delta : \Omega^k_S \to \Omega^{k-1}_S$ determine morphisms of complexes

$$\Delta : R^{\bullet}_i \to R^{\bullet}_{i-1}[-1] .$$

Property (vii) shows that

$$\overline{R}_i^{\bullet} = (\overline{\Omega}_S^k((k-i)N))_k$$

is a subcomplex of $R_i^{\bullet}$ such that

$$\overline{R}_i^{\bullet} = \mathrm{Ker}(R_i^{\bullet} \to R_{i-1}^{\bullet}[-1])$$

$$\overline{R}_{i-1}^{\bullet}[-1] = \mathrm{Im}(R_i^{\bullet} \to R_{i-1}^{\bullet}[-1]) \; .$$

Thus we have the exact sequence of complexes of graded S-modules:

$$0 \to \overline{R}_i^{\bullet} \to R_i^{\bullet} \to \overline{R}_{i-1}^{\bullet}[-1] \to 0\,, \qquad i \in \mathbb{Z}\,.$$

4.1.3. The multiplication by f defines the inclusion of graded S-modules

$$\Omega_S^k \to \Omega_S^k(N)\,, \qquad \overline{\Omega}_S^k \to \overline{\Omega}_S^k(N)\,,$$

which induces the inclusion of complexes

$$R_i^{\bullet} \to R_{i-1}^{\bullet}\,, \qquad \overline{R}_i^{\bullet} \to \overline{R}_{i-1}^{\bullet}\;.$$

Consider the corresponding quotient complexes

$$K_i^{\bullet} = R_{i-1}^{\bullet}/R_i^{\bullet}\,, \qquad \overline{K}_i^{\bullet} = \overline{R}_{i-1}^{\bullet}/\overline{R}_i^{\bullet}\;.$$

The exact sequence of complexes from 4.1.2 induces the exact sequence of complexes of graded S-modules:

$$0 \to \overline{K}_i^{\bullet} \to K_i^{\bullet} \to K_{i-1}^{\bullet}[-1] \to 0\;.$$

4.1.4. Lemma (De Rham). Let A be a commutative ring, $w \in A^{r+1}$ be a regular sequence of elements of A, $h \in \overset{p}{\Lambda}(A^{r+1})$, $p \le r$. Then $w \wedge h = 0$ iff $\exists\, \beta \in \overset{p-1}{\Lambda}(A^{r+1})$ such that $h = w \wedge \beta$.

This is a reformulation of the theorem of acyclicity of the Koszul complex for a regular sequence.

We shall use this lemma in the following situation: A is the coordinate ring of C_X, $A^{r+1} = \Omega_S^1/f\Omega_S^1$, w is the image of df in $\Omega_S^1/f\Omega_S^1$.

Since C_X^* is smooth, the jacobian ideal

$$\theta_f = \left(\frac{\partial f}{\partial T_0}, \dots, \frac{\partial f}{\partial T_0} \right) \subset S(Q)$$

is m_0-primary and hence df determines a regular sequence.

It is clear that the differential of the complex $K_i^{\bullet}$ coincides (up to the multiplication by a constant) with the exterior multiplication by df. Since

$$K_i^s = \overset{s}{\Lambda}(\Omega_S^1/f\Omega_S^1)$$

we may use the De Rham lemma and deduce

Corollary:

$$H^q(K_i^{\bullet}) = 0\,, \quad q \ne r+1\,, \quad \forall i \in \mathbb{Z}\,.$$

4.1.5. Proposition.

$$H^q(\overline{K}_i^\bullet) = 0 , \quad q \geq 0 , \quad i \in \mathbb{Z} .$$

Proof. The above corollary and exact sequence 4.1.3 imply that

$$H^q(\overline{K}_i^\bullet) = H^{q-1}(\overline{K}_{i-1}^\bullet[-1]) = H^{q-2}(\overline{K}_{i-1}^\bullet) , \quad q \leq r .$$

Since for $q < 0$ and $q > r$ $H^q(\overline{K}_i^\bullet) = 0$ we obtain the assertion of the proposition.

4.1.6. Define a graded A-module $\overline{\Omega}_A^i$ by the equality

$$\overline{\Omega}_A^i = \mathrm{Ker}(\overline{K}_i^i \to \overline{K}_i^{i+1}) = \mathrm{Ker}(\overline{\Omega}_S^i(N)/\overline{\Omega}_S^i \xrightarrow{d_f} \overline{\Omega}_S^{i+1}(2N)/\overline{\Omega}_S^{i+1}(N)) .$$

Then we deduce from 4.1.5 that the sequence of graded A-modules

$$0 \to \overline{\Omega}_A^i \to \overline{\Omega}_S^i(N)/\overline{\Omega}_S^i \to \overline{\Omega}_S^{i+1}(2N)/\overline{\Omega}_S^{i+1}(N) \to \cdots \to \overline{\Omega}_S^r((r-1)N)/\overline{\Omega}_S^r((r-i-1)N) \to 0$$

is a resolution of $\overline{\Omega}_A^i$.

Taking associated sheaves on $X = \mathrm{Proj}(A)$, we obtain the resolution of the sheaf $\underline{\overline{\Omega}}_A^i$:

$$0 \to \underline{\overline{\Omega}}_A^i \to \overline{\Omega}_{\mathbb{P}}^i(N)/\overline{\Omega}_{\mathbb{P}}^i \to \cdots \to \overline{\Omega}_{\mathbb{P}}^r((r-1)N)/\overline{\Omega}_{\mathbb{P}}^r((r-i-1)N) \to 0 .$$

4.1.7. We are almost at the goal. It remains to show that the sheaf $\underline{\overline{\Omega}}_A^i$ coincides with the sheaf $\tilde{\Omega}_X^i$ defined as in 2.2.4 by setting $\tilde{\Omega}_X^i = j_*(\Omega_U^i)$, where $U = X - \mathrm{Sing}(X)$.

Let Z be an open set of nonsingular points of X such that $\tilde{\Omega}_X^i | Z = \Omega_Z^i$ and $\tilde{\Omega}_{\mathbb{P}}^i \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z = \Omega_{\mathbb{P}}^i \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z$. We have the exact sequence of locally free sheaves

$$0 \to N_{X/\mathbb{P}} | Z \xrightarrow{d} \Omega_{\mathbb{P}}^1 \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z \to \Omega_Z^1 \to 0$$

where $N_{X/\mathbb{P}} | Z = \mathcal{O}_Z(-N)$ is the normal sheaf of $Z \to \mathbb{P}$.

This sequence determines exact sequences

$$0 \to \Omega_Z^i \to \Omega_{\mathbb{P}}^i(N) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z \xrightarrow{d} \Omega_Z^{i+1}(N)$$

which can be extended to the right to obtain the resolution

$$0 \to \Omega_Z^i \to \Omega_{\mathbb{P}}^i(N) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z \to \Omega_{\mathbb{P}}^{i+1}(2N) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z \to \cdots .$$

Since

$$\Omega_{\mathbb{P}}^{i+k}((1+k)N) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Z \triangleq \Omega_{\mathbb{P}}^{i+k}((1+k)N)/\Omega_{\mathbb{P}}^{i+k}(kN) | Z$$

we see that this resolution is the resolution of $\underline{\overline{\Omega}}_A^i$ (4.1.6) restricted on Z. Hence

$$\underline{\overline{\Omega}}_A^i | Z = \Omega_Z^i$$

and we obtain that

$$\overline{\Omega}^i_A = \tilde{\Omega}^i_X .$$

Thus, we have constructed the resolution of $\tilde{\Omega}^i_X$

$$0 \to \tilde{\Omega}^i_X \to \Omega^i_{\mathbb{P}}(N)/\Omega^i_{\mathbb{P}} \to \cdots \to \Omega^r_{\mathbb{P}}((r-i+1)N)/\Omega^r_{\mathbb{P}}((r-1)N) \to 0 .$$

4.2. The Griffiths theorem.

This theorem generalizes for weighted hypersurfaces a result of [11] and allows to calculate the cohomology $H^i(X,\tilde{\Omega}^i_X)$ as certain quotient spaces of differential forms on $\mathbb{P}$ with poles on X.

4.2.1. Denote by K^p the p^{th} component of the resolution of $\tilde{\Omega}^i_X$ from 4.1.7:

$$K^p = \overline{\Omega}^{i+p}_{\mathbb{P}}((p+1)N)/\overline{\Omega}^{i+p}_{\mathbb{P}}(pN) .$$

Using the exact sequence

$$0 \to \overline{\Omega}^{i+p}_{\mathbb{P}}(pN) \to \overline{\Omega}^{i+p}_{\mathbb{P}}((p+1)N) \to K^p \to 0$$

and the theorem of Bott-Steenbrink (2.3.4) we obtain that

$$H^q(X,K^p) = H^q(\mathbb{P},K^p) = 0 , \quad q > 0 , \quad p > 0$$

$$H^q(X,K^0) = H^q(\mathbb{P},K^0) = H^{q+1}(\mathbb{P},\overline{\Omega}^i_{\mathbb{P}}) = \begin{cases} k, & q=i-1 \\ 0, & q\neq i-1 \end{cases} .$$

Put

$$L = \mathrm{Ker}(K^1 \to K^2) .$$

Then we have the exact sequence of sheaves

$$0 \to \tilde{\Omega}^i_X \to K^0 \to L \to 0$$

which gives the exact cohomology sequence

$$\cdots \to H^{q-1}(X,L) \to H^q(X,\tilde{\Omega}^i_X) \to H^q(X,K^0) \to H^q(X,L) \to \cdots .$$

The sequence

$$0 \to L \to K^1 \to K^2 \to \cdots \to K^{r-i-1} \to 0$$

is an acyclic resolution of L. Thus we have

$$H^q(X,L) = H^q(\Gamma(X,K^{\cdot})) = 0 , \quad q > r-i-2$$

$$H^{r-i-2}(X,L) = \Gamma(X,K^{r-i-1})/\mathrm{Im}\Gamma(X,K^{r-i-2}) =$$

$$= \Gamma(\mathbb{P},\overline{\Omega}^r_{\mathbb{P}}((r-i)N)/\Gamma(\mathbb{P},\overline{\Omega}^r_{\mathbb{P}}((r-i-1)N)) + \mathrm{Im}\Gamma(\overline{\mathbb{P}},\overline{\Omega}^{r-1}_{\mathbb{P}}((r-i-1)N) .$$

4.2.2. Theorem (Weak Lefschetz theorem). The homomorphism

$$H^q(X,\tilde{\Omega}^i_X) \to H^q(X,K^0) \simeq H^{q+1}(\mathbb{P},\Omega^i_{\mathbb{P}})$$

is an isomorphism, if $q > r-i-1$ and an epimorphism if $q=r-i-1$.

Proof. Follows from the exact cohomology sequence 4.2.1 and the above calculation of $H^q(X,L)$.

Corollary. Assume that $k = \mathbb{C}$. Then we have an isomorphism of the Hodge structures

$$H^n(X,\mathbb{C}) \simeq H^{n+1}(\mathbb{P},\mathbb{C})$$

if $n \neq r-1$ and an epimorphism

$$H^{r-1}(X,\mathbb{C}) \to H^r(\mathbb{P},\mathbb{C}) .$$

For $n \geq r-1$ this directly follows from the theorem. For $n < r-1$ we use the Poincare duality for V-varieties (which are rational homology varieties).

Since the Hodge structure of $\mathbb{P}$ is known and very simple (2.3.6) we see that, as in the classic case, only cohomology $H^{r-1}(X)$ are interesting.

4.2.3. Put

$$h_0^{i,r-i-1}(X) = h^{i,r-i-1}(X) - a$$

where

$$a = \begin{cases} 1, & r = 2i \\ 0, & r \neq 2i . \end{cases}$$

Then we obtain that

$$h_0^{i,r-i-1}(X) = \dim_k H^{r-i-2}(X,L) .$$

Hence by calculations of 4.2.1 we obtain

Theorem (Griffiths-Steenbrink).

$$h_0^{i,r-i-1}(X) = \dim_k(\Gamma(\mathbb{P},\Omega^r_{\mathbb{P}}((r-i)N))/\Gamma(\mathbb{P},\Omega^r_{\mathbb{P}}((r-i-1)N)) + \operatorname{Im}\Gamma(\mathbb{P},\Omega^{r-1}_{\mathbb{P}}((r-i-1)N))) .$$

4.3. Explicit calculation.

4.3.1. Let

$$\theta_f = \left(\frac{\partial f}{\partial T_0},\dots,\frac{\partial f}{\partial T_r} \right)$$

be the jacobian ideal with respect to a generator $f \in S(Q)_N$ of the ideal of the affine quasicone C_X of a weighted quasismooth hypersurface $X \subset \mathbb{P}(Q)$.

By the Euler formula 2.1.2 each $\frac{\partial f}{\partial T_i}$ is a homogeneous element of $S(Q)$ of degree $N - q_i$. Hence the ideal θ_f is homogeneous and the quotient space $S(Q)/\theta_f$ has a natural gradation. Since θ_f is m_0-primary this quotient space is finite dimensional.

4.3.2. Theorem (Steenbrink). Assume that char(k) = 0. Then

$$h_0^{i,r-i-1}(X) = \dim_k(S(Q)/\theta_f)_{(r-i)N-|Q|} .$$

Proof. We have a natural isomorphism of graded S(Q)-modules:

$$\Omega_S^{r+1}/\Omega_S^r(-N)\wedge df = (S(Q)/\theta_f)(-|Q|) .$$

Since (see the proof of theorem 2.3.2)

$$\Gamma(\mathbb{P},\bar{\Omega}^i_{\mathbb{P}}(a)) = (\bar{\Omega}^i_S)_a , \quad \forall a \in \mathbb{Z}$$

and the differential $\Gamma(\mathbb{P},\Omega^i_{\mathbb{P}}(a)) \to \Gamma(\mathbb{P}, \Omega^{r+1}_{\mathbb{P}}(a+N))$ corresponds to the operator d_f from 4.1.1, we can reformulate theorem 4.2.3 in the following form:

$$h_0^{i,r-i-1}(X) = \dim_k(\bar{\Omega}^r_S/d_f\bar{\Omega}^{r-1}_S(-N) + f\bar{\Omega}^r_S(-N))_{(r-i)N} .$$

Thus it remains to construct an isomorphism of graded S-modules

$$\bar{\Omega}^r_S/d_f\bar{\Omega}^{r-1}_S(-N) + f\bar{\Omega}^r_S(-N) \cong \Omega^{r+1}_S/\Omega^r_S(-N)\wedge df .$$

By property (iii) of lemma 2.1.3, we obtain that the k-linear map

$$d : \bar{\Omega}^r_S \to \Omega^{r+1}_S$$

is in fact an isomorphism of graded S-modules (here we use that char(k) = 0!). By property (iii) of lemma 4.1.1, we set

$$d(d_f\bar{\Omega}^{r-1}_S(-N)) \subset \Omega^r_S(-N)\wedge df .$$

In fact, we have here an equality. Since d is S-linear it is sufficient to show that all forms

$$dx_{i_1}\wedge\ldots\wedge dx_{i_r}\wedge df \in d(d_g\bar{\Omega}^{r-1}_S(-N))) .$$

But

$$d(d_f(\Delta(dx_{i_1}\ldots dx_{i_r}))) = d(d_f(\sum_{s=1}^{r-1}(-1)^{s+1}x_{i_s}dx_{i_1}\wedge\ldots\wedge\widehat{dx_{i_s}}\wedge\ldots\wedge dx_{i_r})) =$$

$$= d(rfdx_{i_1}\wedge\ldots\wedge dx_{i_r} + (-1)^r a(\Sigma(-1)^{s+1}x_{i_s}dx_{i_1}\wedge\ldots\wedge\widehat{dx_{i_s}}\wedge\ldots\wedge dx_{i_r})\wedge df) =$$

$$= c\ dx_{i_1}\wedge\ldots\wedge dx_{i_r}\wedge df$$

where a,c are some rational numbers which we are too lazy to write down explicitly.

Thus d induces an isomorphism

$$\bar{\Omega}^r_S/d_f\bar{\Omega}^{r-1}_S(-N) \cong \Omega^{r+1}_S/\Omega^r_S(-N)\ df .$$

We have

$$d(f\Omega^r_S) \subset df\wedge\Omega^r_S + f\Omega^{r+1}_S .$$

But, since $f \in \theta_f$ (the Euler formula), $f\Omega^{r+1}_S \subset \Omega^r_S\wedge df$ and hence

$$d(f\Omega^r_S) \subset df\wedge\Omega^r_S .$$

Thus

$$f\overline{\Omega}_S^r(-N) \subset d_f\Omega_S^{r-1}(-N) \ ,$$

and we are through.

4.3.3. The theorem above can be reformulated in the following form. Define a function $\ell : \mathbb{Z}_{\geq 0}^{r+1} \to \mathbb{Z}$ by

$$\ell(a) = \frac{1}{N}\sum_{i=0}^{r}(a_i+1)q_i \ , \qquad a = (a_0,\dots,a_r) \in \mathbb{Z}_{\geq 0}^{r+1} \ .$$

Let $\{T^a\}_{a\in J}$ be a set of monomials of $S(Q)$ whose residues mod θ_f generate the basis of the space $S(Q)/\theta_f$ (such monomials are called basic monomials). Then

$$h_0^{i,r-i-1}(X) = \#\{a \in J : \ell(a) = r-i\} \ .$$

4.4. Examples and supplements.

4.4.1. Suppose $f(T_0,\dots,T_r) \in S$ is of the form

$$f(T_0,\dots,T_r) = T_r^N + g(T_0,\dots,T_{r-1}) \ .$$

If $(T^b)_{b\in J'}$, are basic nomomials for $g(T_0,\dots,T_{r-1})$ considered as elements of $S(q_0,\dots,q_{r-1})$, then the set $\{T^bT_r^{b_r} : b \in J' \ , \ 0 \leq b_r \leq N-2\}$ is the set of basic monomials for f.

This implies that

$$\begin{aligned} h_0^{i,r-i-1}(X) &= \#\{b \in J' : \ell(b) + \frac{b_r+1}{N} = r-i\} \\ &= \#\{b \in J' : r-i-1 < \ell(b) < r-i\} \ . \end{aligned}$$

This formula was obtained in [11] in the homogeneous case.

4.4.2. More generally, if

$$f(T_0,\dots,T_r) = g(T_0,\dots,T_{r-1}) + T_r^m$$

then

$$h_0^{i,r-i-1}(X) = \#\{b \in J' : r-i-1+\frac{m}{N} \leq \ell(b) \leq r-i-\frac{m}{N}\} \ .$$

For example, if g is homogeneous then X is a multiple space (3.5.4) and we obtain

$$h_0^{i,r-1-i}(X) = \#\{b \in \mathbb{Z}^r : (r-i-2)N+m \leq |b| \leq (r-i-1)N-m \ , \quad 0 \leq b_j \leq N-2\}$$

where $|b| = b_0 + \cdots + b_{r-1}$ (cf. III, 8.8).

This can be written in more explicit form

$$h_0^{i,r-i-1} = \sum_{s=(r-i-2)N-m}^{(r-i-1)N-m} c_s$$

where c_s is the coefficient at t^s in $(1+\cdots+t^{N-2})^r$.

4.4.3. Let $Y \subset \mathbb{P}(q_0,\ldots,q_{r-1}) = \mathbb{P}(Q')$ be the hypersurface defined by the polynomial $g(T_0,\ldots,T_r)$ from 4.4.1. Then

$$h_0^{i,r-i-2}(Y) = \#\{b \in J' : \ell(b) = r-i-1\} .$$

Assume now that $k = \mathbb{C}$. The exact sequence of Hodge structures

$$\cdots \to H^i(X) \to H^i(X-Y) \to H^{i-1}(Y)(-1) \to H^{i+1}(X) \to \cdots$$

(dual to the compact cohomology sequence) determines the morphisms of the Hodge structures:

$$i_* : H^{i-1}(Y)(-1) \to H^{i+1}(X)$$

which are obviously induced by the analogous morphisms

$$H^i(\mathbb{P}(Q'))(-1) \to H^{i+2}(\mathbb{P}(Q)) .$$

Applying the Weak Lefschetz theorem (4.2.2) we have that i_* is an isomorphism if $i \neq 0, r-1$. Thus for $U = X - Y$

$$H^i(U) = 0, \quad i \neq 0, r-1 .$$

The Hodge structure on $H^{r-1}(U)$ has the following form

$$Gr_i^W(H^{r-1}(U)) = 0, \quad i \neq r, r-1$$
$$Gr_r^W(H^{r-1}(U)) = H^{r-2}(Y)(-1)_0$$
$$Gr_{r-1}^W(H^{r-1}(U)) = H^{r-1}(H)_0 ,$$

where

$$H^{r-1}(X)_0 = \mathrm{Coker}(H^{r-3}(Y)(-1) \to H^{r-1}(X))$$
$$H^{r-2}(Y)_0 = \mathrm{Ker}(H^{r-2}(Y)(-1) \longrightarrow H^r(X)) .$$

For the Hodge numbers $h^{p,q}(U)$ we obtain (cf. [19])

$$h^{p,q}(U) = 0, \quad \text{if } p+q \neq r-1, r$$
$$h^{i,r-i-1}(U) = h_0^{i,r-i-1}(X) = \#\{b \in J' : r-i-1 < \ell(b) < r-i\}$$
$$h_0^{i,r-i}(U) = h_0^{i-1,r-i-1}(Y) = \#\{b \in J' : \ell(b) = r-i-1\}$$

where we recall that

$$J' = \{b \in \mathbb{Z}^r_{\geq 0} : T^b \text{ are basic monomials for } g(T_0,\ldots,T_{r-1})\}.$$

4.4.4. The calculations of 4.4.3 presents an interest since the open affine subset $U \subset X$ is isomorphic to the nonsingular affine variety in $\mathbb{A}^r$ with the equation

$$g(x_0,\ldots,x_{r-1}) = 1 .$$

This variety plays an important part in the theory of critical points of analytic functions. The cohomology space $H^{r-1}(U)$ is isomorphic to the space of the vanishing cohomology of the isolated critical point $0 \in C^r$ of the analytic function $t = g(z_0,\dots,z_{r-1})$ ([18]). Its dimension (the Milnor number)

$$\mu = \dim_C C[T_0,\dots,T_{r-1}]/\theta_g = \#J' .$$

It can be seen from above as follows:

$$\dim_C H^{r-1}(U) = \sum_{i=0}^{r-1} h_0^{i,r-i-1}(X) + \sum_{i=1}^{r-2} h_0^{i-1,r-i-1}(Y) =$$
$$= \#\{b \in J' : \ell(b) < r\}$$

and so we have to show that for any basic monomial T^b $\ell(b) < r$ or, equivalently,

$$\deg(T^b) < \sum_{i=0}^{r-1} (N-q_i) = rN - |Q'| .$$

Let

$$\mu_k = \#\{b \in J' : \deg(T^b) = k\}$$
$$\chi_g(z) = \sum_k \mu_k z^k .$$

Then ([1])

$$\chi_g(z) = \prod_{i=0}^{r-1} \frac{z^{N-q_i}-1}{z^{q_i}-1} .$$

It is clear that $\chi_g(z)$ is of degree $n = rN-2|Q'|$ and hence for $k > n$ $\mu_k = 0$. This proves the assertion above.

Note that the Hodge numbers of $H^{r-1}(U)$ can be expressed in terms of μ_k as follows

$$h^{i,r-i-1}(U) = \Sigma_{(r-i-1)N-|Q| < k < (r-i)N-|Q|} \mu_k$$
$$h^{i,r-i}(U) = \mu_{(r-i-1)N-|Q|} .$$

The symmetry of the Hodge numbers is in the accord with the symmetry of μ_k:

$$\mu_k = \mu_{n-i} .$$

4.4.5. Let $X = V_n(Q)$ be a quasismooth surface (r=3). We know that

$$h^{0,2}(X) = h^{2,0}(X) = \#\{a \in J : \ell(a) = 1\} = \mu_{N-|Q|}$$
$$h^{1,1}(X) = \#\{a \in J : \ell(a) = 2\} = \mu_{2N-|Q|}$$
$$b_2(X) = 2h^{2,0}(X) + h^{1,1}(X) = 2\mu_{N-|Q|} + \mu_{2N-|Q|} ,$$

where

$$\Sigma\mu_k z^k = \prod_{i=0}^{r} (z^{N-q_i}-1)/(z^{q_i}-1) .$$

It is clear that $\mu_{N-|Q|} = a_{N-|Q|}$ in notations of 3.4.4. In case $\mathbb{P}(Q) = \mathbb{P}^3$ we have

$$\Sigma\mu_k z^k = ((z^{N-1}-1)/(z-1))^4 = (1+z+\cdots+z^{N-2})^4 .$$

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